Zbl 346.10004
Erdős, Paul; Graham, Ronald L.
On products of factorials. (In English)
Bull. Inst. Math., Acad. Sinica 4, 337-355 (1976).

In this paper products of factorials are studied. Let $a_1 < a_2 < \ldots < a_t = n$ be positive integers. First it is proved that the number of distinct integers of the form $\prod_{k=1}^t a_k!$ is

$$\exp\{(1+0(1))n \log\log n/\log n\}.$$

The main part of the paper is devoted to the Diophantine equation

$$(*) \qquad \prod_{k=1}^{t} a_k! = y^2,$$

where t is fixed. Put $F(t) = \{n \mid (*) \text{ is solvable } \}$ and $D(t) = F(t) - F(t-1)$. It is shown that $D(1) = \{1\}$, $D(2) = \{n^2 : n > 1\}$, $D(3) \neq \emptyset$, $D(3)$ has density 0, $D(4)$ has positive lower density, $D(5) \neq \emptyset$, 527 is the smallest element of $D(6)$, $D(t) = \emptyset$ for $t > 6$. Numerous other results are given. For example, $n \in F(6)$ if and only if n is composite. If $n = m^2 r$ with $m > 1$, then $n \in F(4)$. If $p \leq 11$ is a proper prime divisor of n, then $n \in F(5)$. For almost all primes p one has 13 $p \notin F(5)$. Some results from prime number theory are used in the proofs and quite a few open problems are mentioned.

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Classification:
11A41 Elemementary prime number theory
11D57 Multiplicative and norm form diophantine equations
11N05 Distribution of primes