
Zbl 012.01004
Erdős, Pál
On the density of some sequences of numbers. (In English)
J. London Math. Soc. 10, 120-125 (1935).

Let $f(m)$ be a non-negative arithmetical function satisfying

$$f(m_1m_2) = f(m_1) + f(m_2) \text{ if } (m_1, m_2) = 1, \tag{1}$$

$$f(p_1) \neq f(p_2) \tag{2}$$

for two different primes p_1, p_2; and let

$$N(f; c, d) = \sum_{\substack{m \leq n \\ c \leq f(m) \leq d}} 1, \qquad N(f; c) = N(f; c, \infty).$$

The main result of this paper, which constitutes a wide generalization of the author's work on abundant numbers (Zbl 010.10303), is that $\lim_{n\to\infty} N(f; c)/n$ exists and is a continuous function of c. The case of the abundant numbers is obtained by taking $f(m) = \log \frac{\sigma(m)}{m}$, $c = \log 2$.
Suppose first $f(m)$ satisfies the more stringent conditions: (3) $f(p^\alpha) = f(p)$, (4) $\sum_p \frac{f(p)}{p}$ converges. Defining $f_p(m) = \sum_{p|m, p\leq P} f(p)$ it is easily seen that $\lim_{n\to\infty} N(f_p; c)/n = A_p$ exists, and as A_p in non-decreasing and ≤ 1, $\lim_{p\to\infty} A_p = A$ exists. That $\lim_{n\to\infty} N(f; c)/n = A$ follows easily from the two lemmas:
(I) For any $\varepsilon > 0$ there exists a δ such that $N(f; c, c+\delta) < \varepsilon n$ for all sufficiently large n;
(II) For any $\varepsilon, \delta > 0$ there exists a $P(\varepsilon, \delta)$ such that for $P > P(\varepsilon, \delta)$ and all n, the number of integers $m \leq n$ for which $f(m) - f_p(m) > \delta$ is less than εn. The main difficulty lies in the proof of (I), which uses the same idea as the paper already cited.
The author then sketches the proof when (3) is not assumed. As regards (4), he shows that it can be replaced by a weaker condition (4′) and that if (4′) does not hold, then $\lim_{n\to\infty} N(f; c)/n = 1$.

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Classification:
11N60 Distribution functions (additive and positive multipl. functions)