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DUALITY BASED A POSTERIORI ERROR ESTIMATES FOR HIGHER ORDER VARIATIONAL INEQUALITIES WITH POWER GROWTH FUNCTIONALS

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Abstract. We consider variational inequalities of higher order with p-growth potentials over a domain in the plane by the way including the obstacle problem for a plate with power hardening law. Using duality methods we prove a posteriori error estimates of functional type for the difference of the exact solution and any admissible comparision function.

1. Introduction

On a bounded Lipschitz domain $\Omega \subset \mathbf{R}^2$ we consider the minimization problem

$$(\mathscr{P}) \qquad J\,[u] := \int_\Omega \pi_p(\nabla^2 u)\,dx \to \min \ \text{ in } \mathbf{K},$$

where the class $\mathbf{K}$ consists of all functions v from the space $\overset{\circ}{W}{}^2_p(\Omega)$ s.t. $v(x) \geq \Psi(x)$ on Ω, the potential π_p is given by the formula $\pi_p(E) := \frac{1}{p}|E|^p$ for symmetric (2×2)-matrices E and $\nabla^2 u$ represents the matrix of the second generalized derivatives.

It is assumed that $\Psi \in W^2_p(\Omega)$ is a given function s.t. $\Psi|_{\partial\Omega} < 0$ and $\Psi(x_0) > 0$ for some point $x_0 \in \Omega$ and the exponent p is arbitrarily chosen in the interval $1 < p < \infty$. For a definition of the Sobolev spaces $\overset{\circ}{W}{}^2_p(\Omega)$, $W^2_p(\Omega)$ and related classes we refer to [Ad].

We recall that by Sobolev's embedding theorem the functions Ψ and $v \in W^2_p(\Omega)$ have a representative in $C^0(\overline{\Omega})$ and that this observation can be used to show that the class $\mathbf{K}$ is not empty (compare [FLM]), which means that $(\mathscr{P})$ has a unique solution $u \in \mathbf{K}$.

The second order obstacle problem $(\mathscr{P})$ is of some physical relevance: consider a plate which is clamped at the boundary and whose undeformed state is represented by the region Ω. If some outer forces are applied acting in vertical direction, then the

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equilibrium configuration can be found as a minimizer of an energy with principal part

$$\int_\Omega g(\nabla^2 w)\,dx,$$

where the mechanical properties of the plate are characterized by the given convex function g. In the case of linear elastic plates, we have $g = \pi_2$ and since our exponent p is arbitrary we can include any power-hardening law. In particular, for p close to 1 we have an approximation of perfectly plastic plates. The new feature of problem $(\mathscr{P})$ however is that the plate has to respect the given side condition.

In various papers mainly the regularity properties of minimizers (with or without obstacle) for different functions g have been investigated, we refer to [Se] for the case of plastic behaviour, whereas the case of nearly linear growth is studied in [FLM] and [BF1]. For the "classical case" $p = 2$ we refer, e.g., to [Fr].

In the present note we concentrate on a posteriori error estimates of functional type for the solution of problem $(\mathscr{P})$ by combining the methods developed by the third author for first order obstacle problems with quadratic potentials (see [Re] and also [NR1], Chapter 8) with the techniques introduced in [BR] and [BFR] for unconstrained variational problems with a power growth potential. To be more precise, let $u \in \mathbf{K}$ denote the solution of problem $(\mathscr{P})$ and consider any function v from the class of comparison functions. Then our goal is to prove the estimate

$$\|\nabla^2 u - \nabla^2 v\|_{L^p} \le \mathscr{M}(v, \mathscr{D}), \tag{1.1}$$

where $\mathscr{D}$ stands for the set of known data and where $\mathscr{M}$ is a non-negative functional depending on v, on the data of the problem such as p, Ω, Ψ and on "parameters" which are under our disposal. $\mathscr{M}$ should satisfy the following requirements:

i.) the value of $\mathscr{M}$ is easy to calculate for any choice of an admissible function v;

ii.) the estimate is consistent in the sense that

$$\begin{array}{ll} \mathscr{M}(v, \mathscr{D}) = 0 & \text{if and only if } \ v = u, \text{ moreover} \\ \mathscr{M}(v_k, \mathscr{D}) \to 0 & \text{if } \ \|\nabla^2 v_k - \nabla^2 u\|_{L^p} \to 0; \end{array}$$

iii.) $\mathscr{M}$ provides a realistic upper bound for the quantity $\|\nabla^2 u - \nabla^2 v\|_{L^p}$.

Of course iii.) means that for obtaining the bound (1.1) one carefully tries to avoid "over-estimation" so that (1.1) can be used for a reliable verification of approximative solutions obtained by various numerical methods. As already outlined in [BR] and [BFR] the cases $p \ge 2$ and $1 < p < 2$ require different techniques: in Section 3 we will prove an estimate like (1.1) if $p \ge 2$, which—without a priori estimates for the exact solution—we could not verify for the subquadratic case. Therefore, in Section 4, we pass to the dual variational problem $(\mathscr{P})^*$ and discuss a variant of (1.1) involving the dual solution σ^*. We like to remark that just for the case of technical simplicity we did not include terms like $\int_\Omega uf\,dx$ or $\int_\Omega \nabla u \cdot F\,dx$ with functions $f\colon \Omega \to \mathbf{R}$, $F\colon \Omega \to \mathbf{R}^2$ into our variational integral J. These quantities can be added without substantial changes provided f and F satisfy suitable integrability assumptions. In the same spirit we could include the double obstacle problem $\Psi \le u \le \Phi$ combined with different boundary conditions being compatible with the obstacle(s).

It is worth noting that if we start from any function $\Psi \in W_p^2(\Omega)$ and if we require that $\mathbf{K}$ is not empty, then all our calculations remain valid if $\Omega \subset \mathbf{R}^d$, $d \ge 2$, with constants partially depending on d.

2. Preliminaries

Basic facts in duality theory. We recall some facts from duality theory (see, e.g. [ET]) valid for all $1 < p < \infty$: if we let $q := \frac{p}{p-1}$, $Y^* := L^q(\Omega; \mathbf{R}^{2\times 2}_{\mathrm{sym}})$, $Y := L^p(\Omega; \mathbf{R}^{2\times 2}_{\mathrm{sym}})$ and

$$\ell(v, \tau^*) := \int_\Omega \big[\tau^* : \nabla^2 v - \pi_p^*(\tau^*)\big]\, dx,$$

where π_p^* is the conjugate function of π_p, i.e. $\pi_p^*(E) = \frac{1}{q}|E|^q$, then

$$J[v] = \sup_{\tau^* \in Y^*} \ell(v, \tau^*)$$

and

$$J[u] = J^*[\sigma^*],$$

where $J^*[\tau^*] := \inf_{v \in \mathbf{K}} \ell(v, \tau^*)$, $\tau^* \in Y^*$, is the dual functional, σ^* denotes its unique maximizer, and $u \in \mathbf{K}$ is the unique solution of the problem $(\mathscr{P})$.

Clarkson's inequality. If $p \ge 2$, then we will use a version of Clarkson's inequality [Cl] presented in [MM] for tensor-valued functions $\tau_1, \tau_2 \in Y$:

$$\int_\Omega \Big[\Big|\frac{\tau_1 + \tau_2}{2}\Big|^p + \Big|\frac{\tau_1 - \tau_2}{2}\Big|^p\Big]\, dx \;\le\; \frac{1}{2}\,\|\tau_1\|^p_{L^p} + \frac{1}{2}\,\|\tau_2\|^p_{L^p}. \tag{2.1}$$

If $p < 2$, then we apply this inequality to the dual variable.

Basic deviation estimate. In the superquadratic case, i.e. if $p \ge 2$, (2.1) implies for all $v \in \mathbf{K}$

$$\big\|\nabla^2(u - v)\big\|^p_{L^p} \le p\, 2^p \Big[\frac{1}{2}\, J[v] + \frac{1}{2}\, J[u] - J\Big[\frac{u+v}{2}\Big]\Big],$$

and since $\frac{u+v}{2} \in \mathbf{K}$ we deduce from the J-minimality of u that

$$\big\|\nabla^2(u - v)\big\|^p_{L^p} \le p\, 2^{p-1}\Big[J[v] - J[u]\Big]. \tag{2.2}$$

If $p < 2$, then the basic deviation estimate (2.2) takes the form (2.3) and is derived as follows: with $v \in \mathbf{K}$ chosen arbitrarily we have

$$\frac{1}{q}\, 2^{1-q}\|\tau^* - \sigma^*\|^q_{L^q} \le \frac{1}{q} \int_\Omega |\tau^*|^q\, dx + \frac{1}{q} \int_\Omega |\sigma^*|^q\, dx - \frac{2}{q} \int_\Omega \Big|\frac{\tau^* + \sigma^*}{2}\Big|^q\, dx$$

$$
\begin{aligned}
&= \int_\Omega \Big[\frac{1}{q}\,|\tau^*|^q - \tau^* : \nabla^2 v\Big]\,dx + \int_\Omega \Big[\frac{1}{q}\,|\sigma^*|^q - \sigma^* : \nabla^2 v\Big]\,dx \\
&\quad - 2\int_\Omega \Big[\frac{1}{q}\Big|\frac{\tau^*+\sigma^*}{2}\Big|^q - \frac{\sigma^*+\tau^*}{2} : \nabla^2 v\Big]\,dx \\
&= -\ell(v,\tau^*) - \ell(v,\sigma^*) + 2\ell\Big(v, \frac{\tau^*+\sigma^*}{2}\Big) \\
&\le \sup_{v_1\in\mathbf{K}}\Big(-\ell(v_1,\tau^*)\Big) + \sup_{v_2\in\mathbf{K}}\Big(-\ell(v_2,\sigma^*)\Big) + 2\ell\Big(v, \frac{\tau^*+\sigma^*}{2}\Big) \\
&= -\inf_{v_1\in\mathbf{K}}\ell(v_1,\tau^*) - \inf_{v_2\in\mathbf{K}}\ell(v_2,\sigma^*) + 2\ell\Big(v, \frac{\tau^*+\sigma^*}{2}\Big) \\
&= -J^*[\tau^*] - J^*[\sigma^*] + 2\ell\Big(v, \frac{\tau^*+\sigma^*}{2}\Big),
\end{aligned}
$$

and since $v \in \mathbf{K}$ is under our disposal, we may pass to the inf w.r.t. $v \in \mathbf{K}$ on the r.h.s. with the result

$$\frac{1}{q}\,2^{1-q}\|\sigma^*-\tau^*\|^q_{L^q} \le -J^*[\tau^*] - J^*[\sigma^*] + 2J^*\Big[\frac{\sigma^*+\tau^*}{2}\Big],$$

and the J^*-maximality of σ^* implies

$$\frac{1}{q}\,2^{1-q}\|\sigma^*-\tau^*\|^q_{L^q} \le J^*[\sigma^*] - J^*[\tau^*]. \tag{2.3}$$

A modified functional. Following [Re] we introduce a relaxtion of $(\mathscr{P})$: for

$$\lambda \in \Lambda := \{\rho \in L^q(\Omega) : \rho \ge 0 \text{ a.e.}\}$$

we let

$$(\mathscr{P}_\lambda) \qquad J_\lambda[w] := J[w] - \int_\Omega \lambda(w-\Psi)\,dx \to \min \text{ in } \overset{\circ}{W}{}^2_p(\Omega).$$

Clearly $(\mathscr{P}_\lambda)$ is well-posed with unique solution u_λ. Also we note that

$$\sup_{\lambda\in\Lambda} J_\lambda[w] = J[w] - \inf_{\lambda\in\Lambda}\int_\Omega \lambda(w-\Psi)\,dx = \begin{cases} J[w], & \text{if } \ w \in \mathbf{K} \\ +\infty, & \text{if } \ w \notin \mathbf{K}. \end{cases}$$

Letting

$$L(w,\tau^*,\lambda) := \int_\Omega \big[\nabla^2 w : \tau^* - \pi_p^*(\tau^*) - \lambda(w-\Psi)\big]\,dx,\ w \in \overset{\circ}{W}{}^2_p(\Omega),\ \tau^* \in Y^*,\ \lambda \in \Lambda,$$

we define the dual functional

$$J_\lambda^*[\tau^*] := \inf_{w\in\overset{\circ}{W}{}^2_p(\Omega)} L(w,\tau^*,\lambda)$$

with unique maximizer σ_λ^* and get

$$J_\lambda[u_\lambda] = J_\lambda^*[\sigma_\lambda^*]. \tag{2.4}$$

The reader should observe that $J_\lambda^*[\tau^*] > -\infty$ for $\tau^* \in Y^*$ implies that τ^* is in the class

$$Q_\lambda^* := \Big\{\eta \in Y^* : \int_\Omega \big[\eta^* : \nabla^2 w - \lambda w\big]\, dx = 0 \quad \text{for all } w \in \overset{\circ}{W}{}^2_p(\Omega)\Big\},$$

which means that in the distributional sense $\tau^* = (\tau^*_{\alpha\beta})$ satisfies

$$\operatorname{div}(\operatorname{div}\tau^*) := \partial_\alpha(\partial_\beta \tau^*_{\alpha\beta}) = \lambda.$$

In this case we have

$$J_\lambda^*[\tau^*] = \int_\Omega \big[-\pi_p^*(\tau^*) + \psi\lambda\big]\, dx. \tag{2.5}$$

We further note that

$$\begin{aligned}\inf_{\overset{\circ}{W}{}^2_p(\Omega)} J_\lambda \le \inf_{\mathbf{K}} J_\lambda &= \inf_{v\in\mathbf{K}} \Big[\int_\Omega \pi_p(\nabla^2 v)\, dx - \int_\Omega \lambda(v-\Psi)\, dx\Big] \\ &\le \inf_{v\in\mathbf{K}} \int_\Omega \pi_p(\nabla^2 v)\, dx = \inf_{\mathbf{K}} J.\end{aligned} \tag{2.6}$$

3. Estimates for the superquadratic case

Let us now state our first result:

Theorem 3.1. *Let $p \ge 2$. With the notation introduced above we have for any $v \in \mathbf{K}$, for any $\eta^* \in Y^*$, for all $\lambda \in \Lambda$ and for any choice of $\beta > 0$ the estimate*

$$\begin{aligned}\big\|\nabla^2(u-v)\big\|^p_{L^p} \le p2^{p-1}\Big\{& D_p\big[\nabla^2 v, \eta^*\big] + \big[2^{2-q}(3-q) + \frac{1}{q}\beta^{-q}\big] d(\eta^*)^q \\ &+ \frac{1}{p}\beta^p \big\||\eta^*|^{q-2}\eta^* - \nabla^2 v\big\|^p_{L^p} + \int_\Omega \lambda(v-\Psi)\, dx\Big\},\end{aligned} \tag{3.1}$$

where

$$D_p[\rho, \varkappa^*] := \int_\Omega \big[\pi_p(\rho) + \pi_p^*(\varkappa^*) - \rho : \varkappa^*\big]\, dx$$

for $\rho \in Y$, $\varkappa^* \in Y^*$ and

$$d(\varkappa^*) := \inf_{\tau^*\in Q^*_\lambda} \|\varkappa^* - \tau^*\|_{L^q}.$$

If in addition $\operatorname{div}(\operatorname{div}\eta^*) \in L^q(\Omega)$, then we have the inequality

$$d(\eta^*) \le C_p(\Omega)\big\|\lambda - \operatorname{div}(\operatorname{div}\eta^*)\big\|_{L^q}. \tag{3.2}$$

The constant $C_p(\Omega)$ is defined in formula (3.8).

Remark 3.1. Note that all the quantities on the r.h.s. of (3.1) are non-negative, and the r.h.s. of (3.1) vanishes if and only if

$$\nabla^2 v = |\eta^*|^{q-2}\eta^*, \quad \operatorname{div}(\operatorname{div}\eta^*) = \lambda, \quad \lambda(v-\Psi) = 0.$$

Let $w \in \mathbf{K}$. Then the validity of the above equations gives

$$\begin{aligned}\int_\Omega |\nabla^2 v|^{p-2}\nabla^2 v : \nabla^2(w-v)\,dx &= \int_\Omega \eta^* : \nabla^2(w-v)\,dx = \int_\Omega \lambda(w-v)\,dx\\ = \int_\Omega \lambda(w-\Psi)\,dx + \int_\Omega \lambda(\Psi - v)\,dx &= \int_\Omega \lambda(w-\Psi)\,dx \geq 0,\end{aligned}$$

which means that v is the unique solution of Problem $(\mathscr{P})$.

If d is estimated via (3.2) then all the functions on the r.h.s. of (3.1) are either known or in our disposal. Thus (3.1) gives a practical way to measure the accuracy.

Having proved Theorem 3.1 we will give variants of (3.1) by optimizing the function λ.

Proof of Theorem 3.1. We recall (2.6), i.e.

$$J[u] \geq \inf_{\overset{\circ}{W}{}^2_p(\Omega)} J_\lambda,$$

so that according to (2.4)

$$J[u] \geq J^*_\lambda[\sigma^*_\lambda] \geq J^*_\lambda[\tau^*]$$

for all choices of $\lambda \in \Lambda$ and $\tau^* \in Q^*_\lambda$. This gives in combination with (2.2)

$$\big\|\nabla^2(u-v)\big\|^p_{L^p} \leq p\,2^{p-1}\big[J[v] - J^*_\lambda[\tau^*]\big]. \tag{3.3}$$

By (2.5) we find that

$$\begin{aligned}J[v] - J^*_\lambda[\tau^*] &= \int_\Omega \Big[\pi_p(\nabla^2 v) + \pi^*_p(\tau^*) - \Psi\lambda\Big]\,dx\\ &= \int_\Omega \Big[\pi_p(\nabla^2 v) + \pi^*_p(\tau^*) - \tau^* : \nabla^2 v\Big]\,dx + \int_\Omega \lambda(v-\Psi)\,dx\\ &= D_p\Big[\nabla^2 v, \tau^*\Big] + \int_\Omega \lambda(v-\Psi)\,dx,\end{aligned}$$

and according to (3.3) we have shown that

$$\big\|\nabla^2(u-v)\big\|^p_{L^p} \leq p\,2^{p-1}\Big\{D_p[\nabla^2 v, \tau^*] + \int_\Omega \lambda(v-\Psi)\,dx\Big\} \tag{3.4}$$

valid for all $v \in \mathbf{K}$, $\tau^* \in Q^*_\lambda$ and $\lambda \in \Lambda$.

Consider any tensor $\eta^* \in Y^*$. Then (3.4) and the convexity of π^*_p imply

$$\begin{aligned}\big\|\nabla^2(u-v)\big\|^p_{L^p} \leq p2^{p-1}\Big\{D_p[\nabla^2 v, \eta^*] &+ \int_\Omega \big[\pi^*_p(\tau^*) - \pi^*_p(\eta^*) - (\tau^* - \eta^*) : \nabla^2 v\big]\,dx\\ &+ \int_\Omega \lambda(v-\Psi)\,dx\Big\}\end{aligned}$$

$$\begin{aligned}
(3.5)\qquad &\le p2^{p-1}\Big\{D_p[\nabla^2 v,\eta^*]+\int_\Omega\Big[|\tau^*|^{q-2}\tau^*-\nabla^2 v\Big]:(\tau^*-\eta^*)\,dx\\
&\quad+\int_\Omega\lambda(v-\Psi)\,dx\Big\}\\
&= p\,2^{p-1}\Big\{D_p[\nabla^2 v,\eta^*]+\int_\Omega\big[|\tau^*|^{q-2}\tau^*-|\eta^*|^{q-2}\eta^*\big]:(\tau^*-\eta^*)\,dx\\
&\quad+\int_\Omega\big[|\eta^*|^{q-2}\eta^*-\nabla^2 v\big]:(\tau^*-\eta^*)\,dx+\int_\Omega\lambda(v-\Psi)\,dx\Big\}.
\end{aligned}$$

As demonstrated in [BR] we have

$$\int_\Omega\Big[|\tau^*|^{q-2}\tau^*-|\eta^*|^{q-2}\eta^*\Big]:(\tau^*-\eta^*)\,dx\le 2^{2-q}(3-q)\|\tau^*-\eta^*\|^q_{L^q},$$

moreover, from Hölder's inequality it follows that

$$\begin{aligned}
\int_\Omega\big[|\eta^*|^{q-2}\eta^*-\nabla^2 v\big]:(\tau^*-\eta^*)\,dx &\le \big\||\eta^*|^{q-2}\eta^*-\nabla^2 v\|_{L^p}\|\tau^*-\eta^*\|_{L^q}\\
&\le\frac1p\,\beta^p\big\||\eta^*|^{q-2}\eta^*-\nabla^2 v\big\|^p_{L^p}+\frac1q\beta^{-q}\|\tau^*-\eta^*\|^q_{L^q},
\end{aligned}$$

where in the last line we used Young's inequality with some $\beta>0$. Inserting these estimates into (3.5) and taking the inf w.r.t. $\tau^*\in Q^*_\lambda$, inequality (3.1) is proved.

To establish the second part of the theorem, we consider $\eta^*\in Y^*$ with the property $\operatorname{div}(\operatorname{div}\eta^*)\in L^q(\Omega)$. Then $\inf_{\tau^*\in Q^*_\lambda}\big\|\eta^*-\tau^*\big\|_{L^q}$ is attained for some $\tau^*\in Q^*_\lambda$, and $\big\|\eta^*-\tau^*\big\|_{L^q}$ is a measure for the distance from η^* to Q^*_λ. Letting $\overline{\lambda}:=\lambda-\partial_\alpha\partial_\beta\eta^*_{\alpha\beta}$ we find (with an obvious meaning of $Q^*_{\overline{\lambda}}$)

$$(3.6)\qquad \inf_{\rho^*\in Q^*_\lambda}\frac1q\|\rho^*-\eta^*\|^q_{L^q}=-\sup_{\varkappa^*\in Q^*_{\overline{\lambda}}}\Big[-\frac1q\|\varkappa^*\|^q_{L^q}\Big],$$

$$(3.7)\qquad \sup_{\varkappa^*\in Q^*_{\overline{\lambda}}}\Big[-\frac1q\|\varkappa^*\|^q_{L^q}\Big]=\inf_{w\in\overset{\circ}{W}{}^2_p(\Omega)}\int_\Omega\Big[\frac1p|\nabla^2 w|^p-\overline{\lambda}w\Big]\,dx.$$

For $w\in\overset{\circ}{W}{}^2_p(\Omega)$ we have by Poincaré's inequality

$$(3.8)\qquad \|w\|_{L^p}\le C_p(\Omega)\|\nabla^2 w\|_{L^p}$$

with a positive constant $C_p(\Omega)$ depending on p and Ω. Using (3.8) on the r.h.s. of (3.7), we see that the r.h.s. of (3.7) is bounded from below by

$$\inf_{t\ge0}\Big[\frac1p\,t^p-C_p(\Omega)t\|\overline{\lambda}\|_{L^q}\Big],$$

and this inf is attained at

$$t_0:=\Big[\|\overline{\lambda}\|_{L^q}C_p(\Omega)\Big]^{\frac{1}{p-1}}.$$

From (3.6) we therefore get

$$\inf_{\rho^* \in Q^*_\lambda} \|\rho^* - \eta^*\|_{L^q} =: d(\eta^*) \le C_p(\Omega)\|\lambda - \operatorname{div}(\operatorname{div}\eta^*)\|_{L^q}. \tag{3.9}$$ $\square$

Now we discuss two variants of how to choose $\lambda \in \Lambda$ in a suitable way.

Variant 1. Given $v \in \mathbf{K}$ and $\eta^* \in Y^*$ s.t. $\operatorname{div}(\operatorname{div}\eta^*) \in L^q(\Omega)$ we let (following [Re])

$$\begin{cases} \lambda &= 0 \quad \text{on } [v > \Psi], \\ \lambda &= \big[\operatorname{div}(\operatorname{div}\eta^*)\big]_\oplus \quad \text{on } [v = \Psi], \end{cases}$$

where $f_\oplus := \max(0, f)$, $f_\ominus := \min(0, f)$, hence $f = f_\oplus + f_\ominus$ for real-valued functions f. With this choice of λ we get

$$\int_\Omega \lambda(v - \Psi)\, dx = 0,$$

moreover

$$\int_\Omega |\lambda - \operatorname{div}(\operatorname{div}\eta^*)|^q\, dx = \int_{[v>\Psi]} |\operatorname{div}(\operatorname{div}\eta^*)|^q\, dx + \int_{[v=\Psi]} \big|\big[\operatorname{div}(\operatorname{div}\eta^*)\big]_\ominus\big|^q\, dx,$$

and we arrive at

Corollary 3.1. *Let $p \ge 2$. For any $v \in \mathbf{K}$, for all $\eta^* \in Y^*$ s.t.* $\operatorname{div}(\operatorname{div}\eta^*) \in L^q(\Omega)$ *and for all $\beta > 0$ we have with*

$$K_p(\Omega, \beta) := C_p^q(\Omega)\Big[2^{2-q}(3-q) + \frac{1}{q}\,\beta^{-q}\Big]$$

the estimate

$$\begin{aligned} \big\|\nabla^2(v-u)\big\|^p_{L^p} \le p\, 2^{p-1}\Bigg\{ D_p[\nabla^2 v, \eta^*] + K_p(\Omega,\beta)\Bigg[\int_{[v>\Psi]} \big|\operatorname{div}(\operatorname{div}\eta^*)\big|^q\, dx \\ + \int_{[v=\Psi]} \big|\big(\operatorname{div}(\operatorname{div}\eta^*)\big)_\ominus\big|^q\, dx\Bigg] + \frac{1}{p}\,\beta^p \big\||\eta^*|^{q-2}\eta^* - \nabla^2 v\big\|^p_{L^p}\Bigg\}. \end{aligned} \tag{3.10}$$ $\square$

Assume that the r.h.s. of (3.10) is zero for some triple (v, η^*, β). Then $\nabla^2 v = |\eta^*|^{q-2}\eta^*$ together with

$$\begin{aligned} \operatorname{div}(\operatorname{div}\eta^*) &= 0 \text{ on } [v > \Psi], \\ \Big[\operatorname{div}(\operatorname{div}\eta^*)\Big]_\ominus &= 0 \text{ on } [v = \Psi]. \end{aligned}$$

This implies for any $w \in \mathbf{K}$

$$\begin{aligned}\int_\Omega |\nabla^2 v|^{p-2}\nabla^2 v : \nabla^2(w-v)\,dx &= \int_\Omega \eta^* : \nabla^2(w-v)\,dx\\ &= \int_\Omega \operatorname{div}(\operatorname{div}\eta^*)(w-v)\,dx\\ &= \int_{[v=\Psi]} \big[\operatorname{div}(\operatorname{div}\eta^*)\big]_\oplus (w-\Psi)\,dx \geq 0,\end{aligned}$$

and therefore v coincides with the solution u of $(\mathscr{P})$.

Variant 2. As an alternative to (3.10) we again follow ideas of [Re] and estimate $d(\eta^*)^q$ on the r.h.s. of (3.1) by $C_p^q(\Omega)\big\|\lambda - \operatorname{div}(\operatorname{div}\eta^*)\big\|_{L^q}^q$ (recall (3.2)) and then try to find $\lambda \in \Lambda$ s.t.

$$\int_\Omega \lambda(v-\Psi)\,dx + K_p(\Omega,\beta)\int_\Omega \big|\lambda - \operatorname{div}(\operatorname{div}\eta^*)\big|^q\,dx$$

becomes minimal for a fixed triple (v,η^*,β). Of course this can be achieved by pointwise minimization of the function

$$f(t) := t\big(v(x)-\Psi(x)\big) + K\big|t-\delta(x)\big|^q$$

on $[0,\infty)$. Here $K := K_p(\Omega,\beta)$, $\delta := \operatorname{div}(\operatorname{div}\eta^*)$, and in the following we will omit the fixed argument $x \in \Omega$. Note that f is strictly convex on $\mathbf{R}$ and $f(t) \to +\infty$ as $t \to \pm\infty$, thus there is a unique number $t_0 \in \mathbf{R}$ s.t. $f(t_0) = \inf_{\mathbf{R}} f$. From $f'(t_0) = 0$ it follows that

$$0 = v - \Psi + Kq|t_0-\delta|^{q-2}(t_0-\delta),$$

and since $v \geq \Psi$, we must have $t_0 \leq \delta$, hence

$$(\delta - t_0)^{q-1} = \frac{1}{Kq}(v-\Psi), \quad \text{i.e.} \quad t_0 = \delta - \left[\frac{1}{qK}\,(v-\Psi)\right]^{\frac{1}{q-1}}.$$

Case 1. $t_0 < 0$. Since f is strictly increasing on $[t_0,\infty)$, we get

$$\min_{t\geq 0} f(t) = f(0) = K|\delta|^q.$$

Case 2. $t_0 \geq 0$. Then we have

$$\min_{t\geq 0} f(t) = f(t_0)$$

For $v \in \mathbf{K}$, $\eta^* \in Y^*$ s.t. $\operatorname{div}(\operatorname{div}\eta^*) \in L^q(\Omega)$ and $\beta > 0$ we define the sets

$$\begin{aligned}\Omega^+ &:= \left\{x \in \Omega : \operatorname{div}(\operatorname{div}\eta^*)(x) \geq \left[\frac{1}{qK_p(\Omega,\beta)}\,\big(v(x)-\Psi(x)\big)\right]^{\frac{1}{q-1}}\right\},\\ \Omega^- &:= \Omega - \Omega^+\end{aligned}$$

and set $\lambda = 0$ on Ω^- and $\lambda = \big[\frac{1}{qK_p(\Omega,\beta)}(v(x) - \Psi(x))\big]^{\frac{1}{q-1}}$ on Ω^+. We further introduce the quantity

$$\varepsilon(v, \eta^*, \beta) := \int_{\Omega^-} K_p(\Omega, \beta)|\operatorname{div}(\operatorname{div}\eta^*)|^q \, dx$$
$$+ \int_{\Omega^+} \Big[(v - \Psi)\Big(\operatorname{div}(\operatorname{div}\eta^*) - \Big\{\frac{v-\Psi}{qK_p(\Omega,\beta)}\Big\}^{\frac{1}{q-1}}\Big) + K_p(\Omega,\beta)\Big\{\frac{v-\Psi}{qK_p(\Omega,\beta)}\Big\}^p\Big] \, dx.$$

With the above choice of λ it is immediate that

$$\inf_{\mu \in \Lambda} \Big[\int_\Omega \mu(v - \Psi)\, dx + K_p(\Omega,\beta) \int_\Omega \big|\mu - \operatorname{div}(\operatorname{div}\eta^*)\big|^q \, dx\Big] \le \varepsilon(v, \eta^*, \beta),$$

and for $p = 2$ this estimate reduces to the one given in Remark 2 of [Re]. Summing up we arrive at

Corollary 3.2. *With the notation introduced above we have in case* $p \ge 2$

$$\big\|\nabla^2(u - v)\big\|^p_{L^p} \le p\, 2^{p-1}\Big[D_p[\nabla^2 v, \eta^*] + \frac{1}{p}\beta^p\big\||\eta^*|^{q-2}\eta^* - \nabla^2 v\big\|^p_{L^p} + \varepsilon(v, \eta^*, \beta)\Big] \tag{3.11}$$

valid for all $v \in \mathbf{K}$*, for any* $\eta^* \in Y^*$ *such that* $\operatorname{div}(\operatorname{div}\eta^*) \in L^q(\Omega)$ *and for any choice of* $\beta > 0$.

Note that the r.h.s. of (3.11) vanishes only on the exact solution. Estimate (3.11) gives an upper bound on the error related to the approximations of variational inequalities. It should be emphasized that it does not require a priori knowledge of the form of the unknown free boundaries.

Elimination of the quantity $\operatorname{div}(\operatorname{div}\eta^*)$**.** Let us finally reconsider the quantity

$$d(\eta^*) := \inf_{\tau^* \in Q^*_\lambda} \|\eta^* - \tau^*\|_{L^q}$$

for a tensor $\eta^* \in Y^*$ and a function $\lambda \in \Lambda$. We have

$$\inf_{\tau^* \in Q^*_\lambda} \frac{1}{q}\|\eta^* - \tau^*\|^q_{L^q} = \inf_{\rho^* \in Y^*} \sup_{w \in \overset{\circ}{W}{}^2_p(\Omega)} \mathscr{L}(w, \rho^*), \tag{3.12}$$

where

$$\mathscr{L}(w, \rho^*) := \int_\Omega \big[\pi^*_p(\eta^* - \rho^*) + \rho^* : \nabla^2 w - \lambda w\big]\, dx.$$

In fact, it holds

$$\sup_{w \in \overset{\circ}{W}{}^2_p} \mathscr{L}(w, \rho^*) = \begin{cases} +\infty, & \text{if } \rho^* \notin Q^*_\lambda, \\ \displaystyle\int_\Omega \pi^*_p(\eta^* - \rho^*)\, dx, & \text{if } \rho^* \in Q^*_\lambda, \end{cases}$$

and this implies (3.12). Now, using standard results from duality theory, we have

$$\inf_{\rho^* \in Y^*} \sup_{w \in \overset{\circ}{W}{}^2_p} \mathscr{L}(w, \rho^*) = \sup_{w \in \overset{\circ}{W}{}^2_p} \inf_{\rho^* \in Y^*} \mathscr{L}(w, \rho^*). \tag{3.13}$$

Since

$$\inf_{\rho^*\in Y^*} \mathscr{L}(w,\rho^*) = \inf_{\kappa^*\in Y^*} \mathscr{L}(w,\eta^*-\kappa^*),$$

we get from (3.12) and (3.13)

$$(3.14)\qquad \inf_{\tau^*\in Q^*_\lambda} \frac{1}{q}\|\eta^*-\tau^*\|_{L^q} = \sup_{w\in \overset{\circ}{W}{}^2_p} \inf_{\kappa^*\in Y^*} \int_\Omega \big[\pi^*_p(\kappa^*) + (\eta^*-\kappa^*) : \nabla^2 w - \lambda w\big]\, dx.$$

Note that (3.14) corresponds to formula (4.1) established in [NR2] for linear equations related to the biharmonic operator. Proceeding as in this reference, we write

$$\begin{aligned}
&\inf_{\kappa^*\in Y^*} \int_\Omega \big[\pi^*_p(\kappa^*) + (\eta^*-\kappa^*) : \nabla^2 w - \lambda w\big]\, dx\\
&= -\sup_{\kappa^*\in Y^*} \int_\Omega \big[\kappa^* : \nabla^2 w - \pi^*_p(\kappa^*)\big]\, dx + \int_\Omega \big[\eta^* : \nabla^2 w - \lambda w\big]\, dx\\
&= -\int_\Omega \big[\pi_p(\nabla^2 w) - \eta^* : \nabla^2 w + \lambda w\big]\, dx.
\end{aligned}$$

Inserting this into (3.14) we have shown that

$$(3.15)\qquad \inf_{\tau^*\in Q^*_\lambda} \frac{1}{q}\|\eta^*-\tau^*\|^q_{L^q} = -\inf_{w\in \overset{\circ}{W}{}^2_p} \int_\Omega \big[\pi_p(\nabla^2 w) - \eta^* : \nabla^2 w + \lambda w\big]\, dx.$$

If $\operatorname{div}(\operatorname{div}\eta^*) \in L^q(\Omega)$, then (3.15) reduces to (3.7), and we arrive at (3.9). Without further information concerning η^* (3.15) just states that the quantity $d(\eta^*)$ can be obtained by "solving" an auxiliary variational problem, which means to compute a lower bound for the functional

$$w \mapsto \int_\Omega \big[\pi_p(\nabla^2 w) - \eta^* : \nabla^2 w + \lambda w\big]\, dx$$

defined on the space $\overset{\circ}{W}{}^2_p(\Omega)$. Here of course no side condition enters but for each choice of η^* and λ a new problem has to be considered.

A rather natural assumption concerning $\eta^* \in Y^*$ is the requirement that

$$\operatorname{div}\eta^* = \big(\partial_\alpha \eta^*_{\alpha\beta}\big)_{1\le\beta\le 2}$$

is in the space $L^q(\Omega)$. Then

$$\begin{aligned}
&\int_\Omega \big[\pi_p(\nabla^2 w) - \eta^* : \nabla^2 w + \lambda w\big]\, dx\\
&= \int_\Omega \big[\pi_p(\nabla^2 w) + \operatorname{div}\eta^* \cdot \nabla w + \lambda w\big]\, dx\\
&= \int_\Omega \big[\pi_p(\nabla^2 w) + (\operatorname{div}\eta^* - y^*) \cdot \nabla w + (\lambda - \operatorname{div} y^*) w\big]\, dx,
\end{aligned}$$

where y^* is a vector-function from $L^q(\Omega)$ such that $\operatorname{div} y^* \in L^q(\Omega)$. From (3.15) we get by Hölder's inequality

$$\begin{aligned}\frac{1}{q}d(\eta^*)^q \le -\inf_{w\in \overset{\circ}{W}{}^2_p(\Omega)} \Big[\frac{1}{p}\|\nabla^2 w\|^p_{L^p} - \|\operatorname{div}\eta^* - y^*\|_{L^q}\|\nabla w\|_{L^p} \\ - \|\lambda - \operatorname{div} y^*\|_{L^q}\|w\|_{L^p}\Big].\end{aligned} \tag{3.16}$$

From the Poincaré inequality we have for all $v \in \overset{\circ}{W}{}^1_p(\Omega)$

$$\|v\|_{L^p} \le K_p(\Omega)\|\nabla v\|_{L^p} \tag{3.17}$$

and applying (3.17) to $w \in \overset{\circ}{W}{}^2_p(\Omega)$ as well as to the vectorial function $\nabla w \in \overset{\circ}{W}{}^1_p(\Omega)$ we see that the r. h. s. of (3.16) is bounded from above by

$$\begin{aligned}&-\inf_{w\in \overset{\circ}{W}{}^2_p(\Omega)} \Bigg[\frac{1}{p}\|\nabla^2 w\|^p_{L^p} - K_p(\Omega)\|\operatorname{div}\eta^* - y^*\|_{L^q}\|\nabla^2 w\|_{L^p} \\ &\qquad - K_p(\Omega)^2\|\lambda - \operatorname{div} y^*\|_{L^q}\|\nabla^2 w\|_{L^p}\Bigg] \\ &\le -\inf_{t\ge 0}\Big[\frac{1}{p}t^p - K_p(\Omega)\big[\|\operatorname{div}\eta^* - y^*\|_{L^q} + K_p(\Omega)\|\lambda - \operatorname{div} y^*\|_{L^q}\big]t\Big] \\ &= \frac{1}{q}K_p(\Omega)^q\big[\|\operatorname{div}\eta^* - y^*\|_{L^q} + K_p(\Omega)\|\lambda - \operatorname{div} y^*\|_{L^q}\big]^q.\end{aligned}$$

Let us summarize our results:

Theorem 3.2. *Suppose that we are given $\eta^* \in L^q(\Omega)$ and $\lambda \in \Lambda$. Then we have*

$$\inf_{\tau^*\in Q^*_\lambda}\frac{1}{q}\|\eta^* - \tau^*\|^q_{L^q} = -\inf_{w\in \overset{\circ}{W}{}^2_p(\Omega)}\int_\Omega \big[\pi_p(\nabla^2 w) - \eta^* : \nabla^2 w + \lambda w\big]\,dx.$$

If in addition $\operatorname{div}\eta^* \in L^q(\Omega)$, then

$$\inf_{\tau^*\in Q^*_\lambda}\frac{1}{q}\|\eta^* - \tau^*\|_{L^q} \le \frac{1}{q}K_p(\Omega)^q\Big[\|\operatorname{div}\eta^* - y^*\|_{L^q} + K_p(\Omega)\|\lambda - \operatorname{div} y^*\|_{L^q}\Big]^q,$$

where y^* is any vector-function in $L^q(\Omega)$ s.t. $\operatorname{div} y^* \in L^q(\Omega)$ and where $K_p(\Omega)$ is defined according to (3.17).

From the proof it is immediate that the statements of Theorem 3.2 are also valid in the case $1 < p < 2$.

4. Estimates for the subquadratic case

Obviously we have for $\tau^* \in Q^*_\lambda$

$$\begin{aligned} J^*_\lambda[\tau^*] &= \inf_{w\in \overset{\circ}{W}{}^2_p(\Omega)} \int_\Omega \big[l(w,\tau^*) - \lambda(w-\Psi)\big]\,dx \\ &\le \inf_{w\in\mathbf{K}} \int_\Omega \big[l(w,\tau^*) - \lambda(w-\Psi)\big]\,dx \le \inf_{w\in\mathbf{K}} \int_\Omega l(w,\tau^*)\,dx = J^*[\tau^*], \end{aligned}$$

moreover $J^*[\sigma^*] = J[u] \le J[w]$ for any $w \in \mathbf{K}$, hence we get from (2.3)

$$\big\|\tau^* - \sigma^*\|^q_{L^q} \le q\,2^{q-1}\Big[J[w] - J^*_\lambda[\tau^*]\Big]. \tag{4.1}$$

Observe that (4.1) exactly corresponds to (3.3), and as outlined in Section 3 inequality (4.1) can be rewritten as

$$\big\|\tau^* - \sigma^*\|^q_{L^q} \le q\,2^{q-1}\Big\{D_p[\nabla^2 w,\tau^*] + \int_\Omega \lambda(w-\Psi)\,dx\Big\} \tag{4.2}$$

valid for all $\tau^* \in Q^*_\lambda$, $w \in \mathbf{K}$ and $\lambda \in \Lambda$, $D_p[\nabla^2 w,\tau^*]$ having the same meaning as before. If now $\eta^* \in Y^*$ and $\tau^* \in Q^*_\lambda$, then (using (4.2))

$$\begin{aligned} \big\|\eta^* - \sigma^*\big\|^q_{L^q} &\le \Big(\big\|\sigma^* - \tau^*\big\|_{L^q} + \big\|\eta^* - \tau^*\big\|_{L^q}\Big)^q \\ &\le 2^{q-1}\Big(\big\|\sigma^* - \tau^*\big\|^q_{L^q} + \big\|\eta^* - \tau^*\big\|^q_{L^q}\Big) \\ &\le q\,4^{q-1}\Big[D_p[\nabla^2 w,\tau^*] + \int_\Omega \lambda(w-\Psi)\,dx\Big] + 2^{q-1}\big\|\eta^* - \tau^*\big\|^q_{L^q}. \end{aligned} \tag{4.3}$$

The quantity $D_p[\nabla^2 w,\tau^*]$ can be estimated as before:

$$\begin{aligned} D_p[\nabla^2 w,\tau^*] &\le D_p[\nabla^2 w,\eta^*] + \int_\Omega \big[|\tau^*|^{q-2}\tau^* - \nabla^2 w\big] : \big(\tau^* - \eta^*\big)\,dx \\ &= D_p[\nabla^2 w,\eta^*] + \int_\Omega \big[|\tau^*|^{q-2}\tau^* - |\eta^*|^{q-2}\eta^*\big] : (\tau^* - \eta^*)\,dx \\ &\quad + \int_\Omega \big[|\eta^*|^{q-2}\eta^* - \nabla^2 w\big] : (\tau^* - \eta^*)\,dx \\ &=: D_p[\nabla^2 w,\eta^*] + I_1 + I_2. \end{aligned} \tag{4.4}$$

According to the calculations after (3.5) in [BFR] we have

$$I_1 \le (q-1)\big\|\tau^* - \eta^*\big\|^2_{L^q}\Big[\big\|\tau^* - \eta^*\big\|_{L^q} + 2\big\|\eta^*\big\|_{L^q}\Big]^{q-2},$$

and for I_2 we get with Hölder's and Young's inequality

$$I_2 \le \frac{\beta}{2}\,\big\||\eta^*|^{q-2}\eta^* - \nabla^2 w\big\|^2_{L^p} + \frac{1}{2\beta}\,\big\|\tau^* - \eta^*\big\|^2_{L^q},$$

where $\beta > 0$ is arbitrary. If we insert these estimates in (4.4) and return to (4.3) it is shown that

$$\begin{aligned}\big\|\eta^* - \sigma^*\big\|^q_{L^q} \le q\,4^{q-1}\Big[&D_p[\nabla^2 w, \eta^*] \\ &+ (q-1)\big\|\tau^* - \eta^*\big\|^2_{L^q}\Big(\big\|\tau^* - \eta^*\big\|_{L^q} + 2\big\|\eta^*\big\|_{L^q}\Big)^{q-2} \\ &+ \frac{\beta}{2}\,\big\||\eta^*|^{q-2}\eta^* - \nabla^2 w\big\|^2_{L^p} + \frac{1}{2\beta}\,\big\|\tau^* - \eta^*\big\|^2_{L^q} \\ &+ \int_\Omega \lambda(w - \Psi)\,dx\Big] + 2^{q-1}\big\|\eta^* - \tau^*\big\|^q_{L^q}.\end{aligned}$$

Now we may pass to the infimum w.r.t. $\tau^* \in Q^*_\lambda$ with the result:

Theorem 4.1. *Let $1 < p < 2$. Then for all $\eta^* \in Y^*$, for any $w \in \mathbf{K}$ and for all $\lambda \in \Lambda$, $\beta > 0$ the following error estimate holds:*

$$\begin{aligned}\big\|\sigma^* - \eta^*\big\|^q_{L^q} \le\; & q\,4^{q-1}\Big[D_p[\nabla^2 w, \eta^*] + \frac{\beta}{2}\,\big\||\eta^*|^{q-2}\eta^* - \nabla^2 w\big\|^2_{L^p}\Big] \\ &+ q\,4^{q-1}\Big[\frac{1}{2\beta} + (q-1)\big(d(\eta^*) + 2\|\eta^*\|_{L^q}\big)^{q-2}\Big]d(\eta^*)^2 \\ &+ 2^{q-1}d(\eta^*)^q + q\,4^{q-1}\int_\Omega \lambda(w - \Psi)\,dx,\end{aligned} \tag{4.5}$$

where $d(\eta^*) := \inf_{\tau^* \in Q^*_\lambda} \big\|\tau^* - \eta^*\big\|_{L^q}$.

Remark 4.1. i) With exactly the same arguments as used in the proof of Theorem 3.1 we obtain the estimate (3.9) for $p < 2$, i.e.

$$d(\eta^*) \le C_p(\Omega)\big\|\lambda - \operatorname{div}(\operatorname{div}\eta^*)\big\|_{L^q}, \tag{4.6}$$

provided that in addition $\operatorname{div}(\operatorname{div}\eta^*) \in L^q(\Omega)$.

Starting from (4.5) and using (4.6) it is also immediate how to get variants of Corollaries 2.1 and 2.2 for the present situation: the arguments used for proving (3.10) and (3.11) do not depend on the value of p.

ii) If the r.h.s. of (4.5) vanishes for a triple (w, η^*, λ), then $w = u$ and $\eta^* = \sigma^*$.

5. Concluding remarks

Our estimates formally suffer from the fact that we are only allowed to insert functions v from the class $\mathbf{K}$. If $w \in \overset{\circ}{W}{}^2_p(\Omega)$ is arbitrary, then we choose $v \in \mathbf{K}$ s.t.

$$\big\|\nabla^2 v - \nabla^2 w\big\|_{L^p} = \inf_{v' \in \mathbf{K}} \big\|\nabla^2 v' - \nabla^2 w\big\|_{L^p},$$

use v in our estimates, and then we have to find a bound for $\big\|\nabla^2 v - \nabla^2 w\big\|_{L^p}$. In the first order case we have

$$\big\|\nabla v - \nabla w\big\|_{L^p} \le \big\|\nabla w - \nabla \max(w, \Psi)\big\|_{L^p} = \Big(\int_{[w \le \Psi]} |\nabla w - \nabla\Psi|^p\,dx\Big)^{1/p},$$

but unfortunately we did not obtain a comparable result in our situation which is caused by the fact that in general $\max(w, \Psi)$ is not in $\overset{\circ}{W}{}^2_p(\Omega)$. However, from the computational point of view the difficulties arising if $v \notin \mathbf{K}$ are not very serious because one can easily modify (post-process) an approximate solution such that it belongs to the class $\mathbf{K}$.

Our choice of the class $\mathbf{K}$ means that we consider functions v vanishing on $\partial\Omega$ and having in addition zero gradient (in the trace sense) on the boundary. Alternatively we could choose functions v from the class $\overset{\circ}{W}{}^1_p \cap W^2_p(\Omega)$ satisfying $v \geq \Psi$ but the energy J is not coercive on this space, so that one would have to replace J by the functional

$$\widetilde{J}[v] := \frac{1}{p} \int_\Omega |\nabla^2 v|^p \, dx + \frac{1}{p} \int_\Omega |\nabla v|^p \, dx$$

in order to get a well–posed problem. More general, let $1 < p,\ r < \infty$, let Ψ as before and define

$$\mathbf{L} := \Big\{ v \in W^2_p(\Omega) \cap \overset{\circ}{W}{}^1_r(\Omega) : v \geq \Psi \Big\}$$

as well as

$$K[v] := \frac{1}{p} \int_\Omega |\nabla^2 v|^p \, dx + \frac{1}{r} \int_\Omega |\nabla v|^r \, dx.$$

Then the problem $K \to \min$ on $\mathbf{L}$ makes sense and error estimates are available. Moreover, for this new problem in which only the trace of the function itselve is prescribed, it is possible to overcome the difficulty arising in measuring the distance of an arbitrary function to the admissible class of comparison functions. The details will be given in the forthcoming paper [BF2].

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