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AN EXTENSION THEOREM FOR SUPERTEMPERATURES

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Abstract. We present an analogue for supertemperatures of a well-known extension theorem on superharmonic functions.

1. Introduction

We call solutions of the heat equation *temperatures*, and the corresponding supersolutions *supertemperatures*. See [4] and [5] for details. The purpose of this paper is to present an analogue for supertemperatures of the following superharmonic function extension theorem.

Let K be a compact subset of $\mathbf{R}^n$ such that $\mathbf{R}^n\backslash K$ is connected. If u is superharmonic on some open superset of K, then there exists a superharmonic function $\bar{u}$ on $\mathbf{R}^n$ such that $\bar{u} = u$ on a neighbourhood of K.

This result can be found in [1], p. 192.

For the case of supertemperatures on open subsets of $\mathbf{R}^{n+1}$, the condition that the complement of K be connected is still necessary, but is no longer sufficient, as the following example shows.

We need some notation. If $p = (x, t)$ and $p_0 = (x_0, t_0)$ are two points in $\mathbf{R}^n \times \mathbf{R}$, we put

$$W(p_0, p) = (4\pi(t_0 - t))^{-\frac{n}{2}} \exp\left(-\frac{\|x_0 - x\|^2}{4(t_0 - t)}\right)$$

if $t_0 > t$, and $W(p_0, p) = 0$ if $t_0 \le t$.

Example. Let

$$K = \{(x, t) \in \mathbf{R}^n \times \mathbf{R} : \|x\|^2 + t^2 = 1,\ t \le 1/2\}$$

be the part of boundary of the unit ball (centred at the origin) where $t \le 1/2$. Put

$$u(p) = -W(p, 0) \quad \text{for all} \quad p \in \mathbf{R}^{n+1}.$$

Then u is a temperature on $\mathbf{R}^{n+1}\backslash\{0\}$, which is an open superset of K. Suppose that there is a supertemperature $\bar{u}$ on $\mathbf{R}^{n+1}$ such that $\bar{u} = u$ on an open superset D

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of K. Then the function $v = \bar{u} - u$ is a supertemperature on $\mathbf{R}^{n+1}$, and is identically zero on D. Consider v on the set

$$E = \{(x,t) \in \mathbf{R}^{n+1} : \|x\|^2 + t^2 < 1,\ t < 1/2\}.$$

Since $v \equiv 0$ on K, the boundary minimum principle shows that $v \geq 0$ on E. Since D is an open superset of K, we can find a point $p_0 = (x_0, t_0) \in E$ such that $v(p_0) = 0$ and $t_0 > 0$. Now the strong minimum principle implies that $v \equiv 0$ on $E_0 = \{(x,t) \in E : t < t_0\}$, an open set containing the origin. So $\bar{u} = u$ on E_0. But $\bar{u}$ is bounded below on E_0, whereas u is unbounded below, so we have a contradiction.

Before describing our theorem, we collect together the various pieces of notation needed for the remainder of this note. See [4] and [5] for details of these concepts.

The *heat ball* $\Omega(p_0; c)$ is defined for $c > 0$ by

$$\Omega(p_0; c) = \{p \in \mathbf{R}^{n+1} : W(p_0, p) > (4\pi c)^{-\frac{n}{2}}\}.$$

We shall write $\tau(c)$ for $(4\pi c)^{-\frac{n}{2}}$. We shall use the characteristic surface mean values of supertemperatures. For each $x \in \mathbf{R}^n$ and $t > 0$, we put

$$Q(x,t) = \|x\|^2(4\|x\|^2t^2 + (\|x\|^2 - 2nt)^2)^{-1/2}.$$

Then the mean value is defined by

$$\mathscr{M}(u; x_0, t_0; c) = \tau(c) \int_{\partial\Omega(x_0,t_0;c)} Q(x_0 - x, t_0 - t) u(x,t)\, d\sigma$$

for any function u such that the integral exists. Here σ denotes surface area measure.

If E is an open set in $\mathbf{R}^{n+1}$ and $p_0 \in E$, we denote by $\Lambda(p_0, E)$ (respectively $\Lambda^*(p_0, E)$) the set of all points $p \in E \backslash \{p_0\}$ that can be joined to p_0 by a polygonal line in E along which the temporal variable t is strictly increasing (respectively decreasing) as the line is described from p to p_0. In particular, if $B = B(p_0, r)$ is an open ball with centre $p_0 = (x_0, t_0)$ and radius $r > 0$, then $\Lambda(p_0, B)$ is the open half-ball

$$\{(x,t) : \|x - x_0\|^2 + (t - t_0)^2 < r^2,\ t < t_0\}.$$

Furthermore, $\Lambda^*(p_0, \mathbf{R}^{n+1}) = \mathbf{R}^n \times]t_0, \infty[$.

If $q \in \partial E$, and there is an open ball $B = B(q, \epsilon)$ such that $\Lambda(q, B) \subseteq E$, we call q an *abnormal boundary point of* E, and write $q \in \mathrm{ab}(\partial E)$. If ϵ can be chosen so that $\Lambda(q, B) = B \cap E$, then we call q an *abnormal boundary point of the first kind*, and write $q \in \mathrm{ab}_1(\partial E)$. Otherwise, we call q an *abnormal boundary point of the second kind*, and write $q \in \mathrm{ab}_2(\partial E)$. We also put $\mathrm{n}(\partial E) = (\partial E) \backslash \mathrm{ab}(\partial E)$, and call its elements *normal boundary points of* E. The set $\mathrm{ess}(\partial E)$, defined by $\mathrm{ess}(\partial E) = \mathrm{n}(\partial E) \cup \mathrm{ab}_2(\partial E)$, is called the *essential boundary of* E, and is the part of the boundary that is relevant when using the minimum principle, or when considering the Dirichlet problem.

The definition of $\Lambda(p_0, E)$ can be extended in an obvious way to the case where $p_0 \in \mathrm{ab}(\partial E)$. The definition of $\Lambda^*(p_0, E)$ can be extended in a similar way.

If E is a bounded open set, and f is a continuous real-valued function on $\mathrm{ess}(\partial E)$, then there is a unique temperature on E that is associated to f by the PWB method.

It is denoted by H_f^E, and is called the *Dirichlet solution for f on E*. We use the concept of Dirichlet solution in [5] because we need it to be aligned with the strongest form of the boundary minimum principle, also given in [5].

2. The theorem

So a stronger condition than the connectedness of $\mathbf{R}^{n+1}\backslash K$ is required in the present case. To motivate our condition, we first re-write the condition of connectedness of $\mathbf{R}^n\backslash K$ for the superharmonic case. Given x_0 in an open set D, let $\Gamma(x_0, D)$ denote the component of D that contains x_0. Then obviously $K \subseteq \mathbf{R}^n = \Gamma(x_0, \mathbf{R}^n)$, and $\mathbf{R}^n\backslash K$ is connected if and only if there is a point $x_0 \in \mathbf{R}^n\backslash K$ such that $\Gamma(x_0, \mathbf{R}^n\backslash K) = \Gamma(x_0, \mathbf{R}^n)\backslash K$.

Replacing Γ by Λ^* (introduced above), we get the required condition.

Definition. Let K be a compact subset of $\mathbf{R}^{n+1}$. If there is a point p_0 in $\mathbf{R}^{n+1}\backslash K$ such that $K \subseteq \Lambda^*(p_0, \mathbf{R}^{n+1})$ and $\Lambda^*(p_0, \mathbf{R}^{n+1}\backslash K) = \Lambda^*(p_0, \mathbf{R}^{n+1})\backslash K$, then we say that $\mathbf{R}^{n+1}\backslash K$ is *monotonically connected to* p_0.

In general, if $p \in \Lambda^*(p_0, \mathbf{R}^{n+1}\backslash K)$, then $p \in \mathbf{R}^{n+1}\backslash K$ and can be joined to p_0 by a polygonal path in $\mathbf{R}^{n+1}\backslash K$ along which the temporal variable is strictly decreasing. So $p \in \Lambda^*(p_0, \mathbf{R}^{n+1})\backslash K$, and we have the inclusion

$$\Lambda^*(p_0, \mathbf{R}^{n+1}\backslash K) \subseteq \Lambda^*(p_0, \mathbf{R}^{n+1})\backslash K.$$

Equality may fail to hold. If K is as in the above Example, and p_0 is any point such that $K \subseteq \Lambda^*(p_0, \mathbf{R}^{n+1})$, then

$$\Lambda^*(p_0, \mathbf{R}^{n+1}\backslash K) = \Lambda^*(p_0, \mathbf{R}^{n+1})\backslash \bar{E} \subset \Lambda^*(p_0, \mathbf{R}^{n+1})\backslash K.$$

Hence $\mathbf{R}^{n+1}\backslash K$ is not monotonically connected to any point p_0.

Theorem. *Let K be a compact subset of an open set E.*

(a) *If $\mathbf{R}^{n+1}\backslash K$ is monotonically connected to some point, then for each supertemperature u on E there is a lower bounded supertemperature $\bar{u}$ on $\mathbf{R}^{n+1}$ such that $\bar{u} = u$ on a neighbourhood U of K. Furthermore, $\bar{u}$ can be chosen to be the potential of a measure supported in $\bar{U}$, plus a constant.*

(b) *If $\mathbf{R}^{n+1}\backslash K$ is not monotonically connected to any point, then there exists a temperature u on E for which there is no supertemperature $\bar{u}$ on $\mathbf{R}^{n+1}$ that coincides with u on a neighburhood of K.*

Proof. We begin with (b). Suppose that $\mathbf{R}^{n+1}\backslash K$ is not monotonically connected to any point. Choose a point p_0 such that $K \subseteq \Lambda^*(p_0, \mathbf{R}^{n+1})$. There is some point $p_1 \in \Lambda^*(p_0, \mathbf{R}^{n+1})\backslash K$ that does not belong to $\Lambda^*(p_0, \mathbf{R}^{n+1}\backslash K)$, and so the same is true of every point in the set $S = \Lambda(p_1, \mathbf{R}^{n+1}\backslash K)$. Choose a point $p^* \in S$, and put $u = -W(., p^*)$ on $\mathbf{R}^{n+1}$. Then, in particular, u is a temperature on the open superset $\mathbf{R}^{n+1}\backslash\{p^*\}$ of K. Suppose that there is a supertemperature $\bar{u}$ on $\mathbf{R}^{n+1}$ such that $\bar{u} = u$ on an open superset D of K. Note that, by [5] Lemma 1, $\mathrm{ess}(\partial S) \subseteq \mathrm{ess}(\partial(\mathbf{R}^{n+1}\backslash K)) \subseteq \partial(\mathbf{R}^{n+1}\backslash K) = \partial K \subseteq D$.

The function $v = \bar{u} - u$ is a supertemperature on $\mathbf{R}^{n+1}$ and identically zero on D. Since $\text{ess}(\partial S) \subseteq D$, it follows from the minimum principle that $v \geq 0$ on S. Since D is an open superset of K, for each point $p \in S$ there is a point $p' \in \Lambda^*(p, S) \cap D$. Since $v(p') = 0$, the strong minimum principle shows that $v(p) = 0$ also. So $\bar{u} = u$ on S, which is impossible because u is unbounded below on any neighbourhood of p^*, and the supertemperature $\bar{u}$ is locally bounded below on $\mathbf{R}^{n+1}$. So such a function $\bar{u}$ cannot exist if $\mathbf{R}^{n+1}\backslash K$ is not monotonically connected to any point.

The proof of part (a) of the Theorem requires several lemmas. The first of these requires the concept of a *block set.*

3. Block sets

Definition. An open set B in $\mathbf{R}^{n+1}$ will be called a *block set* if it can be written as a union

$$B = \bigcup_{i=1}^{m} R_i$$

of finitely many open rectangles. (By a *rectangle* we mean an $(n+1)$-dimensional interval.)

Note that, if B is a block set and R is a rectangle, then $B\backslash\bar{R}$ is also a block set. To see this, first choose an open rectangle X which contains $B \cup \bar{R}$. Then $X\backslash\bar{R}$ is a block set, because

$$X = \prod_{i=1}^{n+1}]x_i, y_i[, \quad \bar{R} = \prod_{i=1}^{n+1} [a_i, b_i], \quad x_i < a_i < b_i < y_i$$

implies that (with a slight abuse of notation)

$$X\backslash\bar{R} = \bigcup_{k=1}^{n+1} \left(\left(\left(\prod_{i \neq k}]x_i, y_i[\right) \times]x_k, a_k[\right) \bigcup \left(\left(\prod_{i \neq k}]x_i, y_i[\right) \times]b_k, y_k[\right) \right).$$

Now $B\backslash\bar{R} = B \cap (X\backslash\bar{R})$ is an intersection of two block sets, which is itself a block set; because if

$$B = \bigcup_{i=1}^{m} R_i \quad \text{and} \quad X\backslash\bar{R} = \bigcup_{j=1}^{q} S_j,$$

then

$$B\backslash\bar{R} = \left(\bigcup_{i=1}^{m} R_i \right) \bigcap \left(\bigcup_{j=1}^{q} S_j \right) = \bigcup_{i=1}^{m} \bigcup_{j=1}^{q} (R_i \cap S_j),$$

and $R_i \cap S_j$ is a rectangle (or empty) for every i and j.

It follows that, if B and C are both block sets, then $B\backslash\bar{C}$ is also a block set.

In the proof of the superharmonic case given in [1], the relative complement $E\backslash K$ of a compact set K in an open set E, is approximated from within by Dirichlet regular sets. This technique is not available in the present case, and instead we approximate K from without by the closures of block sets. We need to be able to do this in such a way that, if $\mathbf{R}^{n+1}\backslash K$ is monotonically connected to a point p_0, then the approximating block sets are too. This is the purpose of our first lemma.

Lemma 1. *Let E be an open set in $\mathbf{R}^{n+1}$, and let K be a compact subset of E. Then there is a block set B such that $K \subseteq B$ and $\bar{B} \subseteq E$. Furthermore, if $\mathbf{R}^{n+1}\backslash K$ is monotonically connected to some point $p_0 \in \mathbf{R}^{n+1}\backslash K$, then B can be chosen so that $\mathbf{R}^{n+1}\backslash \bar{B}$ is also monotonically connected to p_0.*

Proof. Since K is a compact subset of the open set E, we can cover it with finitely many open rectangles whose closures lie in E. The union B of these rectangles is a block set such that $K \subseteq B$ and $\bar{B} \subseteq E$.

If $\mathbf{R}^{n+1}\backslash K$ is monotonically connected to p_0, then the above choice of B may not suffice to make $\mathbf{R}^{n+1}\backslash \bar{B}$ monotonically connected to p_0. Suppose that there are points p_α in $\Lambda^*(p_0, \mathbf{R}^{n+1})\backslash \bar{B}$ that do not belong to $\Lambda^*(p_0, \mathbf{R}^{n+1}\backslash \bar{B})$. Since $\mathbf{R}^{n+1}\backslash K$ is monotonically connected to p_0, we have $p_\alpha \in \Lambda^*(p_0, \mathbf{R}^{n+1})\backslash K = \Lambda^*(p_0, \mathbf{R}^{n+1}\backslash K)$, so that there is a polygonal path from p_α to p_0 in $\mathbf{R}^{n+1}\backslash K$ along which time is strictly decreasing. But p_α does not belong to $\Lambda^*(p_0, \mathbf{R}^{n+1}\backslash \bar{B})$, so any such path must meet $\bar{B}$. Let $\Gamma(p_\alpha, p_0)$ denote the family of all such paths from p_α to p_0. Then every $\gamma \in \Gamma(p_\alpha, p_0)$ meets $\bar{B}$, and there exists

$$t_{\alpha,\gamma} = \max\{t : (x,t) \in \gamma \cap \bar{B}\}.$$

Put

$$t_\alpha = \inf\{t_{\alpha,\gamma} : \gamma \in \Gamma(p_\alpha, p_0)\}.$$

Because B is a block set, the infimum is attained. Choose a path $\delta \in \Gamma(p_\alpha, p_0)$ such that $t_{\alpha,\delta} = t_\alpha$ and the point $q_\alpha = (y_\alpha, t_\alpha) \in \delta \cap \bar{B}$ is in the relative interior of $(\mathbf{R}^n \times \{t_\alpha\}) \cap \partial B$. Then $\Lambda^*(q_\alpha, \mathbf{R}^{n+1}\backslash \bar{B})$ is defined and contains p_α. Put

$$I(q_\alpha) = \Lambda^*(q_\alpha, \mathbf{R}^{n+1}\backslash \bar{B})\backslash \Lambda^*(p_0, \mathbf{R}^{n+1}\backslash \bar{B}),$$

which is nonempty because it contains p_α.

Take another point q_β, chosen in the same way relative to another point p_β in $\Lambda^*(p_0, \mathbf{R}^{n+1})\backslash \bar{B}$ that does not belong to $\Lambda^*(p_0, \mathbf{R}^{n+1}\backslash \bar{B})$. If q_α and q_β belong to the same component of $(\mathbf{R}^n \times \{t\}) \cap \partial B$ for some t, then $I(q_\alpha) = I(q_\beta)$. Since B is a block set, there are only finitely many different values of t for which $\mathbf{R}^n \times \{t\}$ contains some q_α, and each $(\mathbf{R}^n \times \{t\}) \cap \partial B$ has only finitely many components. So there are only finitely many *distinct* sets $I(q_\alpha)$. We choose a unique point q_k to represent each distinct set $I(q_k)$, and thus obtain a finite set $\{q_1, \ldots, q_m\}$ such that

$$\left(\Lambda^*(p_0, \mathbf{R}^{n+1})\backslash \bar{B}\right) \backslash \Lambda^*(p_0, \mathbf{R}^{n+1}\backslash \bar{B}) \subseteq \bigcup_{k=1}^{m} \Lambda^*(q_k, \mathbf{R}^{n+1}\backslash \bar{B}).$$

Since $q_k \in \mathbf{R}^{n+1}\backslash K$ for $k \in \{1, \ldots, m\}$, and $\mathbf{R}^{n+1}\backslash K$ is monotonically connected to p_0, we can choose a polygonal path γ_k that connects q_k to p_0 along which the temporal variable is strictly decreasing. Since $\bigcup_{k=1}^{m} \gamma_k$ is compact, we can cover it with finitely many open rectangles whose closures do not intersect K. Let U denote the union of the closures of these rectangles. Now $B\backslash U$ is a block set containing K, and $\mathbf{R}^{n+1}\backslash(B\backslash U)$ is monotonically connected to p_0. □

4. Preliminary extension lemmas

The remaining lemmas are all relatively minor extension results. The first is the direct analogue of a result in [1], p. 66.

Lemma 2. *Let v be a supertemperature on an open set E, and let h be a supertemperature on an open subset D of E. If*

$$\liminf_{p\to q,\, p\in D} h(p) \geq v(q) \quad \textit{for all} \quad q\in E\cap\partial D, \tag{1}$$

and w is defined on E by

$$w(p)=\begin{cases}(h\wedge v)(p) & \textit{if} \quad p\in D,\\ v(p) & \textit{if} \quad p\in E\backslash D,\end{cases}$$

then w is a supertemperature on E.

Proof. It is clear that w is a supertemperature on $E\backslash\partial D$, that $w(p)>-\infty$ for all $p\in E$, and that $w<+\infty$ on a dense subset of E. Condition (1) ensures that, for each point $q\in E\cap\partial D$,

$$\liminf_{p\to q} w(p)=\min\left\{\liminf_{p\to q,\, p\in D} h(p),\ \liminf_{p\to q} v(p)\right\}\geq v(q)=w(q),$$

so that w is lower semicontinuous on E. It remains to check that the supertemperature mean value inequality is satisfied at points of $E\cap\partial D$. If $q\in E\cap\partial D$ and $\bar\Omega(q;c)\subseteq E$, then

$$w(q)=v(q)\geq \mathscr{M}(v;q,c)\geq\mathscr{M}(w;q,c).$$

Hence w is a supertemperature on E, by [4] Theorem 15. □

In practice, condition (1) is rarely satisfied when $q\in\mathrm{ab}(\partial D)$, and this limits the usefulness of Lemma 2. We need a substitute result for the case where $D=E\backslash\bar T$ with T a block set such that $\bar T\subseteq E$. In this case, the set of all horizontal edges of T contains $E\cap\mathrm{ab}_2(\partial D)$, and is a closed polar set, in view of [5], p. 280.

Lemma 3. *Let E be an open set, let T be a block set such that $\bar T\subseteq E$, and let $D=E\backslash\bar T$. Let v be a supertemperature on E, and let h be a supertemperature on D. If*

$$\liminf_{p\to q,\, p\in D} h(p)\geq v(q) \quad \textit{for all} \quad q\in E\cap \mathrm{n}(\partial D), \tag{2}$$

$$\liminf_{p\to q,\, p\in D} h(p)>-\infty \quad \textit{for all} \quad q\in E\cap\mathrm{ab}(\partial D), \tag{3}$$

and

$$\liminf_{p\to q,\, p\in D} h(p)\leq v(q) \quad \textit{for all} \quad q\in E\cap\mathrm{ab}_1(\partial D), \tag{4}$$

then the function w, defined on $E\setminus \mathrm{ab}_2(\partial D)$ by

$$w(q)=\begin{cases}(h\wedge v)(q) & \textit{if} \quad q\in D,\\ \liminf_{p\to q,\, p\in D} h(p) & \textit{if} \quad q\in E\cap \mathrm{ab}_1(\partial D),\\ v(q) & \textit{if} \quad q\in E\setminus(D\cup \mathrm{ab}(\partial D)),\end{cases}$$

has a unique extension to a supertemperature on E.

Proof. Let Z denote the closed set of all horizontal edges of T. Then $E\cap \mathrm{ab}_2(\partial D) \subseteq Z$. Clearly w is a supertemperature on $E\setminus\partial D$, and $w>-\infty$ on $E\setminus \mathrm{ab}_2(\partial D)$, which contains $E\setminus Z$. Furthermore, because T is a block set, $E\cap \mathrm{ab}(\partial D)$ is contained in the union of a finite set of hyperplanes of the form $\mathbf{R}^n\times\{t\}$, and so $w<+\infty$ on a dense subset of E.

Next we check the lower semicontinuity. If $q\in E\cap \mathrm{n}(\partial D)$, then

$$\liminf_{p\to q} w(p)=\min\left\{\liminf_{p\to q,\, p\in D} h(p),\ \liminf_{p\to q} v(p)\right\}\geq v(q)=w(q),$$

in view of (2). If $q\in E\cap \mathrm{ab}_1(\partial D)$, then condition (4) and [5] Lemma 12 imply that

$$\liminf_{p\to q,\, p\in D} h(p)\leq \liminf_{p\to q} v(p),$$

so that

$$\liminf_{p\to q} w(p)=\min\left\{\liminf_{p\to q,\, p\in D} h(p),\ \liminf_{p\to q} v(p)\right\}=\liminf_{p\to q,\, p\in D} h(p)=w(q).$$

Hence w is lower semicontinuous on $E\setminus \mathrm{ab}_2(\partial D)$, and in particular on $E\setminus Z$.

We now check that the supertemperature mean value inequality is satisfied at every point of $E\cap(\partial D\setminus \mathrm{ab}_2(\partial D))$. Because T is a block set, $E\cap \mathrm{ab}(\partial D)$ is contained in the union of a *finite* collection of hyperplanes of the form $\mathbf{R}^n\times\{t\}$. Therefore, if $q\in E\cap \mathrm{n}(\partial D)$ we have $\bar\Omega(q;c)\subseteq E\setminus \mathrm{ab}(\partial D)$ for all sufficiently small values of c. For those values,

$$w(q)=v(q)\geq \mathscr{M}(v;q,c)\geq \mathscr{M}(w;q,c).$$

On the other hand, if $q\in E\cap \mathrm{ab}_1(\partial D)$, then condition (3) implies that w is bounded below on some open rectangle R such that $q\in \mathrm{ab}(\partial R)$. Therefore we can use condition (4), Fatou's Lemma, and the lower semicontinuity of $h\wedge v$, to obtain

$$\begin{aligned} w(q)&=\liminf_{p\to q,\, p\in D} h(p)\geq \liminf_{p\to q,\, p\in D}(h\wedge v)(p)\geq \liminf_{p\to q,\, p\in D}\mathscr{M}(h\wedge v;p,c)\\ &\geq \mathscr{M}(h\wedge v;q,c)=\mathscr{M}(w;q,c)\end{aligned}$$

for all sufficiently small values of c. It follows from [4], Theorem 15, that w is a supertemperature on $E\setminus Z$.

Since Z is a closed polar subset of E, we have only to show that w is locally bounded below on E and apply [5], Theorem 29, to complete the proof. Clearly w is bounded below on compact subsets of $E\setminus\partial D$. Condition (3) (along with the lower finiteness of v) implies that w is bounded below on some neighbourhood of any $q \in E \cap \mathrm{ab}(\partial D)$, and condition (2) has a similar implication for $q \in E \cap \mathrm{n}(\partial D)$. So w is locally bounded below on E, and the result follows. □

In the proof of our theorem, we first extend the given supertemperature to a set of the form

$$\Omega^*(p_0; c) = \{p \in \mathbf{R}^{n+1} : W(p, p_0) > \tau(c)\},$$

which is the reflection of $\Omega(p_0; c)$ in the hyperplane $\mathbf{R}^n \times \{t_0\}$, if $p_0 = (x_0, t_0)$. The following lemma then gives an extension to the whole of $\mathbf{R}^{n+1}$.

Lemma 4. *Let u be a supertemperature on $\Omega^* = \Omega^*(p^*; c^*)$, and let S be an open set such that $\bar{S} \subseteq \Omega^*$. Then there is a supertemperature $\bar{u}$ on $\mathbf{R}^{n+1}$, such that $\bar{u} = u$ on S and $\bar{u}$ is lower bounded on $\mathbf{R}^{n+1}$.*

Proof. Let $p^* = (x^*, t^*)$, and choose $t_1 > t^*$ such that $\bar{S} \subseteq \mathbf{R}^n \times]t_1, \infty[$. Choose $\gamma < c^*$ such that $\bar{S} \subseteq \Omega^*(p^*; \gamma)$, and put $\Omega_1^*(\gamma) = \Omega^*(p^*; \gamma) \cap (\mathbf{R}^n \times]t_1, \infty[)$. Then $\bar{\Omega}_1^*(\gamma)$ is a compact subset of Ω^*, so that we can find $k \in \mathbf{R}$ such that $u > k$ on $\Omega_1^*(\gamma)$. Let R_{u-k}^S be the reduction of $u - k$ relative to S in $\Omega_1^*(\gamma)$ (see [2] for details about reductions), and put

$$u_1 = R_{u-k}^S + k \quad \text{on} \quad \Omega_1^*(\gamma).$$

Then u_1 is a supertemperature on $\Omega_1^*(\gamma)$, u_1 is a temperature on $\Omega_1^*(\gamma) \backslash \bar{S}$, $k \le u_1 \le u$ on $\Omega_1^*(\gamma)$, and $u_1 = u$ on S.

Choose α and β such that $0 < \alpha < \beta < \gamma$ and $\bar{S} \subseteq \Omega^*(p^*; \alpha)$. Put $\Omega^*(\alpha) = \Omega^*(p^*; \alpha)$, and $\Omega_1^*(\alpha) = \Omega^*(\alpha) \cap (\mathbf{R}^n \times]t_1, \infty[)$; similarly for β. Since u_1 is continuous on $\Omega_1^*(\gamma) \backslash \bar{S}$, it has a maximum value $M(\alpha) \ge k$ on $\partial\Omega^*(\alpha) \cap (\mathbf{R}^n \times [t_1, \infty[)$. Define u_2 on $\mathbf{R}^{n+1}$ by putting

$$u_2(p) = \frac{M(\alpha) - k}{\tau(\alpha) - \tau(\beta)}(W(p, p^*) - \tau(\beta)) + k.$$

Then u_2 is a supertemperature, $u_2 = M(\alpha)$ on $\partial\Omega^*(\alpha) \backslash \{p^*\}$, and $u_2 = k$ on $\partial\Omega^*(\beta) \backslash \{p^*\}$. Now define u_3 on $\mathbf{R}^n \times]t_1, \infty[$ by

$$u_3 = \begin{cases} u_1 & \text{on} \quad \bar{\Omega}^*(\alpha) \cap (\mathbf{R}^n \times]t_1, \infty[), \\ u_1 \wedge u_2 & \text{on} \quad \Omega_1^*(\beta) \backslash \bar{\Omega}_1^*(\alpha), \\ u_2 & \text{on} \quad (\mathbf{R}^n \times]t_1, \infty[) \backslash \Omega_1^*(\beta). \end{cases}$$

We apply Lemma 2 with $E = \Omega_1^*(\beta)$, $v = u_1$, $D = \Omega_1^*(\beta) \backslash \bar{\Omega}_1^*(\alpha)$, and $h = u_2$, noting that for all $q \in E \cap \partial D = \Omega_1^*(\beta) \cap \partial\Omega_1^*(\alpha)$ we have

$$\liminf_{p \to q,\, p \in D} h(p) \ge u_2(q) = M(\alpha) \ge u_1(q) = v(q).$$

Thus u_3 is a supertemperature on $\Omega_1^*(\beta)$.

A second application of Lemma 2, this time with $E = (\mathbf{R}^n \times]t_1, \infty[) \backslash \bar{\Omega}_1^*(\alpha)$, $v = u_2$, $D = \Omega_1^*(\beta) \backslash \bar{\Omega}_1^*(\alpha)$, and $h = u_1$, so that for all $q \in E \cap \partial D = (\mathbf{R}^n \times]t_1, \infty[) \cap \partial\Omega_1^*(\beta)$ we have

$$\liminf_{p \to q,\, p \in D} h(p) \ge u_1(q) \ge k = u_2(q) = v(q),$$

shows that u_3 is also a supertemperature on $(\mathbf{R}^n\times]t_1,\infty[)\backslash\bar{\Omega}_1^*(\alpha)$, and therefore on the whole of $(\mathbf{R}^n\times]t_1,\infty[)$.

Since $u_1 \geq k$ on $\Omega_1^*(\gamma)$, and

$$u_2 \geq \frac{M(\alpha)-k}{\tau(\alpha)-\tau(\beta)}(-\tau(\beta))+k = \frac{-\tau(\beta)M(\alpha)+\tau(\alpha)k}{\tau(\alpha)-\tau(\beta)}$$

on $\mathbf{R}^{n+1}$, u_3 is lower bounded. Putting

$$\bar{u} = \begin{cases} u_3 & \text{on} \quad \mathbf{R}^n\times]t_1,\infty[, \\ \inf u_3 & \text{on} \quad \mathbf{R}^n\times]-\infty,t_1], \end{cases}$$

we obtain a lower bounded supertemperature $\bar{u}$ on $\mathbf{R}^{n+1}$ such that $\bar{u}=u_3=u_1=u$ on S. □

5. Proof of part (a) of the theorem

Let K be a compact subset of an open set E. We must prove the following statement:

If $\mathbf{R}^{n+1}\backslash K$ is monotonically connected to some point p_0, then for each supertemperature u on E there is a lower bounded supertemperature $\bar{u}$ on $\mathbf{R}^{n+1}$ such that $\bar{u}=u$ on a neighbourhood U of K. Furthermore, $\bar{u}$ can be chosen to be the potential of a measure supported in $\bar{U}$, plus a constant.

Proof. We may suppose that E is bounded, and that $u>0$ on E.

By Lemma 1, we can find an open (block) set S such that $K\subseteq S, \bar{S}\subseteq E$, and $\mathbf{R}^{n+1}\backslash\bar{S}$ is monotonically connected to p_0. Let $v=R_u^S$, the reduction of u relative to S in E. Then v is a supertemperature on E, v is a temperature on $E\backslash\bar{S}$, $0\leq v\leq u$ on E, and $v=u$ on S. Using Lemma 1 again, we can find a block set T such that $\bar{S}\subseteq T$, $\bar{T}\subseteq E$, and $\mathbf{R}^{n+1}\backslash\bar{T}$ is monotonically connected to p_0. Choose $p^*\in\mathbf{R}^{n+1}$ and $c^*>0$ such that $\bar{E}\cup\{p_0\}\subseteq\Omega^*(p^*;c^*)$, and put $\Omega^*=\Omega^*(p^*;c^*)$, $A=\Omega^*\backslash\bar{T}$. We shall extend u to a supertemperature on Ω^*, then use Lemma 4 to further extend u to $\mathbf{R}^{n+1}$.

Put $g_1=v$ on ∂T, $g_1=0$ on $\partial\Omega^*$, $g_2=0$ on ∂T, and $g_2=1$ on $\partial\Omega^*$. Define

$$h_k = H_{g_1}^A - kH_{g_2}^A \quad \text{for all} \quad k\in\mathbf{N}.$$

Note that v is continuous on ∂T, because v is a temperature on $E\backslash\bar{S}$. For each point $(x,t)\in A$ such that $t<\min\{s:(y,s)\in\bar{T}\}$, we have $H_{g_2}^A(x,t)=1$ because $g_2=1$ on $\partial\Omega^*$. In particular, $H_{g_2}^A(p_0)=1$. Since $\mathbf{R}^{n+1}\backslash\bar{T}$ is monotonically connected to p_0, for all $p\in\Lambda^*(p_0,\mathbf{R}^{n+1})\backslash\bar{T}$ we have $p_0\in\Lambda(p,\mathbf{R}^{n+1}\backslash\bar{T})$, and therefore $p_0\in\Lambda(p,A)$ if $p\in A$. Therefore, by the strong minimum principle, $H_{g_2}^A>0$ on A, so that $\{h_k\}$ decreases to $-\infty$ on A as $k\to\infty$.

Our method of extending u to Ω^* requires that $h_j\leq v$ on $E\backslash\bar{T}$ for some j. Because $\{h_k\}$ decreases to $-\infty$ on A, we can find j such that $h_j\leq 0$ on ∂E.

Therefore, for all $q \in \partial E$ we have

$$\liminf_{p\to q,\, p\in E} v(p) \geq h_j(q) = \lim_{p\to q} h_j(p).$$

Consider the points of ∂T as boundary points in the Dirichlet problem on A. Because T is a block set, all points of $\partial T \cap \mathrm{n}(\partial A)$ are regular, by the parabolic tusk test in [3]. All points of $\partial T \cap \mathrm{ab}_1(\partial A)$ can be ignored, because they are irrelevant to both the Dirichlet problem on A and the use of the minimum principle on A. Again because T is a block set, all points of $\partial T \cap \mathrm{ab}_2(\partial A)$ are contained in the union of finitely many sets of the form $\{(x_1,\dots,x_n,t) : t = a, x_j = b \text{ for some } j\}$, each of which is polar by [5], p. 280. So $\partial T \cap \mathrm{ab}_2(\partial A)$ is also polar. It follows that

$$\lim_{p\to q,\, p\in A} h_j(p) = v(q) = \lim_{p\to q,\, p\in E\backslash \bar{T}} v(p),$$

for all $q \in \partial T \cap \mathrm{ess}(\partial A)\backslash Z$ for some polar set Z. Furthermore, because $g_1 \leq \max_{\partial T} v$ and $g_2 \geq 0$ on ∂A, we have $h_j \leq \max_{\partial T} v$ on A, so that $v - h_j$ is lower bounded on $E\backslash\bar{T}$. Applying the minimum principle in [5], p. 284, to $v - h_j$ on $E\backslash\bar{T}$, we obtain $v \geq h_j$.

We now put $D = E\backslash\bar{T}$ and apply Lemma 3 with $h = h_j$, noting that

$$\lim_{p\to q,\, p\in D} h_j(p) = v(q) \quad \text{for all} \quad q \in E \cap \mathrm{n}(\partial D),$$

because $E \cap \mathrm{n}(\partial D) = \partial T \cap \mathrm{n}(\partial A)$ and all such points are regular;

$$\liminf_{p\to q,\, p\in D} h_j(p) > -\infty \quad \text{for all} \quad q \in E \cap \mathrm{ab}(\partial D)$$

because $h_j \geq -j$ on A; and

$$\liminf_{p\to q,\, p\in D} h_j(p) \leq v(q) \quad \text{for all} \quad q \in E \cap \mathrm{ab}_1(\partial D)$$

because $h_j \leq v$ on D, v is continuous on ∂T, and $\mathrm{ab}_1(\partial D) \cap E = \mathrm{ab}_1(\partial D) \cap \partial T$. Thus we see that the function w, defined by

$$w = \begin{cases} h_j = h_j \wedge v & \text{on} \quad D = E\backslash\bar{T} \\ v & \text{on} \quad T, \end{cases}$$

can be extended to a supertemperature $\bar{w}$ on E. Since h_j is a temperature on A, the function $\bar{w}$ can be extended by h_j to a supertemperature on Ω^*.

Next, by Lemma 4, there is a lower bounded supertemperature u_0 on $\mathbf{R}^{n+1}$ such that $u_0 = w = v = u$ on the neighbourhood S of K. Now let U be any open set such that $K \subseteq U \subseteq S$. To show that u_0 can be taken to be the potential of a measure supported in $\bar{U}$, plus a constant, we first put $m = \inf u_0$ and $u_1 = R^U_{u_0-m}$, the reduction of $u_0 - m$ relative to U in $\mathbf{R}^{n+1}$. Since U is open, u_1 is a nonnegative supertemperature on $\mathbf{R}^{n+1}$, and $u_1 = u_0 - m$ on U. In fact, because $\bar{U}$ is compact, u_1 is a potential by [2], p. 319,(m). Furthermore, u_1 is a temperature on $\mathbf{R}^{n+1}\backslash\bar{U}$, and so its Riesz measure is supported in $\bar{U}$, by [5] Theorem 20. The function $\bar{u} = u_1 + m$ has the required form.

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