# Introduction

Erdős' work in graph theory started early and arose in connection with D. König, his teacher in prewar Budapest. The classic paper of Erdős' and Szekeres from 1935 also contains a proof in "graphotheoretic terms." The investigation of the Ramsey function led Erdős to probabilistic methods and seminal papers in 1947, 1958 and 1960. It is perhaps interesting to note that three other very early contributions of Erdős' to graph theory (before 1947) were related to infinite graphs: infinite Eulerian graphs (with Gallai and Vászoni) and a paper with Kakutani On nondenumerable graphs (1943). Although the contributions of Erdős to graph theory are manifold, and he proved (and always liked) beautiful structural results such as the Friendship Theorem (jointly with V. T. Sós and Kövári), and compactness results (jointly with N. G. de Bruijn), his main contributions were in asymptotic analysis, probabilistic methods, bounds and estimates. Erdős was the first who brought to graph theory the experience and rigor of number theory (perhaps being preceded by two papers by V. Jarnik, one of his early coauthors). Thus he contributed in an essential way to lifting graph theory up from the "slums of topology." This chapter contains a "special" problem paper not by Erdős but by his frequent coauthors from Memphis: R. Faudree, C. C. Rousseau and R. Schelp (well, there is actually an Erdős supplement there as well). We encouraged the authors to write this paper and we are happy to include it in this volume. This chapter also includes two papers coauthored by Béla Bollobás, who is one of Erdős' principal disciples. Bollobás contributed to much of Erdős' combinatorial activities and wrote important books about them. (*Extremal Graph Theory, Introduction to Graph Theory, Random Graphs*). His contributions to this chapter (coauthored with his two former students G. Brightwell and A. Thomason) deal with graphs (and thus are in this chapter) but they by and large employ random graph methods (and thus they could be contained in Chapter 3). The main questions there may be considered as extremal graph theory questions (and thus they could fit in Chapter 5). Other contributions to this chapter, which are related to some aspect to Erdős' work or simply pay tribute to him are by N. Alon, Z. Füredi, M. Aigner and E. Triesch, S. Bezrukov and K. Engel, A. Gyárfás, S. Brandt, N. Sauer and H. Wang, H. Fleischner and M. Stiebitz, and D. Beaver, S. Haber and P. Winkler.

Perhaps the main love of Erdős in graph theory is the chromatic number. Let us close this introduction with a few of Erdős' recent problems related mostly to this area in his own words:

Many years ago I proved by the probability method that for every $k$ and $r$ there is a graph of girth $\geq r$ and chromatic number $\geq k$. Lovász when he was still in high school found a fairly difficult constructive proof. My proof still had the advantage that not only was the chromatic number of $G(n)$ large but the largest independent set was of size $< \epsilon n$ for every $\epsilon > 0$ if $n > n_0(\epsilon, r, k)$. Nešetřil and V. Rödl later found a simpler constructive proof.

There is a very great difference between a graph of chromatic number $\aleph_0$ and a graph of chromatic number $\geq \aleph_1$. Hajnal and I in fact proved that if $G$ has chromatic number $\aleph_1$ then $G$ must contain a $C_4$ and more generally $G$ contains the complete bipartite graph $K(n, \aleph_1)$ for every $n < \aleph_0$. Hajnal, Shelah and I proved that every graph $G$ of chromatic number $\aleph_1$ must contain for some $k_0$ every odd cycle of size $\geq k_0$ (for even cycles this was of course contained in our result with Hajnal), but we observed that for every $k$ and every $m$ there is a graph of chromatic number $m$ which contains no odd cycle of length $< k$. Walter Taylor has the following very beautiful problem: Let $G$ be any graph of chromatic number $\aleph_1$. Is it true that for every $m > \aleph_1$ there is a graph $G_m$ of chromatic number $m$ all finite subgraphs of which are contained in $G$? Hajnal and Komjáth have some results in this direction but the general conjecture is still open. If it would have been my problem, I certainly would offer 1000 dollars for a proof or a disproof. (To avoid financial ruin I have to restrict my offers to my problems.)

Let $k$ be fixed and $n \to \infty$. Is it true that there is an $f(k)$ so that if $G(n)$ has the property that for every $m$ every subgraph of $m$ vertices contains an independent set of size $m/2 - k$ then $G(n)$ is the union of a bipartite graph and a graph of $\leq f(k)$ vertices, i.e., the vertex set of $G(n)$ is the union of three disjoint sets $S_1$, $S_2$ and $S_3$ where $S_1$ and $S_2$ are independent and $|S_3| \leq f(k)$. Gyárfás pointed out that even the following special case is perhaps difficult. Let $m$ be even and assume that every $m$ vertices of our $G(n)$ induces an independent set of size at least $m/2$. Is it true then that $G(n)$ is the union of a bipartite graph and a bounded set? Perhaps this will be cleared up before this paper appears, or am I too optimistic?

Hajnal, Szemerédi and I proved that for every $\epsilon > 0$ there is a graph of infinite chromatic number for which every subgraph of $m$ vertices contains an independent set of size $(1 - \epsilon)m/2$ and in fact perhaps $(1 - \epsilon)m/2$ can be replaced by $m/2 - f(m)$ where $f(m)$ tends to infinity arbitrarily slowly. A result of Folkman implies that if $G$ is such that every subgraph of $m$ vertices contains an independent set of size $m/2 - k$ then the chromatic number of $G$ is at most $2k + 2$.

Many years ago Hajnal and I conjectured that if $G$ is an infinite graph whose chromatic number is infinite, then if $a_1 < a_2 < \ldots$ are the lengths of the odd cycles of $G$ we have

$$\sum_i \frac{1}{a_i} = \infty$$

and perhaps $a_1 < a_2 < \ldots$ has positive upper density. (The lower density can be 0 since there are graphs of arbitrarily large chromatic number and girth.)

We never could get anywhere with this conjecture. About 10 years ago Mihók and I conjectured that $G$ must contain for infinitely many $n$ cycles of length $2^n$. More generally it would be of interest to characterize the infinite sequences $A = \{a_1 < a_2 < \ldots\}$ for which every graph of infinite chromatic number must contain infinitely many cycles whose length is in $A$. In particular, assume that the $a_i$ are all odd.

All these problems we unattackable (at least for us). About three years ago Gyárfás and I thought that perhaps every graph whose minimum degree is $\geq 3$ must contain a cycle of length $2^k$ for some $k \geq 2$. We became convinced that the answer almost surely will be negative but we could not find a counterexample. We in fact thought that for every $r$ there must be a $G_r$ every vertex of which has degree $\geq r$ and which contains no cycle of length $2^k$ for any $k \geq 2$. The problem is wide open.

Gyárfás, Komlós and Szemerédi proved that if $k$ is large and $a_1 < a_2 < \ldots$ are the lengths of the cycles of a $G(n, kn)$, that is, an $n$-vertex graph with $kn$ edges, then

$$\sum \frac{1}{a_i} > c \log n \ .$$

The sum is probably minimal for the complete bipartite graphs.

(Erdős-Hajnal) If $G$ has large chromatic number does it contain two (or $k$ if the chromatic number is large) edge-disjoint cycles having the same vertex set? It surely holds if $G(n)$ has chromatic number $> n^\epsilon$ but nothing seems to be known.

Fajtlowicz, Staton and I considered the following problem (the main idea was due to Fajtlowicz). Let $F(n)$ be the largest integer for which every graph of $n$ vertices contains a regular induced subgraph of $\geq F(n)$ vertices. Ramsey's theorem states that $G(n)$ contains a trivial subgraph, i.e., a complete or empty subgraph of $c \log n$ vertices. (The exact value of $c$ is not known but we know $1/2 \leq c \leq 2$.) We conjectured $F(n)/\log n \to \infty$. This is still open. We observed $F(5) = 3$ (since if $G(5)$ contains no trivial subgraph of 3 vertices then it must be a pentagon). Kohayakawa and I worked out the $F(7) = 4$ but the proof is by an uninteresting case analysis. (We found that this was done earlier by Fajtlowicz, McColgan, Reid and Staton, see Ars Combonatoria vol 39.) It would be very interesting to find the smallest integer $n$ for which $F(n) = 5$, i.e., the smallest $n$ for which every $G(n)$ contains a regular induced subgraph of $\geq 5$ vertices. Probably this will be much more difficult than the proof of $F(7) = 4$ since in the latter we could use properties of perfect graphs. Bollobás observed that $F(n) < c\sqrt{n}$ for some $c > 0$.

Let $G(10n)$ be a graph on $10n$ vertices. Is it true that if every index subgraph of $5n$ vertices of our $G(10n)$ has $\geq 2n^2 + 1$ edges then our $G(10n)$

contains a triangle? It is easy to see that $2n^2$ edges do not suffice. A weaker result has been proved by Faudree, Schelp and myself at the Hakone conference (1992, I believe) see also a paper by Fan Chung and Ron Graham (one of the papers in a volume published by Bollobás dedicated to me).

A related forgotten conjecture of mine states that if our $G(10n)$ has more than $20n^2$ edges and every subgraph of $5n$ vertices has $\geqq 2n^2$ edges then our graph must have a triangle. Simonovits noticed that if you replace each vertex of the Petersen graph by $n$ vertices you get a graph of $10n$ vertices, $15\ n^2$ edges, no triangle and every subgraph of $5n$ vertices contains $\geq 2n^2$ edges.