Zbl 359.10038
Alladi, K.; Erdős, Paul
On an additive arithmetic function. (In English)
Pac. J. Math. 71, 275-294 (1977). [0030-8730]

Let n be a positive integer, $n = \prod_{i=1}^{r} p_i^{\alpha_1}$ in canonical form, and let $A(n) = \sum_{i=1}^{r} \alpha_1 p_1$. Clearly A is an additive arithmetic function. Assume the primes p_1 are arranged so that $p_1 \leq p_2 < \cdots < p_r$. Define $P_1(n) = p_r$ and, in general, $P_k(n) = P_1(n/P_1(n) \dots P_{k-1}(n))$ for $k \leq \sum_{i=1}^{r} \alpha_1$ and $P_k(n) = 0$ for $k > \sum_{i=1}^{r} \alpha_1$. If f and g are arithmetic functions such that $\sum_{n \leq x} f(n) \sim \sum_{n<x} g(n)$, and if g is a well behaved function (e.g. polynomial, exponential), then g is referred to as the average order of f. It is proved that for all positive integers, we have

$$\sum_{n \leq x} P_m(n) \sim \sum_{n \leq x} \{A(n) - P_1(n) - \cdots - P_{m-1}(n)\} \sim k_m x^{1+(1/m)}/(\log x)_m$$

where k_m is a positive constant depending only on m. It follows almost immediately from this theorem that the average order of $A(n)$ is $\pi_2 n/6 \log n$. Let $A^*(n) = \sum_{i=1}^{r} p_1$. Then the average order of $A^*(n)$ is also $\pi^2 n/6 \log n$, and the average order of $A(n) - A^*(n)$ is $\log \log n$. For any fixed positive integer M, the set of solutions to $A(n) - A^*(n) = M$ has a positive natural density. Now $A(n) = n$ if and only if n is a prime or $n = 4$. Call n a "special number" if $n \equiv O(\mod A(n))$ and $n \neq A(n)$, and let $\{l_n\}$ be the sequence of special numbers. This paper's first author has previously proved that the sequence $\{l_n\}$ is infinite [Srinivasa Ramanujan Commemoration Volume, Oxford Press, Madras, India, (1974) part II] . Denote by $\mathcal{L}(x)$ the number of $\ell_n \leq x$. It is shown that there exist positive constants c, c' such that

$$\mathcal{L}(x) = O(xe^{-c\sqrt{\log x \log \log x}}) \text{ and } \mathcal{L}(x) \gg xe^{-c'\sqrt{\log x \log \log x}}.$$

Finally, let $\alpha(n) = (-1)^{A(n)}$. It is proved that there exists a positive constant c'' such that

$$\sum_{1 \leq n \leq x} \alpha(n) = O(xe^{-c''\sqrt{\log} x \log \log x}),$$

and that $\sum_{n=1}^{\infty} \alpha(n)/n = 0$.

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Classification:
11N37 Asymptotic results on arithmetic functions
11K65 Arithmetic functions (probabilistic number theory)