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ON BOUNDARY HOMEOMORPHISMS OF TRANS-QUASICONFORMAL MAPS OF THE DISK

Saeed Zakeri

Queens College and Graduate Center of CUNY, Department of Mathematics
65-30 Kissena Blvd., Flushing, New York 11367, U.S.A.; saeed.zakeri@qc.cuny.edu

Abstract. This paper studies boundary homeomorphisms of trans-quasiconformal maps of the unit disk. Motivated by Beurling–Ahlfors's well-known quasisymmetry condition, we introduce the "scalewise" and "pointwise" distortions of a circle homeomorphism and formulate conditions in terms of each that guarantee the existence of a David extension to the disk. These constructions are also used to obtain extension results for maps with subexponentially integrable dilatation as well as BMO-quasiconformal maps of the disk.

1. Introduction

Trans-quasiconformal maps in the plane are generalizations of quasiconformal maps whose dilatation is allowed to grow arbitrarily large in some controlled fashion. They arise as homeomorphic solutions in the Sobolev class $W^{1,1}_{\mathrm{loc}}$ of the Beltrami equation

$$\frac{\partial F}{\partial \bar{z}} = \mu \frac{\partial F}{\partial z}$$

when the measurable function μ satisfies $|\mu| < 1$ a.e. but $\|\mu\|_\infty = 1$. Apart from their intrinsic importance in analysis, they have emerged as useful tools in the study of one-dimensional complex dynamical systems (see [H] and [PZ]).

Various classes of planar trans-quasiconformal maps have been studied in recent years. In fact, their theory can be viewed as part of the much larger theory of "mappings with finite distortion" in Euclidean spaces. In this paper, however, we will only focus on a class of maps introduced by David in 1988 [D] and their spinoffs. These maps are defined in terms of the asymptotic growth of the size of their Beltrami coefficient $\mu_F = (\frac{\partial F}{\partial \bar{z}})/(\frac{\partial F}{\partial z})$ or, more conveniently, their real dilatation

$$K_F = \frac{1+|\mu_F|}{1-|\mu_F|} = \frac{|\frac{\partial F}{\partial z}| + |\frac{\partial F}{\partial \bar{z}}|}{|\frac{\partial F}{\partial z}| - |\frac{\partial F}{\partial \bar{z}}|}.$$

An orientation-preserving homeomorphism $F\colon U \to V$ between planar domains is called a *David map* if $F \in W^{1,1}_{\mathrm{loc}}(U)$ and there are constants $C, \alpha, K_0 > 0$ such that

$$\sigma\{z \in U : K_F(z) > K\} \leq C e^{-\alpha K} \quad \text{for all } K \geq K_0. \tag{1.1}$$

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Here σ denotes the spherical measure on U induced by the metric $|dz|/(1+|z|^2)$. It is not hard to see that (1.1) is equivalent to the exponential integrability condition

$$\exp(K_F) \in L^p(U,\sigma) \quad \text{for some } p>0 \tag{1.2}$$

(compare Lemma 2.2 and its subsequent remark). When U is a bounded domain in the plane, σ in (1.1) or (1.2) can be replaced with Lebesgue measure. According to David's generalization of the measurable Riemann mapping theorem [D], if μ is a Beltrami coefficient in U for which $\frac{1+|\mu|}{1-|\mu|}$ satisfies a condition of the form (1.1) or (1.2), then there is a homeomorphism $F \in W^{1,1}_{\text{loc}}(U)$ which solves the Beltrami equation $\mu_F = \mu$. Moreover, F is unique up to postcomposition with a conformal map of $F(U)$. For basic properties of David maps and how they compare with quasiconformal maps, see [D], [T] or the introduction of [Z].

David's work has been generalized to the case where the exponential function in (1.2) is replaced by functions of slower growth. For instance, [BJ1] and [IM] consider *maps with subexponentially integrable dilatation* for which

$$\Phi \circ K_F \in L^p(U,\sigma) \quad \text{for some } p>0, \tag{1.3}$$

where $\Phi(x) = \exp(x/(1+\log x))$. More generally, we can consider the condition (1.3) for any convex increasing function $\Phi\colon [1,+\infty) \to [1,+\infty)$ such that

$$\lim_{x\to+\infty} \frac{\log \Phi(x)}{x} = 0 \quad \text{but} \quad \lim_{x\to+\infty} \frac{\log\log \Phi(x)}{\log x} = 1. \tag{1.4}$$

This essentially means that the asymptotic growth of Φ is slower than $\exp(\varepsilon x)$ but faster than $\exp(x^\varepsilon)$ for every $0<\varepsilon<1$. Much of the theory of David maps remains true for maps with such subexponentially integrable dilatation as long as we assume $\int_1^{+\infty} x^{-2} \log \Phi(x)\, dx = +\infty$ (compare [BJ2] and [IM]).

Yet another class of trans-quasiconformal maps are those whose dilatation has a majorant of bounded mean oscillation [RSY]. An orientation-preserving homeomorphism $F\colon U \to V$ is called *BMO-quasiconformal* if $F \in W^{1,1}_{\text{loc}}(U)$ and there is a $Q \in BMO(U)$ such that

$$K_F \leq Q \quad \text{a.e. in } U \tag{1.5}$$

(see §5 for definitions). This condition is slightly stronger than David's (1.1), but there are many parallels between the two theories.

The problem of characterizing boundary homeomorphisms of quasiconformal maps of the unit disk $\mathbf{D}$ was first studied in the classical paper of Beurling and Ahlfors [BA]. Transferring the problem to the upper half-plane $\mathbf{H}$, they showed that an orientation-preserving homeomorphism $f\colon \mathbf{R} \to \mathbf{R}$ extends to a quasiconformal map $\mathbf{H} \to \mathbf{H}$ if and only if it is *quasisymmetric*, in the sense that there is a constant $\rho \geq 1$ such that

$$\delta_f(x,t) = \max\left\{ \frac{f(x+t)-f(x)}{f(x)-f(x-t)}, \frac{f(x)-f(x-t)}{f(x+t)-f(x)} \right\} \leq \rho \tag{1.6}$$

for all $x \in \mathbf{R}$ and $t > 0$. In the present paper we address a similar problem for trans-quasiconformal maps $\mathbf{D} \to \mathbf{D}$, i.e., the question of when a circle homeomorphism can be extended to each of the above three classes of trans-quasiconformal maps of the disk. Lifting under the exponential map $\mathbf{e}\colon z \mapsto e^{2\pi i z}$, we may equally work with homeomorphisms of the real line and upper half-plane which commute with the unit translation $z \mapsto z+1$. We denote the groups of all such homeomorphisms by $H_T(\mathbf{R})$ and $H_T(\mathbf{H})$, respectively. Each $F \in H_T(\mathbf{H})$ descends to a homeomorphism $G\colon \mathbf{D} \to \mathbf{D}$ which fixes the origin and satisfies $G \circ \mathbf{e} = \mathbf{e} \circ F$, so $K_G \circ \mathbf{e} = K_F$. Since the derivative of $\mathbf{e}\colon [0,1] \times (0,+\infty) \to \mathbf{D}$ has uniformly bounded spherical norm, it follows that G is David or has subexponentially integrable dilatation whenever F has the corresponding property.

Given $f \in H_T(\mathbf{R})$, we define its *scalewise distortion* $\rho_f = \rho_f(t)$ by taking the supremum over all $x \in \mathbf{R}$ of the quantity $\delta_f(x,t)$ in (1.6). The scalewise distortion is a continuous function of $t > 0$ and we have $\limsup_{t\to 0^+} \rho_f(t) = +\infty$ unless f is quasisymmetric. In §3 we provide conditions for David extendibility of f in terms of the asymptotic behavior of $\rho_f(t)$ as $t \to 0^+$ (Theorem 3.1). In particular, any $f \in H_T(\mathbf{R})$ whose scalewise distortion satisfies

$$\rho_f(t) = O\left(\log \frac{1}{t}\right) \quad \text{as } t \to 0^+ \tag{1.7}$$

extends to a David map in $H_T(\mathbf{H})$. We give two examples which together demonstrate that no optimal condition for David extendibility can be formulated in terms of ρ_f only.

In §4 we suggest a variant of ρ_f which in some respects is a more natural function to look at. More specifically, we define the *pointwise distortion* $\lambda_f = \lambda_f(x)$ by taking the supremum over all $t > 0$ of $\delta_f(x,t)$ in (1.6). This is now a 1-periodic semicontinuous function and may well take the value $+\infty$.

Theorem A. *Suppose the pointwise distortion λ_f of $f \in H_T(\mathbf{R})$ satisfies*

$$\exp(\lambda_f) \in L^p[0,1] \quad \textit{for some } p > 0. \tag{1.8}$$

Then f extends to a David map in $H_T(\mathbf{H})$.

In fact, we show that the dilatation of the Beurling–Ahlfors extension F of f satisfies

$$K_F(x+iy) \le \text{const.} \max\left\{\lambda_f(x), \log\left(\frac{1}{y}\right)\right\}$$

for sufficiently small $y > 0$, from which it easily follows that F is a David map. We also observe that the conditions (1.7) and (1.8) can be unified into a single stronger condition on δ_f that guarantees David extendibility (Theorem 4.3).

In §5 we discuss the extension problem for other classes of trans-quasiconformal maps of the disk. We first prove the analog of Theorem A for maps with subexponentially integrable dilatation:

Theorem B. *Suppose $f \in H_T(\mathbf{R})$ and $\Phi \circ \lambda_f \in L^p[0,1]$ for some $p > 0$, where $\Phi\colon [1,+\infty) \to [1,+\infty)$ is a convex increasing function which satisfies the growth* *conditions* (1.4). *Then* f *extends to a map* $F \in H_T(\mathbf{H})$ *with subexponentially integrable dilatation. In fact,* $\Phi \circ K_F \in L^\nu(\mathbf{H}, \sigma)$ *for some* $\nu > 0$ *depending on* p *and* Φ.

The proof consists of a close adaptation of the estimates involved in the proof of Theorem A, replacing the exponential function with Φ (note however that Theorem A is not a special case of Theorem B since the exponential function does not satisfy the condition (1.4)).

Next, we prove an extension theorem for the class of BMO-quasiconformal maps:

Theorem C. *Consider the following conditions on* $f \in H_T(\mathbf{R})$*:*

(i) *There is a* 1*-periodic function* $q \in BMO(\mathbf{R})$ *such that*

$$\delta_f(x,t) \leq \frac{1}{2t} \int_{x-t}^{x+t} q(s)\, ds \quad \text{for } x \in \mathbf{R}, t > 0.$$

(ii) *The scalewise distortion* ρ_f *has the asymptotic growth*

$$\rho_f(t) = O\left(\log \frac{1}{t}\right) \quad \text{as } t \to 0^+.$$

(iii) *The pointwise distortion* λ_f *has a majorant in* $BMO(\mathbf{R})$.

Then the implications (i) $\Longrightarrow$ (ii) *and* (i) $\Longrightarrow$ (iii) *hold. Under any of these conditions,* f *extends to a* BMO*-quasiconformal map in* $H_T(\mathbf{H})$.

This gives a more general version of the extension result obtained by Sastry in [S]. It also shows that her geometric construction based on the idea of Carleson boxes can be replaced with the familiar Beurling–Ahlfors extension.

We wish to suggest that the pointwise distortion λ_f can be roughly viewed as a "one-dimensional dilatation" for a circle homeomorphism f. Imposing a regularity condition on λ_f would allow a trans-quasiconformal extension of f whose real dilatation satisfies the *same* type of condition as λ_f. This is illustrated in the above four cases: $\|\lambda_f\|_\infty < +\infty$ gives a quasiconformal extension, $\exp(\lambda_f) \in L^p$ gives a David extension, $\Phi \circ \lambda_f \in L^p$ gives an extension with subexponentially integrable dilatation, and λ_f having a BMO majorant gives a BMO-quasiconformal extension.

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2. Preliminaries

Throughout the paper we will adopt the following notations:

- $|X|$ is the n-dimensional Lebesgue measure of $X \subset \mathbf{R}^n$.
- σ is the spherical measure induced by the conformal metric $|dz|/(1+|z|^2)$ on the Riemann sphere.

- $H_T(\mathbf{R})$ and $H_T(\mathbf{H})$ are the groups of orientation-preserving homeomorphisms of the real line and upper half-plane which commute with the translation $z \mapsto z+1$.

Elements of $H_T(\mathbf{R})$ arise as the lifts under the exponential map $z \mapsto e^{2\pi i z}$ of orientation-preserving homeomorphisms of the unit circle. Similarly, elements of $H_T(\mathbf{H})$ arise as the lifts of orientation-preserving homeomorphisms of the unit disk which fix the origin. Blurring the distinction between a map and its lift, we may regard elements of $H_T(\mathbf{R})$ as circle homeomorphisms and those of $H_T(\mathbf{H})$ as disk homeomorphisms.

Functions of logarithmic type. Let $X \subset \mathbf{R}^n$ be Lebesgue measurable and $|X| < +\infty$. A measurable function $\varphi\colon X \to [0,+\infty]$ is said to be of *logarithmic type* if there are constants $C, \alpha > 0$ such that

$$|\{x \in X : \varphi(x) > t\}| \leq Ce^{-\alpha t} \tag{2.1}$$

for all sufficiently large t. The terminology is motivated by the example $\varphi(x) = \log(1/x)$ on $X = [0,1]$ and is meant to suggest that (in the simplest cases) φ has at worst logarithmic singularities. As another example, the real dilatation of a David map of a bounded domain is of logarithmic type.

Lemma 2.1. *Let I_1, I_2 be bounded intervals in* $\mathbf{R}$ *and $a\colon I_1 \to [0,+\infty]$, $b\colon I_2 \to [0,+\infty]$ be functions of logarithmic type. If $\varphi\colon I_1 \times I_2 \to [0,+\infty]$ is a measurable function which satisfies*

$$\varphi(x,y) \leq \max\{a(x), b(y)\},$$

then φ is of logarithmic type.

This simply follows from the inclusion

$$\{(x,y) : \varphi(x,y) > t\} \subset \{(x,y) : a(x) > t\} \cup \{(x,y) : b(y) > t\}.$$

The following characterization will be used frequently:

Lemma 2.2. *A measurable function $\varphi\colon X \to [0,+\infty]$ is of logarithmic type if and only if $\exp(\varphi) \in L^p(X)$ for some $p > 0$.*

Proof. This is quite standard. For a given $p > 0$, set

$$A_t = \{x \in X : \exp(p\varphi(x)) > t\}.$$

First suppose φ is of logarithmic type so that it satisfies (2.1) for all $t \geq t_0$. Set $p = \alpha/2$. Then,

$$|A_t| = \left|\left\{x \in X : \varphi(x) > \frac{2}{\alpha}\log t\right\}\right| \leq \begin{cases} |X| & \text{if } 0 \leq t < t_0, \\ C\,t^{-2} & \text{if } t \geq t_0. \end{cases}$$

Hence

$$\int_X \exp(p\varphi) = \int_0^\infty |A_t|\,dt \leq |X|\,t_0 + C\,t_0^{-1},$$

which shows $\exp(\varphi) \in L^p(X)$.

Conversely, suppose $\exp(\varphi) \in L^p(X)$ for some $p > 0$. Then

$$|A_t| \le t^{-1} \int_{A_t} \exp(p\varphi) \le C\, t^{-1}$$

for some constant $C > 0$. It follows that

$$|\{x \in X : \varphi(x) > t\}| = |A_{\exp(pt)}| \le C\, e^{-pt}. \qquad \square$$

Remark. The definition of functions of logarithmic type and Lemma 2.2 generalize verbatim to every finite measure space.

The Beurling–Ahlfors extension. Let $f\colon \mathbf{R} \to \mathbf{R}$ be an orientation-preserving homeomorphism. Define $\mathscr{E}(f)\colon \mathbf{H} \to \mathbf{H}$ by

$$\mathscr{E}(f)(x+iy) = \frac{1+i}{2}\left(u(x,y) - iv(x,y)\right),$$

where

$$u(x,y) = \frac{1}{y}\int_x^{x+y} f(t)\, dt \quad \text{and} \quad v(x,y) = \frac{1}{y}\int_{x-y}^{x} f(t)\, dt.$$

It is easy to see that $\mathscr{E}(f)$ is a C^1-smooth homeomorphism of $\mathbf{H}$ and $\mathscr{E}(f)(x+iy) \to f(x)$ as $y \to 0$. The map $\mathscr{E}(f)$ is called the *Beurling–Ahlfors extension* of f [BA].

The real dilatation K_F of $F = \mathscr{E}(f)$ satisfies

$$K_F + K_F^{-1} = \frac{(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2}{(\frac{\partial u}{\partial y})(\frac{\partial v}{\partial x}) - (\frac{\partial u}{\partial x})(\frac{\partial v}{\partial y})}, \tag{2.2}$$

where, by the definition of F,

$$\begin{cases} \dfrac{\partial u}{\partial x}(x,y) = \dfrac{1}{y}\left(f(x+y) - f(x)\right) & \dfrac{\partial u}{\partial y}(x,y) = \dfrac{1}{y}\left(f(x+y) - u(x,y)\right) \\ \dfrac{\partial v}{\partial x}(x,y) = \dfrac{1}{y}\left(f(x) - f(x-y)\right) & \dfrac{\partial v}{\partial y}(x,y) = \dfrac{1}{y}\left(f(x-y) - v(x,y)\right). \end{cases} \tag{2.3}$$

The assignment $f \mapsto \mathscr{E}(f)$ is equivariant with respect to the left and right actions of the real affine group $\mathrm{Aut}(\mathbf{C}) \cap \mathrm{Aut}(\mathbf{H}) = \{z \mapsto az+b : a > 0, b \in \mathbf{R}\}$, i.e.,

$$\mathscr{E}(R \circ f \circ S) = R \circ \mathscr{E}(f) \circ S \tag{2.4}$$

for all R, S in this group. In particular, if f commutes with the translation $z \mapsto z+1$, so does its Beurling–Ahlfors extension $\mathscr{E}(f)$. In other words, the operator $\mathscr{E}$ maps $H_T(\mathbf{R})$ into $H_T(\mathbf{H})$.

Lemma 2.3. *Suppose $f \in H_T(\mathbf{R})$ and $F = \mathscr{E}(f) \in H_T(\mathbf{H})$. Then $K_F(x+iy) \to 2$ uniformly in x as $y \to +\infty$.*

Proof. Since F commutes with $z \mapsto z+1$, it suffices to restrict x to the interval $[0,1]$. From the relation $f(x+1) = f(x)+1$ and the definition of F it is easy to see that as $y \to +\infty$,

$$\frac{1}{y}\, f(x+y) \to 1, \quad \frac{1}{y}\, f(x-y) \to -1,$$

and

$$\frac{1}{y}\,u(x,y) \to \frac{1}{2}, \quad \frac{1}{y}\,v(x,y) \to -\frac{1}{2}.$$

Hence by (2.3),

$$\frac{\partial u}{\partial x}(x,y) \to 1, \quad \frac{\partial u}{\partial y}(x,y) \to \frac{1}{2}, \quad \frac{\partial v}{\partial x}(x,y) \to 1 \quad \text{and} \quad \frac{\partial v}{\partial y}(x,y) \to -\frac{1}{2},$$

all limits being uniform in $x \in [0,1]$. It follows from (2.2) that as $y \to +\infty$,

$$K_F(x+iy) + K_F(x+iy)^{-1} \to \frac{5}{2} \quad \text{or} \quad K_F(x+iy) \to 2. \qquad \square$$

Corollary 2.4. *Let $f \in H_T(\mathbf{R})$ and $F = \mathscr{E}(f) \in H_T(\mathbf{H})$. Fix a rectangle $X = [0,1] \times (0,\nu)$.*

(i) *Suppose $\Phi\colon [1,+\infty) \to [1,+\infty)$ is continuous and $p > 0$. Then $\Phi \circ K_F \in L^p(\mathbf{H},\sigma)$ if and only if $\Phi \circ K_F \in L^p(X)$.*

(ii) *F is a David map of $\mathbf{H}$ if and only if K_F is a function of logarithmic type on X.*

Proof. For (i), first note that the spherical and Lebesgue measures are comparable on X, so $\Phi \circ K_F \in L^p(\mathbf{H},\sigma)$ clearly implies $\Phi \circ K_F \in L^p(X)$. Conversely, if $\Phi \circ K_F \in L^p(X)$, then $\Phi \circ K_F \in L^p(X,\sigma)$. Since $K_F(z+n) = K_F(z)$ for each integer n, and since the derivative of $z \mapsto z+n$ on X has spherical norm comparable to $1/n^2$, we must have $\Phi \circ K_F \in L^p(\mathbf{R} \times (0,\nu),\sigma)$. On the other hand, K_F is continuous on $\mathbf{H}$ since F is C^1, so by Lemma 2.3 K_F is bounded on $\mathbf{R} \times [\nu,+\infty)$. It follows that $\Phi \circ K_F \in L^p(\mathbf{H},\sigma)$.

For (ii), apply (i) to $\Phi(x) = \exp(x)$ and make use of Lemma 2.2. $\square$

3. Scalewise distortion of a circle homeomorphism

Basic properties. Let $f \in H_T(\mathbf{R})$ and consider the function $\delta_f(x,t)$ defined for $x \in \mathbf{R}$ and $t > 0$ which measures how much the relative length of the adjacent intervals of size t at x is distorted under f:

$$\delta_f(x,t) = \max\left\{\frac{f(x+t)-f(x)}{f(x)-f(x-t)}, \frac{f(x)-f(x-t)}{f(x+t)-f(x)}\right\}. \tag{3.1}$$

Clearly δ_f is continuous in both variables, $\delta_f \geq 1$ and $\delta_f(x+1,t) = \delta_f(x,t)$. Moreover, it is easy to check that $\delta_f(x,t) \leq 2$ whenever $t \geq 1$.

The *scalewise distortion* of f is the continuous function $\rho_f\colon (0,+\infty) \to [1,+\infty)$ defined by

$$\rho_f(t) = \sup_{x\in\mathbf{R}} \delta_f(x,t).$$

The bound $\rho_f(t) \leq 2$ for $t \geq 1$ shows that the scalewise distortion of a circle homeomorphism can grow large only at small scales, as $t \to 0^+$.

It follows from the definition that if I, I' are adjacent intervals with $|I| = |I'| = t$, then

$$\rho_f(t)^{-1} \leq \frac{|f(I')|}{|f(I)|} \leq \rho_f(t).$$

More generally, if I, I' are adjacent intervals such that $t = |I| \leq |I'| \leq k\,t$ for some positive integer k, an easy induction shows that

$$\rho_f(t)^{-1} \leq \frac{|f(I')|}{|f(I)|} \leq \rho_f(t) + \rho_f(t)^2 + \cdots + \rho_f(t)^k.$$

Thus, if I, I' are adjacent intervals with $t = \min\{|I|, |I'|\} \leq \max\{|I|, |I'|\} \leq k\,t$, then

$$\frac{1}{2}\,\rho_f(t)^{-k} \leq \frac{|f(I')|}{|f(I)|} \leq 2\,\rho_f(t)^k \tag{3.2}$$

provided that $\rho_f(t)$ is large ($\rho_f(t) \geq 2$ will do).

Scalewise distortion and David extensions. The asymptotic behavior of the scalewise distortion can be used to formulate conditions for David extendibility of a circle homeomorphism. To see this, suppose first that $F \in H_T(\mathbf{H})$ is a David map. Let $\mathbf{H}^* = \{z : \mathrm{Im}(z) < 0\}$ denote the lower half-plane and $\iota(z) = \overline{z}$. The Beltrami coefficient

$$\mu = \begin{cases} \mu_F & \text{in } \mathbf{H}, \\ \iota \circ \mu_F \circ \iota & \text{in } \mathbf{H}^* \end{cases}$$

is ι-invariant and $\frac{1+|\mu|}{1-|\mu|}$ satisfies a condition of the form (1.1) in $\mathbf{C}$. It follows from David's theorem (see §1) that there is a unique David map $G\colon \mathbf{C} \to \mathbf{C}$ which solves the Beltrami equation $\mu_G = \mu$ and is normalized so that $G(i) = F(i), G(-i) = \overline{F(i)}$. The David map $\iota \circ G \circ \iota$ satisfies precisely the same conditions, so $\iota \circ G = G \circ \iota$ by uniqueness. In particular, G preserves the real line, maps $\mathbf{H}$ to $\mathbf{H}$ and $\mathbf{H}^*$ to $\mathbf{H}^*$. Now F and (the restriction of) G are David maps in $\mathbf{H}$ with the same Beltrami coefficient. Invoking the uniqueness part of David's theorem, this time on $\mathbf{H}$, it follows that $F = \phi \circ G$ for some conformal automorphism ϕ of $\mathbf{H}$. Since ϕ has two fixed points at $F(i)$ and ∞, we conclude that $\phi = \mathrm{id}$ and $F = G$ in $\mathbf{H}$. In particular, F extends homeomorphically to the boundary.

Let $f \in H_T(\mathbf{R})$ denote the boundary homeomorphism of F. An extremal length estimate gives the inequality

$$\delta_f(x,t) \leq C_1 \exp\left(\frac{C_2}{|D|} \int_D K_G(z)\, |dz|^2 \right) \quad \text{if } 0 \leq x \leq 1,\ 0 < t < 1, \tag{3.3}$$

where $D = \mathbf{D}(x, 2t)$ is the disk of radius $2t$ centered at x and $C_1, C_2 > 0$ are constants (see [S]). On every compact subset X of the plane, the dilatation K_G is a function of logarithmic type. Choose X large enough so that it contains all the disks $D = \mathbf{D}(x, 2t)$ for $0 \leq x \leq 1$ and $0 < t < 1$. By Lemma 2.2, there is a $p > 0$

such that $\exp(K_G) \in L^p(X)$. By Jensen's inequality,

$$\exp\left(\frac{p}{|D|}\int_D K_G(z)\,|dz|^2\right) \le \frac{1}{|D|}\int_D \exp(pK_G(z))\,|dz|^2 \le C_3\,t^{-2}$$

for some $C_3 > 0$. Using this estimate in (3.3) and taking the supremum over all $x \in [0,1]$, we conclude that there are constants $C, \alpha > 0$ such that $\rho_f(t) \le C\,t^{-\alpha}$ for all $0 < t < 1$. (Alternatively, we could arrive at the same result using the general modulus estimates established in [RW].)

Conversely, take any $f \in H_T(\mathbf{R})$ and let $F = \mathscr{E}(f) \in H_T(\mathbf{H})$ be its Beurling–Ahlfors extension. It is shown in [CCH] that the dilatation $K_F(x+iy)$ is bounded above by a constant multiple of $\rho_f(y)$. In particular, if the scalewise distortion $\rho_f(t)$ is dominated by $\log(1/t)$ as $t \to 0^+$, we can find constants $C, \nu > 0$ such that

$$K_F(x+iy) \le C\,\log\frac{1}{y} \quad \text{if } 0 \le x \le 1,\ 0 < y < \nu. \tag{3.4}$$

This, by Corollary 2.4(ii), shows that F is a David map of $\mathbf{H}$.

We collect the above observations in the following

Theorem 3.1. *If $F \in H_T(\mathbf{H})$ is a David map, the scalewise distortion of its boundary homeomorphism $f \in H_T(\mathbf{R})$ satisfies*

$$\log\rho_f(t) = O\left(\log\frac{1}{t}\right) \quad \text{as } t \to 0^+. \tag{3.5}$$

On the other hand, any $f \in H_T(\mathbf{R})$ whose scalewise distortion satisfies

$$\rho_f(t) = O\left(\log\frac{1}{t}\right) \quad \text{as } t \to 0^+ \tag{3.6}$$

extends to a David map in $H_T(\mathbf{H})$.

Two examples. The conditions (3.5) and (3.6) are off by a logarithmic factor. The discrepancy is reminiscent of a similar situation for quasiconformal maps: Every K-quasiconformal mapping of $\mathbf{H}$ restricts to a ρ-quasisymmetric homeomorphism of the real line, with $\rho = (1/16)e^{\pi K}$ [BA]. On the other hand, every ρ-quasisymmetric homeomorphism of $\mathbf{R}$ extends to a K-quasiconformal mapping of $\mathbf{H}$, with $K = 2\rho$ [L].

The question arises as to whether the gap between (3.5) and (3.6) can be filled, i.e., whether there is an optimal condition for David extendability which lies somewhere between (3.5) and (3.6). The following two examples will show that the answer is negative.

Example 3.2. Fix a small $\varepsilon > 0$ and take any $f \in H_T(\mathbf{R})$ which has the following properties: (i) $f(x) = 1/(\log\log 1/x)$ on $0 < x < \varepsilon$; (ii) f is smooth with $f'(x) > 1$ on $0 < x < 1$; (iii) $f(-x) = -f(x)$ for all x. A calculus exercise shows that there is a constant $C > 0$ such that for all small $t > 0$,

$$\rho_f(t) \le C\,\delta_f(t,t) = \frac{C\,f(t)}{f(2t)-f(t)} = \frac{C\,\log\log\frac{1}{2t}}{\log\log\frac{1}{t} - \log\log\frac{1}{2t}}.$$

It follows that

$$\rho_f(t) = O\left(\log\frac{1}{t}\,\log\log\frac{1}{t}\right) \quad \text{as } t \to 0^+. \tag{3.7}$$

This is a much slower growth than (3.5). However, f cannot be extended to a David map in $H_T(\mathbf{H})$ since such an extension would imply a modulus of continuity

$$|f(x) - f(y)| \leq C\left(\log\frac{1}{|x-y|}\right)^{-\alpha} \quad \text{if } |x-y| < 1$$

for some $C, \alpha > 0$ (see [D]) which certainly fails here. *In particular, the condition* (3.5), *which is necessary for David extendibility, is not sufficient.*

Example 3.3. This example comes from complex dynamics (see [PZ] for technical details). Let $g \in H_T(\mathbf{R})$ be real-analytic with a critical point at $x = 0$ and irrational rotation number θ. There exists a unique homeomorphism $f \in H_T(\mathbf{R})$ which fixes 0 and conjugates g to the translation $\tau\colon x \mapsto x + \theta$:

$$f \circ g = \tau \circ f.$$

It is shown in [PZ] that if the partial quotients $\{a_n\}$ of the continued fraction expansion of θ satisfy

$$\log a_n = O(\sqrt{n}) \quad \text{as } n \to +\infty,$$

then f admits a David extension in $H_T(\mathbf{H})$. Fix such a rotation number, for example by letting a_n be the integer part of $e^{\sqrt{n}}$. The scalewise distortion of f can be estimated from below as follows. Suppose $\{p_n/q_n\}$ is the sequence of rational convergents of θ. Let I_n be the closed interval with endpoints 0 and $g^{q_n}(0) - p_n$, and J_n be the closed interval with endpoints 0 and $\tau^{q_n}(0) - p_n$. The pairs (I_n, I_{n-1}) and (J_n, J_{n-1}) are adjacent, i.e., $I_n \cap I_{n-1} = J_n \cap J_{n-1} = \{0\}$. Moreover, the following statements are true for all $n \geq 1$:

(i) I_n and I_{n-1} have comparable lengths, i.e., there is an integer $k \geq 2$ such that

$$k^{-1} \leq \frac{|I_{n-1}|}{|I_n|} \leq k.$$

(ii) There is a constant $C_1 > 0$ such that

$$|I_n| \geq C_1\, k^{-n}.$$

This follows from (i) with $C_1 = |I_0|$.

(iii) $|J_n| = |q_n\theta - p_n|$, so by classical continued fraction theory,

$$\frac{|J_{n-1}|}{|J_n|} = \frac{|q_{n-1}\theta - p_{n-1}|}{|q_n\theta - p_n|} \geq \frac{1}{2}\,a_{n+1}.$$

By (ii), the length $t_n = \min\{|I_{n-1}|, |I_n|\}$ satisfies $t_n \geq C_1\, k^{-n}$, so

$$\log\frac{1}{t_n} \leq C_2\, n \tag{3.8}$$

for some $C_2 > 0$. On the other hand, f maps I_n to J_n for all n, so (i), (iii) and the estimate (3.2) show that

$$\frac{1}{2}\, a_{n+1} \leq \frac{|J_{n-1}|}{|J_n|} = \frac{|f(I_{n-1})|}{|f(I_n)|} \leq 2\,(\rho_f(t_n))^k.$$

Since a_n is the integer part of $e^{\sqrt{n}}$, it follows that

$$\rho_f(t_n) \geq C_4 \exp(C_3 \sqrt{n})$$

for some $C_3, C_4 > 0$. By (3.8), we conclude there is a $C_5 > 0$ such that

$$\rho_f(t_n) \geq C_4 \exp\left(C_5 \sqrt{\log \frac{1}{t_n}} \right). \tag{3.9}$$

This is a much faster growth than (3.6), at least at infinitely many small scales. *In particular, the condition* (3.6), *which is sufficient for David extendibility, is not necessary.*

Since the growth condition in (3.7) is slower than the one in (3.9), we conclude that *no optimal condition for David extendibility can be formulated solely in terms of the asymptotic growth of the scalewise distortion.*

4. Pointwise distortion of a circle homeomorphism

Closely related to the notion of scalewise distortion of $f \in H_T(\mathbf{R})$ is its *pointwise distortion* $\lambda_f \colon \mathbf{R} \to [1, +\infty]$ defined by

$$\lambda_f(x) = \sup_{t>0} \delta_f(x,t),$$

where δ_f is the function introduced in (3.1). Unlike the scalewise distortion, λ_f is only lower semicontinuous and may well take the value $+\infty$. Taking the supremum over all $t > 0$ in the periodicity relation $\delta_f(x+1,t) = \delta_f(x,t)$ gives $\lambda_f(x+1) = \lambda_f(x)$ for all x, which means the pointwise distortion can be viewed as a function on the circle.

Pointwise distortion and David extensions. We first prove Theorem A in §1 that gives a sufficient condition for David extendibility of a circle homeomorphism in terms of its pointwise distortion.

Proof of Theorem A. Let $F = \mathscr{E}(f) \in H_T(\mathbf{H})$ be the Beurling–Ahlfors extension of f. We begin by a standard normalization (compare [BA]). Fix $x_0 + iy_0 \in \mathbf{H}$ with $0 < y_0 < 1$, and consider the real affine maps $R, S \colon \mathbf{H} \to \mathbf{H}$ defined by

$$R(z) = \frac{z - f(x_0)}{f(x_0 + y_0) - f(x_0)} \quad \text{and} \quad S(z) = y_0\, z + x_0.$$

The composition $G = R \circ F \circ S$ is a homeomorphism of $\mathbf{H}$ whose boundary map $g = R \circ f \circ S$ satisfies $g(0) = 0$ and $g(1) = 1$. Note that by (2.4),

$$G = R \circ \mathscr{E}(f) \circ S = \mathscr{E}(R \circ f \circ S) = \mathscr{E}(g).$$

Evidently the dilatation $K_F(x_0+iy_0)$ is equal to $K_G(i)$. To estimate the latter, use (2.2), (2.3) and the conditions $g(0)=0, g(1)=1$ to deduce that

$$K_G(i)+K_G(i)^{-1}=\frac{r^{-1}(1+a^2)+r(1+b^2)}{a+b}, \tag{4.1}$$

where

$$r=-g(-1), \quad a=1-\int_0^1 g(t)\,dt \quad \text{and} \quad b=1+r^{-1}\int_{-1}^0 g(t)\,dt.$$

Clearly $0<a,b<1$. By replacing $g(x)$ with $-\frac{1}{r}g(-x)$ if necessary, we may assume that $r\geq 1$. The rest of the proof consists essentially of estimating the right side of (4.1).

The definition of λ_f shows that for $0\leq x<1$,

$$\frac{g(x)-g(2x-1)}{g(1)-g(x)}\leq \lambda_f(x_0+xy_0), \tag{4.2}$$

or

$$g(x)-g(2x-1)\leq \lambda_f(x_0+xy_0)\,(1-g(x)).$$

Integrating from 0 to 1, we obtain

$$\int_0^1 g(x)\,dx-\frac{1}{2}\int_{-1}^1 g(x)\,dx\leq \int_0^1 \lambda_f(x_0+xy_0)(1-g(x))\,dx. \tag{4.3}$$

The left side of (4.3) is

$$\frac{1}{2}\int_0^1 g(x)\,dx-\frac{1}{2}\int_{-1}^0 g(x)\,dx=\frac{1}{2}(1-a)-\frac{1}{2}r(b-1). \tag{4.4}$$

Let us estimate the right side of (4.3). By the assumption $\exp(\lambda_f)\in L^p[0,1]$ for some $p>0$. Jensen's inequality (applied to the probability measure $\frac{1}{a}(1-g(x))\,dx$ on $[0,1]$) then shows

$$\begin{aligned}\exp\left(\frac{1}{a}\int_0^1 p\,\lambda_f(x_0+xy_0)(1-g(x))\,dx\right) &\leq \frac{1}{a}\int_0^1 \exp(p\,\lambda_f(x_0+xy_0))(1-g(x))\,dx\\ &\leq \frac{1}{a}\int_0^1 \exp(p\,\lambda_f(x_0+xy_0))\,dx\\ &=\frac{1}{ay_0}\int_{x_0}^{x_0+y_0}\exp(p\,\lambda_f(x))\,dx\\ &\leq \frac{N^p}{ay_0},\end{aligned}$$

where N is the L^p-norm of $\exp(\lambda_f)$ on $[0,1]$. Set

$$C=\max\{3,N^p\} \tag{4.5}$$

and take the logarithm of the last inequality to obtain

$$\int_0^1 \lambda_f(x_0 + xy_0)(1 - g(x))\,dx \le \frac{a}{p}\,\log\left(\frac{C}{ay_0}\right). \tag{4.6}$$

Putting (4.3), (4.4) and (4.6) together, it follows that

$$\frac{1}{2}(1-a) - \frac{1}{2}\,r(b-1) \le \frac{a}{p}\,\log\left(\frac{C}{ay_0}\right),$$

which can be written in the form

$$b \ge -\frac{1}{r}\,a\left(1 + \frac{2}{p}\log\left(\frac{C}{ay_0}\right)\right) + \frac{r+1}{r}. \tag{4.7}$$

This suggests that we consider the function

$$\beta = B(\alpha) = -\frac{1}{r}\,\alpha\left(1 + \frac{2}{p}\log\left(\frac{C}{\alpha y_0}\right)\right) + \frac{r+1}{r}, \quad 0 < \alpha \le 1. \tag{4.8}$$

Since $C \ge 3$ by (4.5), it is easily seen that B is strictly decreasing and convex. Moreover,

$$B(1) = 1 - \frac{2}{rp}\log\left(\frac{C}{y_0}\right) < 1 < B(0^+) = \frac{r+1}{r}$$

(see Figure 1). It follows that there exists a unique $0 < \varepsilon < 1$ such that $B(\varepsilon) = 1$. In other words, ε is the unique solution of the equation

$$\frac{1}{\varepsilon} = \frac{2}{p}\log\left(\frac{C}{\varepsilon y_0}\right) + 1. \tag{4.9}$$

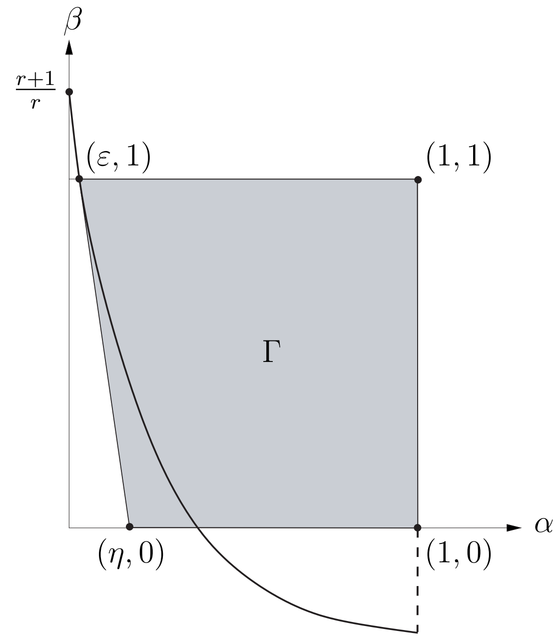

Figure 1. Graph of the function $\beta = B(\alpha)$ in (4.8). Here $B(1) < 0$ but depending on the size of the parameters, we may have $B(1) \ge 0$.

We need an estimate for how small ε can be. Using the inequality $\log x \leq \sqrt{x}$ for $x > 0$, we see that

$$\frac{1}{\varepsilon} = \frac{2}{p} \log\left(\frac{C}{\varepsilon y_0}\right) + 1 \leq \frac{2}{p}\left(\frac{C}{\varepsilon y_0}\right)^{\frac{1}{2}} + 1 \leq \left(\frac{C_1}{\varepsilon y_0}\right)^{\frac{1}{2}}$$

for some $C_1 > 0$, which gives the inequality $1/\varepsilon \leq C_1/y_0$. Putting this back into (4.9), we obtain

$$\frac{1}{\varepsilon} \leq \frac{2}{p} \log\left(\frac{CC_1}{y_0^2}\right) + 1,$$

which yields the improved estimate

$$\frac{1}{\varepsilon} \leq C_2 \log\left(\frac{C_3}{y_0}\right) \tag{4.10}$$

for some $C_2, C_3 > 0$. Let $(\eta, 0)$ be the point where the tangent line to the graph of $\beta = B(\alpha)$ at $(\varepsilon, 1)$ meets the horizontal axis (see Figure 1). By (4.9),

$$B'(\varepsilon) = -\frac{1}{r}\left(1 + \frac{2}{p}\log\left(\frac{C}{\varepsilon y_0}\right)\right) + \frac{2}{rp} = -\frac{1}{r\varepsilon} + \frac{2}{rp} > -\frac{1}{r\varepsilon},$$

so

$$\eta = \varepsilon - \frac{1}{B'(\varepsilon)} > \varepsilon + r\varepsilon > r\varepsilon. \tag{4.11}$$

Now consider the quadrilateral Γ in the (α, β)-plane with vertices $(1, 0)$, $(1, 1)$, $(\varepsilon, 1)$, and $(\eta, 0)$ as in Figure 1. By (4.7) and the convexity of B, the point (a, b) must belong to Γ. Beurling and Ahlfors observe in [BA] that the quantity

$$L(\alpha, \beta) = \frac{r^{-1}(1+\alpha^2) + r(1+\beta^2)}{\alpha + \beta}$$

is a convex function of (α, β). Hence its maximum on Γ must occur at one of the vertices. The assumption $r \geq 1$ and the inequality (4.11) show that

$$\begin{aligned}
&L(1,0) = 2r^{-1} + r \leq 3r,\\
&L(1,1) = r^{-1} + r \leq 2r,\\
&L(\varepsilon,1) = \left(\frac{\varepsilon^2+1}{\varepsilon+1}\right) r^{-1} + \left(\frac{2}{\varepsilon+1}\right) r \leq 2(r^{-1}+r) \leq 4r,\\
&L(\eta,0) = r^{-1}\eta + (r + r^{-1})\,\eta^{-1} \leq 2(r+r^{-1})\eta^{-1} \leq 4r\eta^{-1} \leq 4\varepsilon^{-1}.
\end{aligned}$$

It follows from (4.1) that

$$K_G(i) < K_G(i) + K_G(i)^{-1} = L(a,b) \leq 4\,\max\{r, \varepsilon^{-1}\}.$$

Substituting $x = 0$ in (4.2) gives $r \leq \lambda_f(x_0)$. Together with (4.10) and the fact that $K_F(x_0 + iy_0) = K_G(i)$, this gives the estimate

$$K_F(x_0 + iy_0) \leq 4\,\max\left\{\lambda_f(x_0), C_2 \log\left(\frac{C_3}{y_0}\right)\right\} \quad \text{if } 0 \leq x_0 \leq 1,\ 0 < y_0 < 1. \tag{4.12}$$

By Lemma 2.2, λ_f is of logarithmic type on $[0,1]$. So is $\log(C_3/y)$ on $(0,1)$ trivially. Hence, Lemma 2.1 shows the same must be true of K_F on $[0,1]\times(0,1)$. It follows from Corollary 2.4(ii) that F is a David map of $\mathbf{H}$. □

I do not know how the condition (1.8) of Theorem A and (3.6) of Theorem 3.1 compare in general. However, the following is true:

Theorem 4.1. *Suppose $f \in H_T(\mathbf{R})$ and $\exp(\lambda_f) \in L^p[0,1]$ for some $p > 0$. Then*

$$\rho_f(t) = O\left(\left(\log \frac{1}{t}\right)^2\right) \quad \textit{as } t \to 0^+.$$

In view of Example 3.3, we conclude that (1.8) is not a necessary condition for David extendibility of a circle homeomorphism.

The proof of Theorem 4.1 is based on the following a priori estimate:

Lemma 4.2. *Suppose $f \in H_T(\mathbf{R})$ and $\delta_f(x_0,t) = \delta > 1$. Then*

$$\left|\left\{x \in [x_0 - t, x_0 + t] : \lambda_f(x) > \frac{1}{2}\sqrt{\delta}\right\}\right| \geq \frac{1}{8}t.$$

Proof. Without losing generality, we may assume $f(x_0 - t) = 0$, $f(x_0 + t) = 1$, and $f(x_0) = \delta/(\delta+1)$. If $\lambda_f(x) > \frac{1}{2}\sqrt{\delta}$ for all $x \in [x_0, x_0 + \frac{1}{8}t]$ there is nothing to prove. Otherwise, we can find $y \in [x_0, x_0 + \frac{1}{8}t]$ such that $\lambda_f(y) \leq \frac{1}{2}\sqrt{\delta}$. Set $s = x_0 + t - y$. Since

$$\frac{f(y) - f(y-s)}{1 - f(y)} = \delta_f(y,s) \leq \lambda_f(y) \leq \frac{1}{2}\sqrt{\delta},$$

we have

$$f(y-s) \geq f(y)(\frac{1}{2}\sqrt{\delta}+1) - \frac{1}{2}\sqrt{\delta} \geq f(x_0)(\frac{1}{2}\sqrt{\delta}+1) - \frac{1}{2}\sqrt{\delta} = \frac{2\delta - \sqrt{\delta}}{2(\delta+1)}.$$

Clearly, $x_0 - t \leq y - s \leq x_0 - \frac{3}{4}t$. Moreover, for all $x \in [y-s, x_0 - \frac{1}{2}t]$,

$$\begin{aligned} \lambda_f(x) \geq \delta_f(x, x - x_0 + t) &= \frac{f(x)}{f(2x - x_0 + t) - f(x)} \geq \frac{f(y-s)}{f(x_0) - f(y-s)} \\ &\geq \left(\frac{2\delta - \sqrt{\delta}}{2(\delta+1)}\right) \Big/ \left(\frac{\delta}{\delta+1} - \frac{2\delta - \sqrt{\delta}}{2(\delta+1)}\right) = 2\sqrt{\delta} - 1 > \tfrac{1}{2}\sqrt{\delta}. \end{aligned}$$

This proves the result since the length of $[y-s, x_0 - \frac{1}{2}t]$ is at least $\frac{1}{4}t$. □

Proof of Theorem 4.1. For any small $t > 0$, find x_0 so that $\delta = \rho_f(t) = \delta_f(x_0,t) > 1$. Combining Lemma 2.2 and Lemma 4.2, we obtain

$$\frac{1}{8}t \leq \left|\left\{x \in [x_0 - t, x_0 + t] : \lambda_f(x) > \frac{1}{2}\sqrt{\delta}\right\}\right| \leq Ce^{-\alpha\sqrt{\delta}}$$

for some constants $C, \alpha > 0$. It follows that $\delta \leq C_1(\log 1/t)^2$ for some $C_1 > 0$, as required. □

A unified condition for David extendibility. Below we show that the conditions (1.8) on λ_f and (3.6) on ρ_f are both implied by a single condition on the function δ_f in (3.1):

Theorem 4.3. *Consider the following conditions on $f \in H_T(\mathbf{R})$:*

(i) *There is a Borel measure μ on $\mathbf{R}$, invariant under $x \mapsto x+1$ and finite on $[0,1]$, and a constant $\alpha > 0$ such that*

$$\exp(\alpha\, \delta_f(x,t)) \leq \frac{1}{2t}\, \mu([x-t, x+t]) \quad \text{if } x \in \mathbf{R},\ t > 0. \tag{4.13}$$

(ii) *The scalewise distortion ρ_f has the asymptotic growth*

$$\rho_f(t) = O\left(\log \frac{1}{t}\right) \quad \text{as } t \to 0^+.$$

(iii) *The pointwise distortion λ_f satisfies $\exp(\lambda_f) \in L^p[0,1]$ for some $p > 0$.*

Then the implications (i) $\Longrightarrow$ (ii) *and* (i) $\Longrightarrow$ (iii) *hold. In particular, any of these conditions implies that f extends to a David map in $H_T(\mathbf{H})$.*

Proof. Assuming (i), first note that there is a $C > 0$ such that $\mu([x-t, x+t]) \leq C$ for all $x \in \mathbf{R}$ and $0 < t < 1$. Taking the supremum over all x in (4.13), we obtain

$$\exp(\alpha\rho_f(t)) \leq \frac{C}{2t} \quad \text{if } 0 < t < 1$$

which implies (ii).

Again assuming (i), take the supremum over all $t > 0$ in (4.13) to get

$$\exp(\alpha\lambda_f) \leq M(\mu),$$

where $M(\mu)$ is the Hardy–Littlewood maximal function of μ. It is well-known that $M(\mu)$ is in weak L^1 so that

$$|\{x \in [0,1] : M(\mu)(x) > t\}| \leq \frac{C}{t}$$

for some $C > 0$. It follows that

$$|\{x \in [0,1] : \lambda_f(x) > t\}| = |\{x \in [0,1] : \exp(\alpha\lambda_f(x)) > e^{\alpha t}\}| \leq C\, e^{-\alpha t},$$

which means λ_f is of logarithmic type on $[0,1]$. This, by Lemma 2.2, implies (iii).

That either of the conditions (ii) or (iii) implies a David extension follows from Theorem 3.1 and Theorem A. $\square$

5. Extensions for other trans-quasiconformal maps

The preceding results yield analogous extension theorems for other classes of trans-quasiconformal maps introduced in §1. Let us first prove Theorem B quoted in §1 on extensions with subexponentially integrable dilatation.

Proof of Theorem B. The argument is a close adaptation of the proof of Theorem A, so we only give a quick sketch. Since Φ satisfies (1.4), the inverse function $\Psi = \Phi^{-1}$ grows faster than $\log x$ but slower than $(\log x)^\kappa$ for any $\kappa > 1$. The proof of

Theorem A can thus be repeated with obvious modifications, e.g., by replacing exp and log by Φ and Ψ everywhere and defining an appropriate analog of the function $\beta = B(\alpha)$. Tracing all the steps to the end this way, we obtain constants $C_1, C_2 > 0$ such that the dilatation of $F = \mathscr{E}(f)$ satisfies

$$K_F(x+iy) \le C_1 \max\left\{\lambda_f(x), \Psi\left(\frac{C_2}{y}\right)\right\} \quad \text{if } 0 \le x \le 1,\ 0 < y < 1.$$

Choose $\kappa > 0$ so that $\Phi(C_1 x) \le (\Phi(x))^\kappa$ and without losing generality assume $0 < p < 1$. It follows that $\Phi \circ K_F \in L^\nu([0,1] \times (0,1))$, where $\nu = p/\kappa$. Thus, by Corollary 2.4(i), $\Phi \circ K_F \in L^\nu(\mathbf{H}, \sigma)$. $\square$

Next, we discuss BMO-quasiconformal maps and Theorem C. We start by recalling a few basic facts about BMO functions.

Let $J \subset \mathbf{R}$ be an open interval and $q \in L^1_{\mathrm{loc}}(J)$. We say q has *bounded mean oscillation* on J and write $q \in BMO(J)$ if

$$\|q\|_* = \sup_{I \subset J} \frac{1}{|I|} \int_I |q(x) - q_I|\, dx < +\infty.$$

Here the supremum is taken over all compact intervals I in J and $q_I = (1/|I|) \int_I q$ is the average value of q over I.

The space $BMO(J)$ contains $L^\infty(J)$ properly. More generally, according to John and Nirenberg [JN], $q \in BMO(J)$ if and only if there are constants $C, \alpha > 0$ such that

$$\int_I \exp(\alpha |q(x) - q_I|)\, dx \le C\, |I| \tag{5.1}$$

for every compact interval $I \subset J$. In particular, it follows from Lemma 2.2 that *if $I \subset J$ is compact, every positive function $q \in BMO(J)$ is of logarithmic type on I.*

Functions of bounded mean oscillation in higher dimensional Euclidean spaces are defined similarly by replacing compact intervals I in the above definition with compact cubes or round balls.

We will need the following analog of Lemma 2.1 for BMO functions:

Lemma 5.1. *Let I_1, I_2 be open intervals in $\mathbf{R}$ and consider positive functions $a \in BMO(I_1)$ and $b \in BMO(I_2)$. Then the function $\varphi\colon I_1 \times I_2 \to [0, +\infty]$ defined by*

$$\varphi(x,y) = \max\{a(x), b(y)\}$$

is in $BMO(I_1 \times I_2)$.

Proof. In view of

$$\varphi(x,y) = \frac{1}{2}\,(a(x) + b(y)) + \frac{1}{2}\,|a(x) - b(y)|$$

it suffices to prove that the function $\psi(x,y) = |a(x) - b(y)|$ is in $BMO(I_1 \times I_2)$. Take any compact cube $I \times J \subset I_1 \times I_2$ and set $c = |a_I - b_J|$. The inequality

$$|\psi(x,y) - c| \le |a(x) - a_I| + |b(y) - b_J|$$

gives

$$\int_I |\psi(x,y)-c|\,dx \leq \|a\|_* |I| + |b(y)-b_J|\,|I|.$$

Hence, by Fubini,

$$\begin{aligned}\int_{I\times J} |\psi(x,y)-c|\,dx\,dy &= \int_J \left(\int_I |\psi(x,y)-c|\,dx\right) dy\\ &\leq |I| \int_J (\|a\|_* + |b(y)-b_J|)\,dy\\ &\leq |I|\,|J|\,(\|a\|_* + \|b\|_*).\end{aligned}$$

Since it is easy to check that

$$\int_{I\times J} |\psi(x,y)-\psi_{I\times J}|\,dx\,dy \leq 2\int_{I\times J} |\psi(x,y)-c|\,dx\,dy,$$

we obtain $\psi \in BMO(I_1 \times I_2)$. □

We are now ready to prove Theorem C in §1.

Proof of Theorem C. Assuming (i), use John–Nirenberg's inequality (5.1) to deduce $\exp(q) \in L^p[0,1]$ for some $p>0$. By Jensen's inequality, if $0<t<1$,

$$\exp(p\,\delta_f(x,t)) \leq \frac{1}{2t}\int_{x-t}^{x+t} \exp(p\,q(s))\,ds \leq \frac{C_1}{t}$$

for some $C_1>0$. Taking the supremum over all x then gives

$$\exp(p\,\rho_f(t)) \leq \frac{C_1}{t} \quad \text{if } 0<t<1,$$

which implies (ii).

Again assuming (i), take the supremum over all $t>0$ to obtain

$$\lambda_f \leq M(q),$$

where $M(q)$ is the Hardy–Littlewood maximal function of q. According to Bennett, DeVore and Sharpley, $M(q) \in BMO(\mathbf{R})$ whenever $q \in BMO(\mathbf{R})$ [BDS]. This gives (iii).

Finally, let us check that either of the conditions (ii) or (iii) implies f has a BMO-quasiconformal extension in $H_T(\mathbf{H})$. In the case of (ii), by the proof of Theorem 3.1, the dilatation of $F=\mathscr{E}(f)$ satisfies

$$K_F(x+iy) \leq C\,\log\frac{1}{y} \quad \text{if } 0<y<\nu$$

for some $C,\nu>0$ (see (3.4)). By Lemma 2.3, the quantity

$$K_0 = \sup\{K_F(x+iy) : x\in\mathbf{R},\ y\geq\nu\}$$

is finite. The function

$$h(y) = \begin{cases} C\,\log(1/y) & 0<y<\nu,\\ K_0 & y\geq\nu\end{cases} \tag{5.2}$$

is easily seen to be in $BMO(0,+\infty)$ and it follows that the majorant of K_F defined by $Q(x+iy)=h(y)$ is in $BMO(\mathbf{H})$.

In the case of (iii), the assumption is that $\lambda_f \le g$ for some $g \in BMO(\mathbf{R})$. As before, John–Nirenberg's inequality implies $\exp(\lambda_f) \in L^p[0,1]$ for some $p>0$. Theorem A then shows that the dilatation of $F=\mathscr{E}(f)$ satisfies

$$K_F(x+iy) \le C\,\max\left\{\lambda_f(x), \log\frac{1}{y}\right\} \quad \text{if } 0<y<\nu$$

for some $C,\nu>0$ (compare (4.12)). By Lemma 5.1, the majorant of K_F defined by

$$Q(x+iy)=\max\{C\,g(x),h(y)\}$$

with $h(y)$ defined as in (5.2) is in $BMO(\mathbf{H})$. □

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