Zbl 070.29601

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On the number of zeros of successive derivatives of analytic functions. (In English. RU summary)

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Let $N_k(f(z),r)$ denote the number of zeros of $f^{(k)}(z)$ in $|z| \leq r < R$. The following theorems, which include and extend results of *G.Pólya* [Bull. Am. Math. Soc. 49, 178-191 (1943; Zbl 061.11510)] and *Evgrafov* (Interpolationsaufgabe von Abel-Gončarov, Moskau 1954) are proved.
Theorem 1. If $f(z)$ is regular in $|z| < 1$ and $0 < r < 1$, then $\lim_{k\to+\infty} k^{-1}N_k(f(z),r) \leq K(r)$, where $K(r)$ is the only positive root of $K = r(1+K)^{1+1/K}$.
Theorem 2. Let $g(r) \uparrow +\infty$ in $0 < r < +\infty$. Let $x = h(y)$ denote the inverse function of $y = g(x)$. Then, if $f(z)$ is an integral function which satisfies

$$\underline{\lim}_{r\to+\infty} \{g(r)\}^{-1} \log M(r) < 1,$$

we have

$$\underline{\lim}_{k\to+\infty} k^{-1}N_k(f(z),1)h(k) \leq e^2.$$

Theorem 3. If $f(z)$ is an integral function and z_k is the zero of $f^{(k)}(z)$ which is nearest the origin $(k = 1,2,3,...)$, and if $x = H(y)$ is the inverse function of $y = \log M(x)$, then $\underline{\lim}_{k\to+\infty}\{k|z_k|\}^{-1}H(k) \leq e(\log 2)^{-1}$.
Theorem 4. If $f(z)$ is regular in $|z| < R$ and is not a polynomial, then $\overline{\lim}_{k\to+\infty} k|z_k| \geq R\log 2$. The proof are based on Jensen's formula and Rouché's theorem.

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Classification:
30C15 Zeros of polynomials, etc. (one complex variable)