
Zbl 129.31402
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An elementary inequality between the probabilities of events (In English)
Math. Scand. 13, 90-104 (1963). [0025-5521]

The reviewer (Zbl 064.13005) has proved that for any n events $A_1, A_2, ..., A_n$ such that $\Pr(A_i) = \omega_1$ for $i = 1, 2, ..., n$ and $\Pr(A_i \cap A_j) = \omega_2$ for $i \neq j$ we have the inequality

$$(1) \qquad \omega_2 \geq \omega_1^2 - \frac{\omega_1(1-\omega_1)}{n-1} + \frac{(n\omega_1 - [n\omega_1])(1 - n\omega_1 + [n\omega_1])}{n(n-1)}$$

with $[n\omega_1]$ denoting the integral part of $n\omega_1$, and that this inequality is an equality for some collection of events $A_1, A_2, ..., A_n$ whatever ω_1 and n.
Here the authors consider the closely related more general problem of the determination, for any natural n and $a \in (0,1)$, of the constant $\varepsilon_n(\alpha)$ defined as the least real number ε such that for any collection of events $A_1, A_2, ..., A_n$ subject to the only condition (2) $\Pr(A_i \cap A_j) \leq \alpha^2$ for $i \neq j$ we have the inequality $\sum \Pr(A_i) \leq n\alpha + \varepsilon$. With ν denoting the largest integer such that $\nu(\nu-1) \leq n(n-1)\alpha^2$ the constant sought for is found to be given by

$$\varepsilon_n(\alpha) = \frac{1}{2}(1-\alpha) + (n\alpha - \nu)((n-1)\alpha - \nu)/2\nu.$$

The second term in this formula vanishes if $n\alpha$ or $(n-1)\alpha$ is an integer; otherwise for $n \to \infty$ it is of the order of $1/n$. An explicit extremal collection of events $A_1, A_2, ..., A_n$ is constructed in the case of $\alpha = \frac{1}{2}$ and $n \equiv 3 \pmod 4$ by the use of the method of quadratic residues.

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Classification:
60C05 Combinatorial probability
60E05 General theory of probability distributions