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MAPPINGS OF FINITE DISTORTION: COMPOSITION OPERATOR

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Abstract. We give sharp integrability conditions on the distortion function of a homeomorphism f of finite distortion, under which f induces a composition operator between two Sobolev spaces.

1. Introduction

It is well-known that the composition operator $T_f\colon\ T_f(u) = u\circ f$ maps $W^{1,n}_{\mathrm{loc}}(\Omega_2)$ into $W^{1,n}_{\mathrm{loc}}(\Omega_1)$ if $f\colon \Omega_1 \to \Omega_2$ is a quasiconformal mapping ([11, 15, 20]). Here quasiconformality requires that f be a homeomorphism with $f \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbf{R}^n)$ and that

$$|Df(x)|^n \leq K J_f(x) \text{ a.e.} \tag{1.1}$$

for some constant $K \geq 1$. Recently, the class of more general homeomorphisms of finite distortion, for which one allows K above to depend on x has been under intense study [1, 2, 5, 7, 8, 9, 10, 13, 16]. To be more precise, we say that a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbf{R}^n)$ is of finite distortion if (1.1) holds for f with some measurable function $K(x) \geq 1$ which is finite almost everywhere. In these studies, one typically assumes some integrability condition on the distortion function K. It is then natural to inquire if a suitable integrability condition on K would still guarantee that T_f maps $W^{1,n}_{\mathrm{loc}}(\Omega_2)$ into $W^{1,p}_{\mathrm{loc}}(\Omega_1)$ for some $1 \leq p \leq n$. Our first result gives a precise integrability criteria for f to induce such a composition operator.

Theorem 1.1. *Let $f\colon \Omega_1 \to \Omega_2$ be a homeomorphism of finite distortion K and let $p \in [1,n]$. Then T_f maps $W^{1,n}_{\mathrm{loc}}(\Omega_2)$ into $W^{1,p}_{\mathrm{loc}}(\Omega_1)$ if $K \in L^{\frac{p}{n-p}}_{\mathrm{loc}}(\Omega_1)$. Moreover, given $\varepsilon > 0$, one can find Ω_1, Ω_2 and a homeomorphism $f\colon \Omega_1 \to \Omega_2$ of finite distortion K so that $K \in L^{\frac{p}{n-p}}_{\mathrm{loc}}(\Omega_1)$ but $T_f(W^{1,n}_{\mathrm{loc}}(\Omega_2)) \not\subset W^{1,p+\varepsilon}_{\mathrm{loc}}(\Omega_1)$.*

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Let us make a couple of comments on the claim of Theorem 1.1. First of all, $T_f(u)$ could in principle depend on the choice of the representative for u. However, this turns out not to be the case: $T_f(u)$ belongs to $W^{1,p}_{\rm loc}(\Omega_1)$ for each (representative of) $u \in W^{1,n}_{\rm loc}(\Omega_1)$ and $T_f(u) = T_f(\hat{u})$ a.e. in Ω_1 if $\hat{u}$ is some other representative of u. Secondly, our proof in fact gives the estimate

$$\|\nabla T_f(u)\|_{L^p(G)} \le \|K\|^{1/n}_{L^{p/(n-p)}(G)} \|\nabla u\|_{L^n(f(G))}$$

for $G \subset\subset \Omega_1$ and $u \in W^{1,n}_{\rm loc}(\Omega_2)$.

By applying Theorem 1.1 to the projections $(x_1, \cdots, x_n) \mapsto x_j$, one concludes that $f \in W^{1,p}_{\rm loc}(\Omega_1, \mathbf{R}^n)$ under the assumptions of Theorem 1.1. Alternatively, this conclusion can also be easily deduced by means of the Hölder inequality, applying the distortion inequality (1.1) and the local integrability of the Jacobian of a Sobolev-homeomorphism. In the proof of Theorem 1.1 we actually show that this conclusion is essentially sharp by constructing, for each given $\varepsilon > 0$, a homeomorphism f of finite distortion K so that $K^{p/(n-p)}$ is locally integrable but $|Df|^{p+\varepsilon}$ fails to be locally integrable. Thus, it may happen that $T(W^{1,q}_{\rm loc}(\Omega_2)) \not\subset W^{1,p+\varepsilon}_{\rm loc}(\Omega_1)$ for each $q \ge n$ under the assumptions of Theorem 1.1.

Suppose then that we consider a homeomorphism f whose regularity is better then what guaranteed by Theorem 1.1. One could expect that $T_f(W^{1,n}_{\rm loc}(\Omega_2)) \subset W^{1,p+\varepsilon}_{\rm loc}(\Omega_1)$ for some $\varepsilon > 0$ depending on the regularity of f. This turns out not to be the case. For example, given $\varepsilon > 0$ and $p \ge 1$, one can find a homeomorphism f with finite distortion K so that both $K^{1/(n-1)}$ and $|Df|^p$ are locally integrable but $T(W^{1,n}_{\rm loc}(\Omega_2)) \not\subset W^{1,1+\varepsilon}_{\rm loc}(\Omega_1)$. On the other hand, our next result shows that the target space can be improved on, provided we consider the image of $W^{1,q}_{\rm loc}(\Omega_2)$ for some $q > n$.

Theorem 1.2. *Suppose that $\Omega_1, \Omega_2 \subset \mathbf{R}^n$, $n \ge 2$, are domains. Let $p \in [1, \infty)$, $q \in (n, \infty)$ and $s \in [1, \infty)$. Suppose that $s(q-p) - p(q-n) \ge 0$ and set*

$$a = \frac{ps}{s(q-p) - p(q-n)}. \tag{1.2}$$

Suppose that $f \in W^{1,s}_{\rm loc}(\Omega_1, \Omega_2)$ is a homeomorphism of finite distortion such that $K \in L^a_{\rm loc}(\Omega_1)$. Then T_f maps $W^{1,q}_{\rm loc}(\Omega_2)$ into $W^{1,p}_{\rm loc}(\Omega_1)$. Moreover, given $\varepsilon > 0$, $s \ge p$, q and $a \ge 1/(n-1)$ as above, one can find Ω_1, Ω_2 and a homeomorphism $f\colon \Omega_1 \to \Omega_2$ of finite distortion K so that $K \in L^a_{\rm loc}(\Omega_1)$ and $f \in W^{1,s}_{\rm loc}(\Omega_1, \Omega_2)$ but $T_f(W^{1,q}_{\rm loc}(\Omega_2)) \not\subset W^{1,p+\varepsilon}_{\rm loc}(\Omega_1)$.

Above, the mapping property of T_f means that each $u \in W^{1,q}_{\rm loc}(\Omega_2)$ has a repesentative $\hat{u}$ so that $T_f(\hat{u}) \in W^{1,p}_{\rm loc}(\Omega_1)$. In fact, this will always be the case for the continuous representative $\hat{u}$ and actually for every representative when $a \ge 1/(n-1)$. When $a < 1/(n-1)$, this is not necessarily the case. Indeed, then there is a Lipschitz mapping f of finite distortion K with $K^a \in L^1_{\rm loc}(\Omega_1)$ and so that f maps a compact Cantor-type set of positive volume to a set of volume zero (cf. [10]). By defining $u = \chi_{f(E)}$ we see that $T_f(u)$ may fail even to be in $W^{1,1}_{\rm loc}(\Omega_1)$.

The sharpness of our formula is only claimed for $a \geq 1/(n-1)$. We do however expect this assumption to be superfluous. The asserted examples are constructed relying on a general scheme initiated in [7] and further refined in [8].

Notice that we have not considered the action of the composition operator T_f on $W^{1,p}_{\mathrm{loc}}(\Omega_2)$ for $1 \leq p < n$. There is a simple reason for this: in this case one can easily give examples of quasiconformal f (so, $K \in L^\infty(\Omega_1)$) so that $T_f(W^{1,p}_{\mathrm{loc}}(\Omega_2)) \notin W^{1,1}_{\mathrm{loc}}(\Omega_1)$.

Our motivation for the study of the composition operator T_f partially arose from the following question: when is the composition of two homeomorphisms of finite distortion also of finite distortion? For the consequences of our work on this problem we refer the reader to Section 6 below.

2. Preliminaries

2.1. Notation. The euclidean norm of $x \in \mathbf{R}^n$ is denoted by $\|x\|$. We use the notation sgn for the sign function, i.e. $\operatorname{sgn}(t) = 1$ if $t > 0$ and $\operatorname{sgn}(t) = -1$ if $t < 0$. Given two functions $h, g\colon \Omega \to \mathbf{R}$ we write $h \sim g$ if there is constant $A \geq 1$ such that $\frac{1}{A} f(x) \leq g(x) \leq A f(x)$ for every $x \in \Omega$.

We say that a mapping $f\colon \Omega \to \mathbf{R}^n$ is Lipschitz continuous (or Lipschitz for short) if there is a constant $L > 0$ such that $\|f(x) - f(y)\| \leq L\|x - y\|$ for all $x, y \in \Omega$.

A mapping $f :: \Omega \to \mathbf{R}^n$ is said to satisfy the Lusin condition (N) if $\mathscr{L}_n(f(A)) = 0$ for every $A \subset \Omega$ such that $\mathscr{L}_n(A) = 0$. Analogously, f is said to satisfy the Lusin condition (N^{-1}) if $\mathscr{L}_n(f^{-1}(A)) = 0$ for every $A \subset \mathbf{R}^n$ such that $\mathscr{L}_n(A) = 0$.

2.2. Area formula. Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbf{R}^n)$ be a homeomorphism and let η be a non-negative Borel-measurable function on $\mathbf{R}^n$. Without any additional assumption we have

$$\int_\Omega \eta(f(x))|J_f(x)|\,dx \leq \int_{\mathbf{R}^n} \eta(y)\,dy. \tag{2.1}$$

This follows from the area formula for Lipschitz mappings and from the fact that Ω can be exhausted up to a set of measure zero by sets, the restriction to which of f is Lipschitz continuous (see [3, Theorem 3.1.4 and Theorem 3.1.8]).

2.3. Differentiability of radial functions. The following lemma can be verified by an elementary calculation.

Lemma 2.1. *Let $\rho\colon (0,\infty) \to (0,\infty)$ be a strictly monotone, differentiable function. Then for the mapping*

$$f(x) = \frac{x}{\|x\|}\rho(\|x\|), \quad x \neq 0,$$

we have for almost every x

$$Df(x) \sim \max\Big\{\frac{\rho(\|x\|)}{\|x\|}, |\rho'(\|x\|)|\Big\}, \quad J_f(x) \sim \rho'(\|x\|)\Big(\frac{\rho(\|x\|)}{\|x\|}\Big)^{n-1}.$$

2.4. Adjugate. The adjugate $\operatorname{adj} B$ of an invertible square matrix B is defined by the formula

$$B \operatorname{adj} B = I \det B,$$

where $\det B$ denotes the determinant of B and I is the identity matrix. The operator adj is then continuously extended to $\mathbf{R}^{n\times n}$.

2.5. Auxiliary inequality. Let $\alpha > 0$. Then

$$ab \le C(\alpha)\exp(2a^{\frac{1}{\alpha}}) + b\log^{\alpha}(e+b) \tag{2.2}$$

for every $a > 0$ and $b > 0$. Indeed, if the second term is not bigger than the left-hand side, then $a > \log^{\alpha}(e+b)$, which implies that

$$ab \le a\exp(a^{\frac{1}{\alpha}}) \le C(\alpha)\exp(2a^{\frac{1}{\alpha}}).$$

2.6. Lorentz space. If $f\colon \Omega \to \mathbf{R}$ is a measurable function, we define its distributional function $m(\cdot, f)$ by

$$m(\sigma, f) = \mathscr{L}_n(\{x : |f(x)| > \sigma\}), \quad \sigma > 0,$$

and the non-increasing rearrangement $f^{\star}$ of f by

$$f^{\star}(t) = \inf\{\sigma : m(\sigma, f) \le t\}.$$

The Lorentz space $L^{n-1,1}(\Omega)$ is defined as the class of all measurable functions $f\colon \Omega \to \mathbf{R}$ for which

$$\int_0^{\infty} t^{\frac{1}{n-1}} f^{\star}(t)\frac{dt}{t} < \infty,$$

and the local space $L^{n-1,1}_{\mathrm{loc}}(\Omega)$ is then obtained as usual. For an introduction to Lorentz spaces see e.g. [17]. Recall that, for $n = 2$, we have $L^{1,1}_{\mathrm{loc}}(\Omega) = L^{1}_{\mathrm{loc}}(\Omega)$ and that

$$\bigcap_{p>n-1} L^{p}_{\mathrm{loc}}(\Omega) \subset L^{n-1,1}_{\mathrm{loc}}(\Omega) \subset L^{n-1}_{\mathrm{loc}}(\Omega).$$

3. Proof of the first part of Theorem 1.1

The first part of Theorem 1.1 could be reduced to a result in [18]. However, the proof there seems to have a gap and thus we, for the sake of completeness, present a simple proof below. The argument below should also help the reader in understanding the further reasoning regarding the composition operator.

The inequality in the following lemma is well-known; the proof relies on an argument due to Hedberg [6].

Lemma 3.1. *Let $B \subset \mathbf{R}^n$ be an open ball and let $u \in W^{1,q}(3B)$, $1 < q < \infty$. Suppose that $x, y \in B$ are Lebesgue points of f. Then*

$$|u(x) - u(y)| \le C(n)|x-y|\big(M(|\nabla u|)(x) + M(|\nabla u|)(y)\big)$$

where

$$Mh(x) = \sup_{B(x,r)\subset 3B} \frac{1}{|B(x,r)|}\int_{B(x,r)} |h(z)|\,dz$$

is the Hardy–Littlewood maximal function of $h\colon 3B \to \mathbf{R}$.

Proof of the first part of Theorem 1.1. Fix $u \in W^{1,n}_{\mathrm{loc}}(\Omega_2)$, and let $x_0 \in \Omega_1$. We can clearly find a ball B and $r > 0$ such that $3B \subset\subset \Omega_2$ and $f(B(x_0, r)) \subset B$. We want to prove that $T_f(u) := u \circ f \in W^{1,1}(B(x_0, r))$ and that $|Df| \in L^p(B(x_0, r))$. For $\lambda > 0$, set

$$F_\lambda = \{x \in B : M(|\nabla u|)(x) \le \lambda\} \cap \{x \in B : x \text{ is a Lebesgue point of } u\}.$$

In view of Lemma 3.1, we obtain that u is Lipschitz-continuous on F_λ with Lipschitz-constant $C\lambda$. By the classical McShane extension theorem, there is a $C\lambda$-Lipschitz function $u_\lambda\colon B \to \mathbf{R}$ such that $u_\lambda = u$ on F_λ.

Set $g_j = u_j \circ f$ for $j \in \mathbf{N}$. Since u_j is Lipschitz, we obtain that $g_j \in W^{1,1}(B(x_0, r))$. We want to show that $\{\nabla g_j\}_{j\in\mathbf{N}}$ is a Cauchy sequence in $L^p(B(x_0, r), \mathbf{R}^n)$. From $|\nabla u| \in L^n(3B)$, we conclude that $M(\nabla u) \in L^n(B)$, and therefore

$$|B \setminus F_j| = o(j^{-n}). \tag{3.1}$$

Now let $i \le j$. Then

$$\begin{aligned} \int_B |\nabla u_i - \nabla u_j|^n &\le C\Big(\int_{B\setminus F_i} |\nabla u_i|^n + \int_{F_j\setminus F_i} |\nabla u_j|^n + \int_{B\setminus F_j} |\nabla u_j|^n\Big) \\ &\le o(i^{-n})i^n + C\int_{B\setminus F_i} |\nabla u|^n + o(j^{-n})j^n \overset{i\to\infty}{\to} 0. \end{aligned} \tag{3.2}$$

Set $q = \frac{n}{p}$. From the chain rule, the definition of mappings of finite distortion and Hölder's inequality we obtain

$$\begin{aligned} \int_{B(x_0,r)} |\nabla g_i - \nabla g_j|^p &\le \int_{B(x_0,r)} |Df(x)|^p |\nabla u_i(f(x)) - \nabla u_j(f(x))|^p\,dx \\ &\le \int_{B(x_0,r)} K(x)^{\frac{p}{n}} J_f(x)^{\frac{p}{n}} |\nabla u_i(f(x)) - \nabla u_j(f(x))|^p\,dx \\ &\le \| K^{\frac{p}{n}} \|_{L^{q'}(B(x_0,r))} \| J_f^{\frac{p}{n}} |\nabla u_i(f) - \nabla u_j(f)|^p \|_{L^q(B(x_0,r))}\,. \end{aligned}$$

Since $\frac{p}{n}q' = \frac{p}{n-p}$ and $K \in L^{\frac{p}{n-p}}_{\mathrm{loc}}(\Omega_1)$, we know that the first norm is finite. Thanks to (2.1) and (3.2) we have

$$\begin{aligned} \| J_f^{\frac{p}{n}} |\nabla u_i(f) - \nabla u_j(f)|^p \|^q_{L^q(B(x_0,r))} &= \int_{B(x_0,r)} J_f(x) |\nabla u_i(f(x)) - \nabla u_j(f(x))|^n\,dx \\ &\le \int_B |\nabla u_i(y) - \nabla u_j(y)|^n\,dy \overset{i\to\infty}{\to} 0. \end{aligned}$$

Therefore the sequence $\{\nabla g_j\}$ is a Cauchy sequence in L^p, and hence we can find $g \in L^p(B(x_0, r), \mathbf{R}^n)$ such that $\nabla g_j \to g$ in $L^p(B(x_0, r), \mathbf{R}^n)$.

Since f satisfies the Lusin condition (N^{-1}) [10], according to which f^{-1} maps sets of volume zero to sets of volume zero, and $|B \setminus F_j| \to 0$ we obtain that the sets $A_j := B(x_0, r) \cap f^{-1}(F_j)$ satisfy $|A_j| \to |B(x_0, r)|$. Thus we can find j_0 such that $|A_{j_0}| > \frac{1}{2}|B(x_0, r)|$. It follows from the definition of g_j that $g_j(x) = u \circ f(x)$ for every

$x \in A_{j_0}$ and $j \geq j_0$. Fix $i, j \geq j_0$. Since $g_i - g_j = 0$ on A_{j_0} and $|A_{j_0}| \geq \frac{1}{2}|B(x_0, r)|$ we can use the Poincaré inequality to obtain

$$\int_{B(x_0,r)} |g_i - g_j| = \int_{B(x_0,r)} \Big| g_i(x) - g_j(x) - \frac{1}{|A_{j_0}|} \int_{A_{j_0}} \big(g_i(y) - g_j(y)\big)\, dy \Big|\, dx$$
$$\leq Cr \int_{B(x_0,r)} |\nabla g_i - \nabla g_j|.$$

Since $\{\nabla g_i\}$ is a Cauchy sequence in $L^1(B(x_0, r), \mathbf{R}^n)$, we obtain that $\{g_i\}$ is also a Cauchy sequence in $L^1(B(x_0, r))$. Hence $g_j \to u \circ f$ in $L^1(B(x_0, r))$ because $g_j = u \circ f$ on A_j and $|B(c, r) \setminus A_j| \to 0$.

Clearly

$$\int_{B(x_0,r)} \nabla g_j(x)\phi(x)\, dx = -\int_{B(x_0,r)} g_j(x)\nabla\phi(x)\, dx$$

for every test function $\phi \in C_c^\infty(B(x_0, r), \mathbf{R}^n)$. Since $g_j \to u \circ f$ in L^1 and $\nabla g_j \to g$ in L^p we obtain, after passing to a limit, that

$$\int_{B(x_0,r)} g(x)\phi(x)\, dx = -\int_{B(x_0,r)} u \circ f(x)\nabla\phi(x)\, dx$$

which means that $g \in L^p(B(x_0, r))$ is a weak gradient of $u \circ f$ on $B(x_0, r)$. It then follows from the L^p-Poincaré inequality that $u \circ f \in W^{1,p}(B(x_0, r))$. □

4. Proof of the first part of Theorem 1.2

Proof of the first part of Theorem 1.2. Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega_2)$. Pick a sequence u_i of functions in $C^\infty(\Omega_2)$ so that $u_i \to u$ in $W^{1,q}_{\mathrm{loc}}(\Omega_2)$. Then $u_i \to \hat{u}$ locally uniformly in Ω_2 for the continuous representative $\hat{u}$ that coincides with u almost everywhere. By a simple modification to the reasoning at the end of the proof of the first part of Theorem 1.1, in order to prove that $\hat{u} \circ f \in W^{1,p}_{\mathrm{loc}}(\Omega_1)$, it suffices to show that the sequence $\nabla(u_i \circ f)$ is Cauchy in $L^p(A)$ whenever A is a ball compactly contained in Ω_1.

Let $i \leq j$. Fix a ball $A \subset\subset \Omega_1$ and set $G = \{x \in A : |Df(x)| > 0\}$. We can use the fact that $J_f > 0$ on G, apply Hölder's inequality and use (2.1) to obtain

$$\int_A |\nabla(u_i \circ f) - \nabla(u_j \circ f)|^p \leq \int_A |\nabla u_i(f(x)) - \nabla u_j(f(x))|^p |Df(x)|^p\, dx$$
$$= \int_G |\nabla u_i(f(x)) - \nabla u_j(f(x))|^p J_f(x)^{\frac{p}{q}} \frac{|Df(x)|^p}{J_f(x)^{\frac{p}{q}}}\, dx$$
$$\leq \Big(\int_{f(A)} |\nabla u_i - \nabla u_j|^q\Big)^{\frac{p}{q}} \Big(\int_G \Big(\frac{|Df(x)|^p}{J_f(x)^{\frac{p}{q}}}\Big)^{\frac{q}{q-p}}\, dx\Big)^{\frac{q-p}{q}}.$$

This clearly shows that $\nabla(u_i \circ f)$ is Cauchy in L^p if the last integral is finite. By Hölder's inequality and (1.2) we have

$$\int_G \Big(\frac{|Df(x)|^p}{J_f(x)^{\frac{p}{q}}}\Big)^{\frac{q}{q-p}}\,dx = \int_G \Big(\frac{|Df|^n}{J_f}\Big)^{\frac{p}{q-p}} |Df|^{\frac{q-n}{q-p}p}$$
$$\leq C\Big(\int_G K^a\Big)^{\frac{p}{a(q-p)}}\Big(\int_G |Df|^s\Big)^{\frac{p(q-n)}{s(q-p)}} < \infty.$$

When $a \geq 1/(n-1)$, f satisfies the Lusin condition (N^{-1}) (cf. [10]) and it follows that $u\circ f = \hat{u}\circ f$ almost everywhere and consequently that also $u\circ f \in W^{1,p}_{\mathrm{loc}}(\Omega_1)$. $\square$

5. Construction of examples

The following general construction of examples of mappings of finite distortion was introduced in [8] (see also [7]). Here we give only the brief overview of the construction, for details see [8, Section 5].

5.1. Canonical transformation. If $c \in \mathbf{R}^n$, $a, b > 0$, we use the notation

$$Q(c,a,b) := [c_1 - a, c_1 + a] \times \cdots \times [c_{n-1} - a, c_{n-1} + a] \times [c_n - b, c_n + b]$$

for the interval with center at c and halfedges a in the first $n-1$ coordinates and b in the last coordinate. If $Q = Q(c,a,b)$, the affine mapping

$$\varphi_Q(y) = (c_1 + ay_1, \ldots, c_{n-1} + ay_{n-1}, c_n + by_n)$$

is called the *c*anonical parametrization of the interval Q. Let P, P' be concentric intervals, $P = Q(c,a,b)$, $P' = Q(c,a',b')$, where $0 < a < a'$ and $0 < b < b'$. We set

$$\varphi_{P,P'}(t,y) = (1-t)\varphi_P(y) + t\varphi_{P'}(y), \quad t \in [0,1],\ y \in \partial Q_0.$$

This mapping is called the *c*anonical parametrization of the rectangular annulus $P' \setminus P^\circ$, where P° is the interior of P.

Now, we consider two rectangular annuli, $P' \setminus P^\circ$, and $\tilde{P}' \setminus \tilde{P}^\circ$, where $P = Q(c,a,b)$, $P' = Q(c,a',b')$, $\tilde{P} = Q(\tilde{c},\tilde{a},\tilde{b})$ and $\tilde{P}' = Q(\tilde{c},\tilde{a}',\tilde{b}')$, The mapping

$$h = \varphi_{\tilde{P},\tilde{P}'} \circ (\varphi_{P,P'})^{-1}$$

is called the *canonical transformation* of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$.

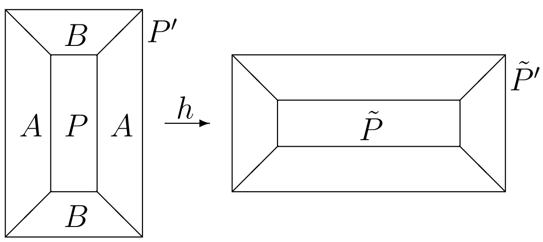

Figure 1. The canonical transformation of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$ for $n = 2$.

We will need an estimate of the derivate of h on $P' \setminus P^\circ$. For $t \in [0,1]$ fixed we denote

$$a'' = (1-t)a + ta', \quad b'' = (1-t)b + tb',$$
$$\tilde{a}'' = (1-t)\tilde{a} + t\tilde{a}', \quad \tilde{b}'' = (1-t)\tilde{b} + t\tilde{b}'.$$

It is possible to compute the derivative of $\varphi_{P,P'}(t,y)$ in one of the sides $\{y_i = \pm 1\}$. The image of the side has the shape of a pyramidal frustum. We must distinguish two cases, according to the position of the first variable.

Case A. We will represent the possibilities

$$\begin{aligned}&\varphi_{P,P'}(t,1,z_2,\dots,z_n),\ \varphi_{P,P'}(t,-1,z_2,\dots,z_n),\\ &\dots\\ &\varphi_{P,P'}(t,z_1,\dots z_{n-2},1,z_n),\ \varphi_{P,P'}(t,z_1,\dots z_{n-2},-1,z_n)\end{aligned}$$

by

$$\varphi(t,z) = \varphi_{P,P'}(t,1,z), \quad z = (z_2,\dots,z_n).$$

Then it can be computed (see [8, Section 5] for details) that

$$Dh(\varphi(t,z)) = \begin{pmatrix} \frac{\tilde{a}'-\tilde{a}}{a'-a}, & 0, & 0, & \dots, & 0\\ \big(\frac{\tilde{a}'-\tilde{a}}{a'-a} - \frac{\tilde{a}''}{a''}\big)z_2, & \frac{\tilde{a}''}{a''}, & 0, & \dots, & 0\\ \big(\frac{\tilde{a}'-\tilde{a}}{a'-a} - \frac{\tilde{a}''}{a''}\big)z_3, & 0, & \frac{\tilde{a}''}{a''}, & \dots, & 0\\ \dots\\ \big(\frac{\tilde{b}'-\tilde{b}}{a'-a} - \frac{\tilde{b}''}{b''}\frac{b'-b}{a'-a}\big)z_n, & 0, & 0, & \dots, & \frac{\tilde{b}''}{b''}\end{pmatrix}. \tag{5.1}$$

Case B. A representative is

$$\begin{aligned}\varphi(t,z) = \Big(&(\varphi_{P,P'})_n(t,z,1),\ (\varphi_{P,P'})_1(t,z,1),\dots,(\varphi_{P,P'})_{n-1}(t,z,1)\Big),\\ &z = (z_1,\dots,z_{n-1}).\end{aligned}$$

The purpose of the permutation of coordinates is that this leads to a triangular matrix which is easier to handle. Then

$$Dh(\varphi(t,z)) = \begin{pmatrix} \frac{\tilde{b}'-\tilde{b}}{b'-b}, & 0, & 0, & \dots, & 0\\ \big(\frac{\tilde{a}'-\tilde{a}}{b'-b} - \frac{\tilde{a}''}{a''}\frac{a'-a}{b'-b}\big)z_1, & \frac{\tilde{a}''}{a''}, & 0, & \dots, & 0\\ \big(\frac{\tilde{a}'-\tilde{a}}{b'-b} - \frac{\tilde{a}''}{a''}\frac{a'-a}{b'-b}\big)z_2, & 0, & \frac{\tilde{a}''}{a''}, & \dots, & 0\\ \dots\\ \big(\frac{\tilde{a}'-\tilde{a}}{b'-b} - \frac{\tilde{a}''}{a''}\frac{a'-a}{b'-b}\big)z_{n-1}, & 0, & 0, & \dots, & \frac{\tilde{a}''}{a''}\end{pmatrix}. \tag{5.2}$$

5.2. Construction of a mapping. By $\mathbf{V}$ we denote the set of 2^n vertices of the cube $[-1,1]^n =: Q_0$. The sets $\mathbf{V}^k = \mathbf{V} \times \dots \times \mathbf{V}$, $k \in \mathbf{N}$, will serve as the sets of indices for our construction. If $\boldsymbol{w} \in \mathbf{V}^k$ and $v \in \mathbf{V}$, then the concatenation of $\boldsymbol{w}$ and v is denoted by $\boldsymbol{w}^\wedge v$. The following two results are proven in [8].

Lemma 5.1. *Let $n \geq 2$. Suppose that we are given two sequences of positive real numbers $\{a_k\}_{k\in\mathbf{N}_0}$, $\{b_k\}_{k\in\mathbf{N}_0}$,*

$$a_0 = b_0 = 1; \tag{5.3}$$

$$a_k < a_{k-1},\ b_k < b_{k-1},\ \text{for } k \in \mathbf{N}. \tag{5.4}$$

Then there exist unique systems $\{Q_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbf{N}}\mathbf{V}^k}$, $\{Q'_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbf{N}}\mathbf{V}^k}$ of intervals

$$Q_{\boldsymbol{v}} = Q(c_{\boldsymbol{v}}, 2^{-k}a_k, 2^{-k}b_k), \quad Q'_{\boldsymbol{v}} = Q(c_{\boldsymbol{v}}, 2^{-k}a_{k-1}, 2^{-k}b_{k-1}) \tag{5.5}$$

such that

$$Q'_{\boldsymbol{v}},\ \boldsymbol{v} \in \mathbf{V}^k, \quad \text{are nonoverlaping for fixed } k \in \mathbf{N}, \tag{5.6}$$

$$Q_{\boldsymbol{w}} = \bigcup_{v\in\mathbf{V}} Q'_{\boldsymbol{w}\wedge v} \text{ for each } \boldsymbol{w} \in \mathbf{V}^k,\ k \in \mathbf{N}, \tag{5.7}$$

$$c_v = \frac{1}{2}v, \qquad v \in \mathbf{V}, \tag{5.8}$$

$$c_{\boldsymbol{w}\wedge v} = c_{\boldsymbol{w}} + \sum_{i=1}^{n-1} 2^{-k}a_k v_i \mathbf{e}_i + 2^{-k}b_k v_n \mathbf{e}_n, \tag{5.9}$$

$$\boldsymbol{w} \in \mathbf{V}^k,\ k \in \mathbf{N},\ v = (v_1, \dots, v_n) \in \mathbf{V}.$$

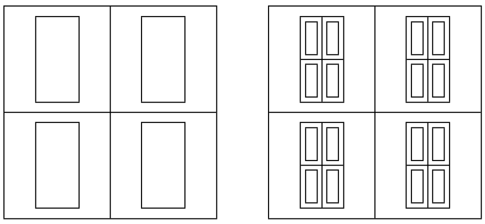

Figure 2. Intervals $Q_{\boldsymbol{v}}$ and $Q'_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbf{V}^1$ and $\boldsymbol{v} \in \mathbf{V}^2$ for $n = 2$.

Theorem 5.2. *Let $n \geq 2$. Suppose that we are given four sequences of positive real numbers $\{a_k\}_{k\in\mathbf{N}_0}$, $\{b_k\}_{k\in\mathbf{N}_0}$, $\{\tilde{a}_k\}_{k\in\mathbf{N}_0}$, $\{\tilde{b}_k\}_{k\in\mathbf{N}_0}$,*

$$a_0 = b_0 = \tilde{a}_0 = \tilde{b}_0 = 1; \tag{5.10}$$

$$a_k < a_{k-1},\ b_k < b_{k-1},\ \tilde{a}_k < \tilde{a}_{k-1},\ \tilde{b}_k < \tilde{b}_{k-1},\ \text{for } k \in \mathbf{N}. \tag{5.11}$$

Let the systems $\{Q_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbf{N}}\mathbf{V}^k}$, $\{Q'_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbf{N}}\mathbf{V}^k}$ of intervals be as in Lemma 5.1, and similarly systems $\{\tilde{Q}_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbf{N}}\mathbf{V}^k}$, $\{\tilde{Q}'_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbf{N}}\mathbf{V}^k}$ of intervals be associated with the sequences $\{\tilde{a}_k\}$ and $\{\tilde{b}_k\}$. Then there exists a unique sequence $\{f^k\}$ of bilipschitz homeomorphisms of Q_0 onto itself such that

(a) *f^k maps each $Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbf{V}^m$, $m = 1, \dots, k$, onto $\tilde{Q}'_{\boldsymbol{v}} \setminus \tilde{Q}_{\boldsymbol{v}}$ canonically,*

(b) *f^k maps each $Q_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbf{V}^k$, onto $\tilde{Q}_{\boldsymbol{v}}$ affinely.*

Moreover,

$$|f^k - f^{k+1}| \lesssim 2^{-k}, \quad |(f^k)^{-1} - (f^{k+1})^{-1}| \lesssim 2^{-k}. \tag{5.12}$$

The sequence f^k converges uniformly to a homeomorphism f of Q_0 onto Q_0.

5.3. Completion of the proofs of theorems 1.1 and 1.2.

Construction for Theorem 1.1. It is a well-known fact that for every $\varepsilon > 0$ there is a quasiconformal mapping f such that $f \notin W^{1,n+\varepsilon}_{\mathrm{loc}}$. Therefore we can assume that $p < n$. Choose $\delta > 0$ such that

$$\delta < \frac{n}{p} - 1 \quad \text{and} \quad \delta(n-1+p+\varepsilon) < \varepsilon\frac{n-1}{p}. \tag{5.13}$$

Set

$$\alpha = 1 - \frac{1}{p} + \delta, \quad \beta = \frac{n-1}{p} \quad \text{and} \quad \gamma = \delta.$$

With the help of (5.13) it is not difficult to verify that

$$\begin{aligned} (n-1)\alpha + \beta + p(\gamma-\beta) &= (n-1+p)\delta > 0 \\ (n-1)\alpha + \beta + (p+\varepsilon)(\gamma-\beta) &= (n-1+p+\varepsilon)\delta - \varepsilon\frac{n-1}{p} < 0 \\ (n-1)\alpha + \beta + \frac{p(n-1)}{n-p}(\alpha-\beta) &= \big(n-1+\frac{p(n-1)}{n-p}\big)\delta > 0. \end{aligned} \tag{5.14}$$

Use Theorem 5.2 for

$$a_k = \frac{1}{(k+1)^\alpha}, \quad b_k = \frac{1}{(k+1)^\beta}, \quad \tilde{a}_k = \frac{1}{(k+1)^\gamma} \quad \text{and} \quad \tilde{b}_k = \frac{1}{(k+1)^\gamma}$$

to obtain the sequence $\{f^k\}$ and a limit mapping mapping f.

For fixed $t \in [0,1]$ we denote

$$\begin{aligned} a''_k &= (1-t)a_k + ta_{k-1}, \quad b''_k = (1-t)b_k + tb_{k-1}, \\ \tilde{a}''_k &= (1-t)\tilde{a}_k + t\tilde{a}_{k-1}, \quad \tilde{b}''_k = (1-t)\tilde{b}_k + t\tilde{b}_{k-1}. \end{aligned}$$

Since $\frac{1}{k^\omega} - \frac{1}{(k+1)^\omega} \sim \frac{1}{k^{\omega+1}}$ for every $\omega > 0$, it is easy to check that

$$\begin{aligned} \frac{\tilde{a}_{k-1}-\tilde{a}_k}{a_{k-1}-a_k} &\sim \frac{\tilde{a}''_k}{a''_k} \sim k^{\alpha-\gamma}, \quad & \frac{\tilde{b}_{k-1}-\tilde{b}_k}{b_{k-1}-b_k} &\sim \frac{\tilde{b}''_k}{b''_k} \sim k^{\beta-\gamma}, \\ \frac{\tilde{b}_{k-1}-\tilde{b}_k}{a_{k-1}-a_k} &\sim \frac{\tilde{b}''_k}{a''_k} \sim k^{\alpha-\gamma}, \quad & \frac{\tilde{a}_{k-1}-\tilde{a}_k}{b_{k-1}-b_k} &\sim \frac{\tilde{a}''_k}{b''_k} \sim k^{\beta-\gamma}, \\ \frac{b_{k-1}-b_k}{a_{k-1}-a_k} &\sim \frac{b''_k}{a''_k} \sim k^{\alpha-\beta}. \end{aligned}$$

From (5.13) we obtain that $\alpha < \beta$ and therefore it is not difficult to deduce from (5.1) and (5.2) that

$$\begin{aligned} |Df^k(x)| &= |Df(x)| \sim k^{\beta-\gamma} \quad \text{and} \\ K(x) &= \frac{|Df(x)|^n}{J_f(x)} \sim \frac{k^{n(\beta-\gamma)}}{k^{(n-1)(\alpha-\gamma)+(\beta-\gamma)}} = k^{(n-1)(\beta-\alpha)} \end{aligned} \tag{5.15}$$

for almost every $x \in \tilde{Q}'_{\boldsymbol{v}} \setminus \tilde{Q}_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbf{V}^k$. It is also not difficult to find out from the construction that

$$\mathscr{L}_n(\tilde{Q}'_{\boldsymbol{v}} \setminus \tilde{Q}_{\boldsymbol{v}}) \sim \frac{1}{2^{kn} k^{(n-1)\alpha+\beta+1}} \quad \text{for every } \boldsymbol{v} \in \mathbf{V}^k. \tag{5.16}$$

Let $k < m$. From (5.15), (5.16) and (5.14) we obtain

$$\begin{aligned} \int_{Q_0} |Df^k - Df^m|^p \, dx &\lesssim \int_{\{f^k \neq f^m\}} \big(|Df^k|^p + |Df^m|^p\big)\, dx \\ &\lesssim \sum_{\boldsymbol{v}\in\mathbf{V}^k} \int_{Q_{\boldsymbol{v}}} |Df^k|^p \, dx + \sum_{j=k+1}^{m} \sum_{\boldsymbol{v}\in\mathbf{V}^j} \int_{Q'_{\boldsymbol{v}}\setminus Q_{\boldsymbol{v}}} |Df|^p \, dx + \sum_{\boldsymbol{v}\in\mathbf{V}^m} \int_{Q_{\boldsymbol{v}}} |Df^m|^p \, dx \\ &\lesssim \sum_{j=k}^{m} 2^{kn} \frac{\big(j^{\beta-\gamma}\big)^p}{2^{kn} j^{(n-1)\alpha+\beta+1}} \lesssim k^{-(n-1+p)\delta} \to 0. \end{aligned}$$

It follows that the sequence $\{f^k\}$ converges to f in $W^{1,p}(Q_0, \mathbf{R}^n)$ and, in particular, $f \in W^{1,p}(Q_0, \mathbf{R}^n)$. From (5.14) and (5.15) we also have

$$\int_{Q_0} |Df|^{p+\varepsilon} \sim \sum_{k\in\mathbf{N}} \frac{k^{(p+\varepsilon)(\beta-\gamma)}}{k^{1+(n-1)\alpha+\beta}} = \infty \quad \text{and}$$

$$\int_{Q_0} K^{\frac{p}{n-p}} \sim \sum_{k\in\mathbf{N}} \frac{\big(k^{(n-1)(\beta-\alpha)}\big)^{\frac{p}{n-p}}}{k^{1+(n-1)\alpha+\beta}} < \infty.$$

By considering the functions $u_i(x) = x_i$, we see that $T_f(W^{1,n}(Q_0)) \not\subset W^{1,p+\varepsilon}_{\mathrm{loc}}(Q_0)$. □

Construction for Theorem 1.2. Since $s(q-p) - p(q-n) > 0$ (i.e. $a > 0$) we can clearly find $\eta > 0$ small enough such that

$$\begin{aligned} &\eta < qs + np - qp - \frac{1}{n-1}(qp + nsp - np - qs) \quad \text{and} \\ &(n-1)\eta + \varepsilon\big(-ns + \eta\big) + \eta p < 0. \end{aligned} \tag{5.17}$$

Set

$$\begin{aligned} &\alpha = \frac{1}{n-1}(qp + nsp - np - qs) + \eta, \quad \beta = qs + np - qp, \\ &\gamma = q(s-p) + \eta \text{ and } \delta = (q-n)(s-p) + \eta. \end{aligned}$$

It is easy to check that β, γ and δ are positive. From $a \geq \frac{1}{n-1}$ we obtain that also α is positive and (5.17) implies $\beta > \alpha$. With the help of the definition of a and (5.17) it is not difficult to verify that

$$\begin{aligned} (n-1)\alpha + \beta + s(\gamma - \beta) &= (n-1)\eta + s\eta > 0 \\ (n-1)\alpha + \beta + a(n-1)(\alpha - \beta) &= (n-1)\eta + a(n-1)\eta > 0 \\ n\gamma + q(\delta - \gamma) &= n\eta > 0 \\ (n-1)\alpha + \beta + (p+\varepsilon)(\delta - \beta) &= (n-1)\eta + \varepsilon\big(-ns + \eta\big) + \eta p < 0. \end{aligned} \tag{5.18}$$

Use Theorem 5.2 for the sequences

$$a_k = \frac{1}{(k+1)^\alpha}, \quad b_k = \frac{1}{(k+1)^\beta}, \quad \tilde{a}_k = \frac{1}{(k+1)^\gamma} \quad \text{and} \quad \tilde{b}_k = \frac{1}{(k+1)^\gamma}$$

to obtain the sequence $\{f^k\}$ and a limit mapping mapping f. Analogously to the proof of the second part of Theorem 1.1 we obtain that $f \in W^{1,s}$ and thanks to $\beta > \alpha$ and (5.18) we have

$$\int_{Q_0} |Df|^s \sim \sum_{k\in\mathbf{N}} \frac{k^{s(\beta-\gamma)}}{k^{1+(n-1)\alpha+\beta}} < \infty \quad \text{and}$$

$$\int_{Q_0} K^a \sim \sum_{k\in\mathbf{N}} \frac{\left(k^{n(\beta-\gamma)-(n-1)(\alpha-\gamma)-(\beta-\gamma)}\right)^a}{k^{1+(n-1)\alpha+\beta}} < \infty.$$

Analogously, we can use Theorem 5.2 for the sequences

$$\tilde{a}_k = \frac{1}{(k+1)^\gamma}, \quad \tilde{b}_k = \frac{1}{(k+1)^\gamma}, \quad \tilde{\tilde{a}}_k = \frac{1}{(k+1)^\delta} \quad \text{and} \quad \tilde{\tilde{b}}_k = \frac{1}{(k+1)^\delta}$$

to obtain a limit mapping g such that

$$\int_{Q_0} |Dg|^q \sim \sum_{k\in\mathbf{N}} \frac{k^{q(\gamma-\delta)}}{k^{1+n\gamma}} < \infty.$$

From [8, Remark 5.6] we know that the mapping $h = g \circ f$ can be obtained as a limit mapping from Theorem 5.2 applied to the sequences

$$a_k = \frac{1}{(k+1)^\alpha}, \quad b_k = \frac{1}{(k+1)^\beta}, \quad \tilde{\tilde{a}}_k = \frac{1}{(k+1)^\delta} \quad \text{and} \quad \tilde{\tilde{b}}_k = \frac{1}{(k+1)^\delta}.$$

Therefore (5.18) yields

$$\int_{Q_0} |Dh|^{p+\varepsilon} \sim \sum_{k\in\mathbf{N}} \frac{k^{(p+\varepsilon)(\beta-\delta)}}{k^{1+(n-1)\alpha+\beta}} = \infty.$$

To obtain a real-valued function u as indicated in the second part of Theorem 1.2, simply consider the coordinate functions of g. □

6. Integrability of the distortion of $f_2 \circ f_1$

In this section we give conditions which quarantee nice integrability of the distortion of $f_2 \circ f_1$. The following example shows that even if f_1 and f_2 and their distortions are very nice it does not follow that the distortion of their composition is nice.

Example 6.1. *Let $n \geq 2$ and $p \geq 1$. There exist homeomorphisms $f_1, f_2 \colon B(0,1) \to B(0,1)$ of finite distortion such that f_1 and f_2 are Lipschitz, $\exp(K_1^p) \in L^1(B(0,1))$ and $K_2 \in L^p(B(0,1))$, but $K \notin L^\delta_{\mathrm{loc}}(B(0,1))$ for any $\delta > 0$, where K denotes the distortion of the mapping $f = f_2 \circ f_1$.*

Proof. Set

$$f_1(x) = e\frac{x}{\|x\|}\exp\Big(-\log^{1+\frac{1}{2(n-1)p}}\frac{e}{\|x\|}\Big) \text{ for } x \in B(0,1)\setminus\{0\},$$
$$f_2(x) = e\frac{x}{\|x\|}\exp\big(-\|x\|^{-\frac{1}{p}}\big) \text{ for } x \in B(0,1)\setminus\{0\}$$

and $f_1(0) = f_2(0) = 0$. From Lemma 2.1 we easily obtain that f_2 is Lipschitz and that

$$\int_{B(0,1)} K_2^p(x)\,dx \sim \int_{B(0,1)} \frac{1}{\|x\|^{n-1}}\,dx < \infty.$$

Analogously we obtain that f_1 is Lipschitz and

$$\int_{B(0,1)} \exp(K_1^p(x))\,dx \sim \int_{B(0,1)} \exp\Big(C\log^{1/2}\frac{e}{\|x\|}\Big)\,dx < \infty.$$

Since for every $x \neq 0$ we have

$$f(x) = e\frac{x}{\|x\|}\exp\Big(-e^{-1/p}\exp\big(\frac{1}{p}\log^{1+\frac{1}{2(n-1)p}}\frac{e}{\|x\|}\big)\Big)$$

one can use Lemma 2.1 to obtain that

$$K(x) \sim \exp\big(\frac{n-1}{p}\log^{1+\frac{1}{2(n-1)p}}\frac{e}{\|x\|}\big)\log^{\frac{1}{2p}}\frac{e}{\|x\|}$$

and it is easy to check that K^δ is not integrable for any $\delta > 0$. □

Lemma 6.2. *Let $n \geq 2$, $p > n-1$ and let $\Omega \subset \mathbf{R}^n$ be a domain. Suppose that $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ is a homeomorphism of finite distortion such that $|Df| \in L^{n-1,1}_{\mathrm{loc}}(\Omega)$ and $K \in L^p_{\mathrm{loc}}(\Omega)$. Then $|Df^{-1}|^n \log^{\frac{p-n+1}{p}}(e + |Df^{-1}|) \in L^1_{\mathrm{loc}}(f(\Omega))$.*

Proof. Fix a compact set $E \subset \Omega$. The fact that, under our assumptions, we have $f^{-1} \in W^{1,n}_{\mathrm{loc}}(f(\Omega), \mathbf{R}^n)$ and moreover that f is mapping of finite distortion follows from [8, Theorem 1.2 and Theorem 4.1]. Therefore, analogously to [8, Proof of Theorem 4.1], we obtain

$$\begin{aligned}(6.1)\qquad &\int_{f(E)} |Df^{-1}(y)|^n \log^{\frac{p-n+1}{p}}(e + |Df^{-1}(y)|)\,dy \\ &\leq \int_E K(x)^{n-1}\log^{\frac{p-n+1}{p}}\big(e + \frac{K(x)}{|Df(x)|}\big)\,dx.\end{aligned}$$

Set $S = \{x \in E : \frac{K(x)}{|Df(x)|} \leq \exp(K^p(x))\}$. For every $x \in E \setminus S$ we have

$$K^p(x) \leq C(p)\log\big(e + \frac{1}{|Df(x)|}\big)$$

and therefore we can split the integral in (6.1) into two parts and prove that it is no greater than

$$\int_E K(x)^{n-1}\big(C(p) + K^{p-n+1}(x)\big)\,dx + C(p)\int_E \log\big(e + \frac{1}{|Df(x)|}\big)\,dx.$$

The finiteness of the first integral follows from $K \in L^p(E)$. Analogously to [7, Theorem 6.1], one can prove that $\log(1+\frac{1}{|J_f|}) \in L^1_{\mathrm{loc}}(\Omega)$ for every $n \geq 2$, which implies that the second integral is also finite. □

It follows from the Example 6.1 that if we want to prove the integrability of some power of the distortion of $f_2 \circ f_1$, we must require some stronger condition than the integrability of some power of the distortion of f_2.

Theorem 6.3. *Let $n \geq 2$, $\Omega \subset \mathbf{R}^n$ be a domain, $p > n-1$ and $r > 0$ and set $q = \frac{pr(p-n+1)}{r(p-n+1)+p(p+1)}$. Suppose that $f_1 \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ and $f_2 \in W^{1,n}_{\mathrm{loc}}(f_1(\Omega), \mathbf{R}^n)$ are homeomorphisms with finite distortion such that $|Df_1| \in L^{n-1,1}(\Omega)$, $K_1 \in L^p(\Omega)$ and $\exp(2K_2^r) \in L^1_{\mathrm{loc}}(f_1(\Omega))$. Then $f = f_2 \circ f_1$ is a mapping of finite distortion and its distortion satisfies $K \in L^q_{\mathrm{loc}}(\Omega)$.*

Proof. From Theorem 1.1 we know that $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$. We claim that for almost every $x \in \Omega$ we have

$$Df(x) = Df_2(f_1(x))Df_1(x) \quad \text{and} \quad J_f(x) = J_{f_2}(f_1(x))J_{f_1}(x). \tag{6.2}$$

From [3] we know that we can find a Borel partition of $f_1(\Omega)$, $\{A_k\}$, such that $|A_0| = 0$ and f_2 is Lipschitz on A_k, $k > 0$. We know that f_1 is differentiable almost everywhere (see [14]) and that f_2 restricted to A_k is differentiable almost everywhere. Since f_1 satisfies the Lusin (N^{-1}) condition (see [10, Theorem 1.2]) it is not difficult to deduce that (6.2) holds almost everywhere on $f_1^{-1}(A_k)$ for every $k > 0$. The Lusin (N^{-1}) condition also gives us $|f_1^{-1}(A_0)| = 0$ and therefore (6.2) holds almost everywhere. Since f_1 and f_2 are mappings of finite distortion and f_2 satisfies the Lusin (N^{-1}) condition, we can deduce from (6.2) that f is also a mapping of finite distortion.

Let $A \subset\subset \Omega$ be a fixed Borel set such that f_1 is differentiable at A (recall that this happens almost everywhere in Ω [14]) and that $|Df(x)| > 0$ for every $x \in A$. Set

$$s = \frac{p^2}{r(p-n+1)+p(p+1)} \tag{6.3}$$

and check that clearly $0 < s < 1$. The definition of distortion, (6.2) and the Hölder's inequality give us

$$\begin{aligned}\int_A K^q(x)\,dx &\leq \int_A \frac{|Df_2(f_1(x))|^{nq}}{J_{f_2}(f_1(x))^q}\frac{J_{f_1}(x)^s}{|Df_1(x)|^{ns}}\frac{|Df_1(x)|^{n(q+s)}}{J_{f_1}(x)^{q+s}}\,dx\\ &\leq \Big(\int_A K_2^{\frac{q}{s}}(f_1(x))\frac{J_{f_1}(x)}{|Df_1(x)|^n}\,dx\Big)^s\Big(\int_A K_1^{\frac{q+s}{1-s}}(x)\,dx\Big)^{1-s}.\end{aligned}$$

Clearly $\frac{q+s}{1-s} = p$, which implies that the second integral is finite and therefore it is enough to prove the finiteness of the first integral. By (2.1), $Df_1(f_1^{-1}(y))Df_1^{-1}(y) =$

I and (2.2) for $\alpha = \frac{p-n+1}{p}$ we have

$$\begin{aligned} \int_A K_2^{\frac{q}{s}}(f_1(x))\frac{J_{f_1}(x)}{|Df_1(x)|^n}\,dx &\leq \int_{f_1(A)} K_2^{\frac{q}{s}}(y)|Df_1^{-1}(y)|^n\,dy \\ &\leq\ C\int_{f_1(A)} \exp\bigl(2K_2^{\frac{qp}{s(p-n+1)}}(y)\bigr) \\ &\quad + C\int_{f_1(A)} |Df_1^{-1}(y)|^n \log^{\frac{p-n+1}{p}}(e+|Df_1^{-1}(y)|)\,dy. \end{aligned} \tag{6.4}$$

The boundedness of the first integral follows from (6.3) and our assumptions and the boundedness of the second follows from Lemma 6.2. □

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