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Turán-Ramsey theorems and simple asymptotically extremal structures. (In English)

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Let $L_1, L_2, \ldots, L_r$ be given graphs ("forbidden" graphs), and let n be a positive integer and f a given function; $\alpha(G)$ denotes the maximum number of independent vertices in a graph G; $\mathrm{RT}(n, L_1, L_2, \ldots, L_r, f(n))$ denotes the maximum number of edges in a graph G_n on n vertices, having $\alpha(G_n) \leq f(n)$, whose edges may be coloured in r colours so that the subgraph of the ith colour contains no L_i $(i = 1, 2, \ldots, r)$; the results of this paper generally apply to the case $f(n) = o(n)$, and the maximum is usually denoted by $\mathrm{RT}(n, L_1, L_2, \ldots, L_r, o(n))$. A sequence (S_n) of graphs for which $\alpha(S_n) \leq f(n)$ and S_n has $\mathrm{RT}(n, L_1, L_2, \ldots, L_r, f(n)) + o(n^2)$ edges, is asymptotically extremal for $\mathrm{RT}(n, L_1, L_2, \ldots, L_r, f(n))$ if the edges of S_n may be r- coloured so that the subgraph of the ith colour contains no L_i $(i = 1, \ldots, r)$. In Theorem 2 a construction of *B. Bollobás* and *P. Erdős* [On a Ramsey-Turán type problem, J. Comb. Theory, Series B 21, 166-168 (1976; Zbl 337.05134)] used to prove that $\mathrm{RT}(n, K_4, o(n)) \geq \frac{1}{8}n^2 - o(n^2)$ is generalized to prove the existence of a sequence of graphs that is asymptotically extremal for $\mathrm{RT}(n, K_{k_1}, K_{k_2}, \ldots, K_{k_r}, o(n^2))$, where $k_1, k_2, \ldots, k_r$ are integers each exceeding 2. Let $\vartheta(L_1, L_2, \ldots, L_r)$ denote the minimum real number such that $\mathrm{RT}(n, L_1, L_2, \ldots, L_r, f(n)) \leq \vartheta(L_1, L_2, \ldots, L_r)n^2 + o(n^2)$; in Theorem 3 the values of $\vartheta(K_3, K_3)$, $\vartheta(K_3, K_4)$, $\vartheta(K_3, K_5)$, $\vartheta(K_4, K_4)$ are determined, as well as an asymptotically extremal sequence for each case; it is shown that the distance between two such sequences - - i.e. the minimum number of edge additions/deletions needed to transform one such sequence into another – is $o(n^2)$ in each case. Theorem 4: If p and q are odd integers, then $\mathrm{RT}(n, C_p, C_q, o(n)) = \frac{1}{4}n^2 + o(n^2)$.
The paper concludes with a list of open problems.

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05C55 Generalized Ramsey theory
05C38 Paths and cycles

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