Introduction

It is difficult to estimate the relative impact of Erdős' research in different areas of mathematics. But it is a fact that Erdős started with number theory (e.g., out of his first 60 papers only 2 are not related to number theory) and that among his publications, the number theory papers have highest frequency. His achievements are well known and are amply mirrored by contributions to this chapter (which is the largest of all the chapters of these volumes).

The papers by Ahlswede and Khachatrian, Konyagin and Pomerance, Nathanson, Nicolas, Schinzel, Shorey and Tijdeman, Sárközy and Sós, and Tenenbaum survey and relate to various parts of Erdős' research, and they complement in various respects his own recollections in Chapter 1.

Some of these papers are research articles, such as the papers by Ahlswede and Cai, Sárközy, Tenenbaum and Bergelson et al. (which includes Erdős himself as a coauthor).

Although we believe this is a representative sample of Erdős' activities in this area, many problems and particular research directions are not covered. The reader should bear in mind that Erdős himself considered the probabilistic methods in number theory together with his work on prime numbers as his main contributions to number theory. Probabilistic methods are covered by the next section as well. But we cannot resist to close this introduction with a few more recent Erdős problems in his own words:

Here is a purely computational problem (this problem cannot be attacked by other means at present). Call a prime p *good* if every even number $2r \leq p-3$ can be written in the form $q_1 - q_2$ where $q_1 \leq p$, $q_2 \leq p$ are primes. Are there infinitely many good primes? The first bad prime is 97 I think. Selfridge and Blecksmith have tables of the good primes up to 10^{37} at least, and they are surprisingly numerous.

I proved long ago that every $m < n!$ is the distinct sum of $n-1$ or fewer divisors of $n!$. Let $h(m)$ be the smallest integer, if it exists, for which every integer less than m is the distinct sum of $h(m)$ or fewer divisors of m. Srinivasan called the numbers for which $h(m)$ exists *practical*. It is well known and easy to see that almost all numbers m are not practical. I conjectured that there is a constant $c \geq 1$ for which for infinitely many m we have $h(m) <$

$(\log\log m)^c$. M. Vose proved that $h(n!) < cn^{1/2}$. Perhaps $h(n!) < c(\log n)^{c_2}$. I would be very glad to see a proof of $h(n!) < n^\epsilon$.

A practical number n is called a *champion* if for every $m > n$, we have $h(m) > h(n)$. For instance, 6 and 24 are champions, as $h(6) = 2$, the next practical number is 24, $h(24) = 3$, and for every $m > 24$, we have $h(m) > 3$. It would be of some interest to prove some results about champions. A table of the champions $< 10^6$ would be of some interest. I conjecture that $n!$ is not a champion for $n > n_0$.

The study of champions of various kinds was started by Ramanujan (Highly composite numbers, *Collected Papers of Ramanujan*). See further my paper with Alaoglu on highly composite and similar numbers and many papers of J. L. Nicolas and my joint papers with Nicolas.

The following related problem is perhaps of some mild interest, in particular, for those who are interested in numerical computations. Denote by $g_r(n)$ the smallest integer which is not the distinct sum of r or fewer divisors of n. A number n is an r-champion if for every $t < n$ we have $g_r(n) > g_r(t)$. For $r = 1$ the least common multiple M_m of the integers $\leq m$ is a champions for any m, and these are all the 1-champions. Perhaps the M_m are r-champions too, but there are other r-champions; e.g., 18 is a 2-champion.

Let $f_k(n)$ be the largest integer for which you can give $f_k(n)$ integers $a_i \leq n$ for which you cannot find $k+1$ of them which are relatively prime. I conjectured that you get $f_k(n)$ by taking the multiple $\leq n$ of the first k primes. This has been proved for small k by Ahlswede, and Khachatrian disproved it for $k \geq 212$. Perhaps if $n \geq (1+\epsilon)p_k^2$, where p_k is the kth prime, the conjecture remains true.

Let $n_1 < n_2 < \ldots$ be an arbitrary sequence of integers. Besicovitch proved more than 60 years ago that the set of the multiples of the n_i does not have to have a density. In those prehistoric days this was a great surprise. Davenport and I proved that the set of multiples of the $\{n_i\}$ have a logarithmic density and the logarithmic density equals the lower density of the set of multiples of the $\{n_i\}$. Now the following question is perhaps of interest: Exclude one or several residues mod n_i (where only the integers $\geq n_i$ are excluded). Is it true that the logarithmic density of the integers which are not excluded always exists? This question seems difficult even if we only exclude one residue mod n_i for every n_i.

For a more detailed explanation of these problems see the excellent books by Halberstam and Roth, *Sequences*, Springer-Verlag, and by Hall and Tenenbaum, *Divisors*, Cambridge University Press.

Tenenbaum and I recently asked the following question: let $n_1 < n_2 < \ldots$ be an infinite sequence of positive integers. Is it then true that there always is a positive integer k for which almost all integers have a divisor of the form $n_i + k$? In other words, the set of multiples of the $n_i + k$ $(1 \leq i < \infty)$ has density 1. Very recently Ruzsa found a very ingenious counterexample.

Tenenbaum thought that perhaps for every $\epsilon > 0$ there is a k for which the density of the multiples of the $n_i + k$ has density $> 1 - \epsilon$.

In a paper (Proc. London Math. Soc. (1970) dedicated to the memory of Littlewood) Sárközy and I state the following problem: Let $1 \le a_i < a_2 < \cdots < a_{n+2} \le 3n$ be $n+2$ integers. Prove that there always are three of them $a_i < a_j < a_k$ for which $a_j + a_k \equiv 0 \pmod{a_i}$. The integers $2n \le t \le 3n$ show that $n+1$ integers do not suffice.

Perhaps a proof or disproof will be easy. As far as I know, the problem has been rather forgotten.

Many more problems are contained in the book P. Erdős and R. L. Graham, Old and New Problems and Results in Combinatorial Number Theory, the second edition of which should appear soon.