Ann. Funct. Anal. 1 (2010), no. 2, 28–45
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

FURUTA INEQUALITY AND ITS RELATED TOPICS

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Communicated by M. S. Moslehian

ABSTRACT. This article is devoted to a brief survey of Furuta inequality and its related topics. It consists of 4 sections: 1. From Löwner-Heinz inequality to Furuta inequality, 2. Ando–Hiai inequality, 3. Grand Furuta inequality, and 4. Chaotic order.

1. FROM LÖWNER-HEINZ INEQUALITY TO FURUTA INEQUALITY.

The noncommutativity of operators appears in the fact that t^2 is not order-preserving. That is, there is a pair of positive operators A and B such that $A \geq B$ and $A^2 \ngeq B^2$. The following is a quite familiar example;

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that t^p is not order-preserving for $p > 1$ by assuming the following fact, see [20, 23, 24]:

Theorem 1.1 (Löwner-Heinz inequality (LH))**.** *The fuction t^p is order-preserving for $0 \leq p \leq 1$, i.e.,*

$$A \geq B \geq 0 \implies A^p \geq B^p.$$

The essense of the Löwner-Heinz inequality is the case $p = \frac{1}{2}$:

$$A \geq B \geq 0 \implies A^{\frac{1}{2}} \geq B^{\frac{1}{2}}.$$

It is rephrased as follows: For $A,\ B \geq 0$,

$$AB^2A \leq 1 \implies A^{\frac{1}{2}}BA^{\frac{1}{2}} \leq 1.$$

Date: Received: 1 December 2010; Accepted: 21 December 2010.
2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 47A64.
Key words and phrases. Löwner-Heinz inequality, Furuta inequality, Ando–Hiai inequality, Grand Furuta inequality, chaotic order and operator geometric mean.

The assumption $AB^2A \le 1$ is equivalent to $\|AB\| \le 1$. Thus, noting the commutativity of the spectral radius, $r(XY) = r(YX)$, we have

$$\|A^{\frac{1}{2}}BA^{\frac{1}{2}}\| = r(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = r(AB) \le \|AB\| \le 1.$$

The above discussion goes to Pedersen's proof of the Löwner-Heinz inequality. As a matter of fact, the following statement is proved: Let I be the set of all $p \in [0, \frac{1}{2}]$ such that $A \ge B \ge 0$ implies $A^{2p} \ge B^{2p}$. Then I is convex.

So suppose that $A^pB^{2p}A^p \le 1$ and $A^qB^{2q}A^q \le 1$, or equivalently $\|A^pB^p\| \le 1$ and $\|B^qA^q\| \le 1$. Then

$$\begin{aligned}\|A^{\frac{p+q}{2}}B^{p+q}A^{\frac{p+q}{2}}\| &= r(A^{\frac{p+q}{2}}B^{p+q}A^{\frac{p+q}{2}}) = r(A^{p+q}B^{p+q}) = r(A^pB^pB^qA^q) \\ &\le \|A^pB^p\|\|B^qA^q\| \le 1.\end{aligned}$$

This implies that if $2p, 2q \in I$, then $p + q \in I$, that is, I is convex.

Related to the case $p = \frac{1}{2}$ in the Löwner-Heinz inequality, Chan-Kwong [4] conjectured that

$$A \ge B \ge 0 \implies (AB^2A)^{\frac{1}{2}} \le A^2.$$

Moreover, if it is true, then the following inequality holds;

$$A \ge B \ge 0 \implies (BA^2B)^{\frac{1}{2}} \ge B^2.$$

Here we cite a useful lemma on exponent.

Lemma 1.2. *For $p \in \mathbb{R}$, $(X^*A^2X)^p = X^*A(AXX^*A)^{p-1}AX$ for $A > 0$ and invertible X.*

Proof. It is easily checked that $Y^*(YY^*)^nY = Y^*Y(Y^*Y)^n$ for any $n \in \mathbb{N}$. This implies that $Y^*f(YY^*)Y = Y^*Yf(Y^*Y)$ for any polynomials f and so it holds for continuous functions f on a suitable interval. Hence we have the conclusion by applying it to $f(x) = x^p$ and $Y = AX$. □

Consequently, Chan-Kwong conjecture is modified in the following sense: If it is true, then

$$A \ge B \ge 0 \implies (AB^2A)^{\frac{3}{4}} \le A^3.$$

As a matter of fact, we have

$$(AB^2A)^{\frac{3}{4}} = AB(BA^2B)^{-\frac{1}{4}}BA = AB((BA^2B)^{-\frac{1}{2}})^{\frac{1}{2}}BA \le ABB^{-1}BA = ABA \le A^3.$$

Based on this consideration, the Furuta inequality was established as follows:

Theorem 1.3 (Furuta inequality (FI))**.** *If $A \ge B \ge 0$, then for each $r \ge 0$,*

$$\text{(i)} \qquad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \ge (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$\text{(ii)} \qquad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \ge (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \ge 0$ and $q \ge 1$ with

$$(*) \qquad (1 + r)q \ge p + r.$$

See [14, 15, 5, 21, 25, 19]

Professor Berberian said that the figuare determined by (*) is "Rosetta Stone" in (FI). Incidentally it is notable that the figuare (*) is expressed by qp-axix: Berberian's interesting comment might contain it.

Proof of (FI). It suffices to show that if $A \geq B > 0$, then

$$(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} \leq A^{1+r}.$$

It is proved for arbitrary $p \geq 1$ by the induction on r. First of all, we take $r \in [0, 1]$.

$$(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} = A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{\frac{p}{2}} A^r B^{\frac{p}{2}})^{\frac{1-p}{p+r}} B^{\frac{p}{2}} A^{\frac{r}{2}}$$

$$\leq A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{\frac{p}{2}} B^r B^{\frac{p}{2}})^{\frac{1-p}{p+r}} B^{\frac{p}{2}} A^{\frac{r}{2}} = A^{\frac{r}{2}} B A^{\frac{r}{2}} \leq A^{\frac{r}{2}} A A^{\frac{r}{2}} = A^{1+r}.$$

Next we suppose that it is true for some $r_1 > 0$, i.e.,

$$B_1 = (A^{\frac{r_1}{2}} B^p A^{\frac{r_1}{2}})^{\frac{1+r_1}{p+r_1}} \leq A^{1+r_1} = A_1.$$

Then for $r \in (0, 1]$

$$(A_1^{\frac{r}{2}} B_1^{\frac{p+r_1}{1+r_1}} A_1^{\frac{r}{2}})^{\frac{1+r}{p_1+r}} \leq A_1^{1+r},$$

where $p_1 = \frac{p+r_1}{1+r_1}$. Putting $s = r_1 + (1+r_1)r = (1+r_1)(1+r) - 1$, we have

$$(A^{\frac{s}{2}} B^p A^{\frac{s}{2}})^{\frac{1+s}{p+s}} \leq A^{1+s},$$

This means that it is true for $s \in [r_1, 1+2r_1]$. Hence the proof is complete. $\square$

To make clear the structure of (FI), we give a mean theoretic approach to (FI). The Löwner-Heinz inequality says that the function t^α is operator monotone for $\alpha \in [0, 1]$. It induces the α-geometric operator mean defined for $\alpha \in [0, 1]$ as

$$A \sharp_\alpha B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$$

if $A > 0$, i.e., A is invertible, by the Kubo-Ando theory [22], see also [1].

For the sake of convenience, we cite a useful lemma which we will use frequently in the below.

Lemma 1.4. *For $X,\ Y > 0$ and $a,\ b \in [0, 1]$,*
(i) *monotonicity: $X \leq X_1$ and $Y \leq Y_1 \implies X \sharp_a Y \leq X_1 \sharp_a Y_1$,*
(ii) *transformer equality: $T^*XT \sharp_a T^*YT = T^*(X \sharp_a Y)T$ for invertible T,*
(iii) *transposition: $X \sharp_a Y = Y \sharp_{1-a} X$,*
(iv) *multiplicativity: $X \sharp_{ab} Y = X \sharp_a (X \sharp_b Y)$.*

Proof. First of all, (iii) follows from Lemma 1.2 and (iv) does from a direct computation under the assumption of invertibility of operators.

To prove (i), we may assume that $X, Y > 0$. If $Y \leq Y_1$, then $X \sharp_a Y \leq X \sharp_a Y_1$ is assured by (LH) (and the formula of $\sharp_a$). Moreover the monotonicity of the other is shown by the use of (iii).

Finally we prove (ii). We put $Z = X^{\frac{1}{2}}T = U|Z|$, the polar decomposition of Z, where U is unitary. Then it follows that

$$\begin{aligned} T^*XT \ \sharp_a \ T^*YT &= Z^*Z \ \sharp_a \ T^*YT \\ &= |Z|(|Z|^{-1}T^*YT|Z|^{-1})^a|Z| \\ &= |Z|(|Z|^{-1}Z^*(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})Z|Z|^{-1})^a|Z| \\ &= |Z|(U^*(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})U)^a|Z| \\ &= |Z|U^*(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^aU|Z| \\ &= Z^*(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^aZ \\ &= T^*X^{\frac{1}{2}}(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^aX^{\frac{1}{2}}T \\ &= T^*(X \ \sharp_a \ Y)T. \end{aligned}$$

□

By using the mean theoretic notation, the Furuta inequality has the following expression:

(FI) *If* $A \geq B > 0$, *then*

$$A^{-r} \ \sharp_{\frac{1+r}{p+r}} \ B^p \leq A \quad \textit{for} \quad p \geq 1 \ \textit{and} \ \ r \geq 0. \tag{1.1}$$

Related to this, we have to mention the following more presice expression of it, see [21]. We say it a satellite inequaqlity of (FI), simply (SF):

Theorem 1.5 (Satellite inequality (SF))**.** *If* $A \geq B > 0$*, then*

$$A^{-r} \ \sharp_{\frac{1+r}{p+r}} \ B^p \leq B \leq A \quad \textit{for} \quad p \geq 1 \ \textit{and} \ \ r \geq 0. \tag{1.2}$$

Proof. As the first stage, we assume that $0 \leq r \leq 1$. Then the monotonicity of $\sharp_\alpha$ $(\alpha \in [0,1])$ implies that

$$A^{-r} \ \sharp_{\frac{1+r}{p+r}} \ B^p \leq B^{-r} \ \sharp_{\frac{1+r}{p+r}} \ B^p = B.$$

Next we assume that for some $r > 0$,

$$A \geq B > 0 \ \Rightarrow \ A^{-r} \ \sharp_{\frac{1+r}{p+r}} \ B^p \leq B \leq A$$

holds for all $p \geq 1$. So we prove that it is true for $s = 1 + 2r$. Since $A \geq B > 0$ is assumed, we have

$$A^{-1} \ \sharp_{\frac{2}{p+1}} \ B^p \leq B,$$

so that

$$B_1 = (A^{\frac{1}{2}}B^pA^{\frac{1}{2}})^{\frac{2}{p+1}} \leq A^{\frac{1}{2}}BA^{\frac{1}{2}} \leq A^2 = A_1.$$

By the assumption, it follows that for $p_1 \geq 1$

$$A_1^{-r} \ \sharp_{\frac{1+r}{p_1+r}} \ B_1^p \leq B_1 \leq A^{\frac{1}{2}}BA^{\frac{1}{2}}.$$

Arranging this for $p_1 = \frac{p+1}{2}$, we have

$$A^{-2r} \ \sharp_{\frac{2(1+r)}{p+1+2r}} \ A^{\frac{1}{2}}B^pA^{\frac{1}{2}} \leq B_1 \leq A^{\frac{1}{2}}BA^{\frac{1}{2}}.$$

Furthermore multiplying $A^{-\frac{1}{2}}$ on both sides, it follows that for $s = 2r + 1$

$$A^{-s} \natural_{\frac{1+s}{p+s}} B^p \leq B,$$

as desired. $\square$

2. Ando–Hiai Inequality

Ando and Hiai [3] proposed a log-majorization inequality, whose essential part is the following operator inequality. We say it the Ando–Hiai inequality, simply (AH).

Theorem 2.1 (Ando–Hiai Inequality (AH))**.** *If $A \natural_\alpha B \leq I$ for $A, B > 0$, then $A^r \natural_\alpha B^r \leq I$ for $r \geq 1$.*

Proof. It suffices to show that $A^r \natural_\alpha B^r \leq I$ for $1 \leq r \leq 2$. Put $p = r - 1 \in [0, 1]$ and $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$. Then, since the assumption $A \natural_\alpha B \leq I$ is equivalent to $C^\alpha \leq A^{-1}$ and so $C^{-\alpha} \geq A$, it follows from Lemma 1.2 that

$$\begin{aligned} A^{-\frac{1}{2}} B^r A^{-\frac{1}{2}} &= A^{-\frac{1}{2}} (A^{\frac{1}{2}} C A^{\frac{1}{2}})^r A^{-\frac{1}{2}} = C^{\frac{1}{2}} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^p A^{-\frac{1}{2}} \\ &\leq C^{\frac{1}{2}} (C^{\frac{1}{2}} C^{-\alpha} C^{\frac{1}{2}})^p C^{\frac{1}{2}} = C^{1+(1-\alpha)p}. \end{aligned}$$

Hence we have

$$\begin{aligned} A^r \natural_\alpha B^r &= A^{\frac{1}{2}} (A^p \natural_\alpha A^{-\frac{1}{2}} B^r A^{-\frac{1}{2}}) A^{\frac{1}{2}} \leq A^{\frac{1}{2}} (C^{-\alpha p} \natural_\alpha C^{1+(1-\alpha)p}) A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} C^{(1+p)\alpha - \alpha p} A^{\frac{1}{2}} = A^{\frac{1}{2}} C^\alpha A^{\frac{1}{2}} \leq A^{\frac{1}{2}} A^{-1} A^{\frac{1}{2}} = I. \end{aligned}$$

$\square$

Based on an idea of Furuta inequality, we propose two variables version of Ando–Hiai inequality, see [11, 12]:

Theorem 2.2 (Generalized Ando–Hiai inequality (GAH))**.** *For $A,\ B > 0$ and $\alpha \in [0, 1]$, if $A \natural_\alpha B \leq I$, then*

$$A^r \natural_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} B^s \leq I \quad \text{for} \quad r, s \geq 1.$$

It is obvious that the case $r = s$ in Theorem 2.2 is just Ando–Hiai inequality. Now we consider two one-sided versions of Theorem 2.2:

Proposition 2.3. *For $A,\ B > 0$ and $\alpha \in [0,\ 1]$, if $A \natural_\alpha B \leq I$, then*

$$A^r \natural_{\frac{\alpha r}{\alpha r + 1 - \alpha}} B \leq I \quad \text{for} \quad r \geq 1.$$

Proposition 2.4. *For $A,\ B > 0$ and $\alpha \in [0,\ 1]$, if $A \natural_\alpha B \leq I$, then*

$$A \natural_{\frac{\alpha}{\alpha + (1-\alpha)s}} B^s \leq I \quad \text{for} \quad s \geq 1.$$

Next we investigate relations among them and Theorem 2.2.

Theorem 2.5. (1) *Propositions 2.3 and 2.4 are equivalent.*
(2) *Theorem 2.2 follows from Propositions 2.3 and 2.4*

Proof. (1) We first note the transposition formula $X \natural_\alpha Y = Y \natural_\beta X$ for $\beta = 1-\alpha$. Therefore Proposition 2.3 (for β) is rephrased as follows:

$$B \natural_\beta A \le I \quad \Rightarrow \quad B^s \natural_{\frac{\beta s}{\beta s+\alpha}} A \le I \quad \text{for} \quad s \ge 1.$$

Using the transposition formula again, it coincides with Proposition 2.4 because

$$1 - \frac{\beta s}{\beta s + \alpha} = \frac{\alpha}{\beta s + \alpha} = \frac{\alpha}{(1-\alpha)s + \alpha}.$$

(2) Suppose that $A \natural_\alpha B \le I$ and $r, s \ge 1$ are given. Then it follows from Proposition 2.3 that $A^r \natural_{\alpha_1} B \le I$ for $\alpha_1 = \frac{\alpha r}{\alpha r + 1 - \alpha}$. We next apply Proposition 2.4 to it, so that we have

$$1 \ge A^r \natural_{\frac{\alpha_1}{\alpha_1 + (1-\alpha_1)s}} B^s = A^r \natural_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} B^s,$$

as desired. □

We now point out that Proposition 2.3 is an equivalent expression of Furuta inequality of Ando–Hiai type:

Theorem 2.6. *Proposition 2.3 is equivalent to the Furuta inequality.*

Proof. For a given $p \ge 1$, we put $\alpha = \frac{1}{p}$. Then $A \ge B (\ge 0)$ if and only if

$$A^{-1} \natural_\alpha B_1 \le 1, \text{ for } B_1 = A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}}. \tag{2.1}$$

If $A \ge B > 0$, then (2.1) holds for $A,\ B > 0$, so that Proposition 2.3 implies that for any $r \ge 0$

$$1 \ge A^{-(r+1)} \natural_{\frac{\frac{r+1}{p}}{(1-\frac{1}{p}) + \frac{r+1}{p}}} B_1 = A^{-(r+1)} \natural_{\frac{1+r}{p+r}} B_1 = A^{-(r+1)} \natural_{\frac{1+r}{p+r}} A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}}.$$

Hence we have (FI);

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \le A.$$

Conversely suppose that (FI) is assumed. If $A^{-1} \natural_\alpha B_1 \le 1$, then $A \ge (A^{\frac{1}{2}} B_1 A^{\frac{1}{2}})^\alpha = B$, where $p = \frac{1}{\alpha}$. So (FI) implies that for $r_1 = r - 1 \ge 0$

$$A \ge A^{-r_1} \natural_{\frac{1+r_1}{p+r_1}} B^p = A^{-(r-1)} \natural_{\frac{r}{p+r-1}} A^{\frac{1}{2}} B_1 A^{\frac{1}{2}}.$$

Since $\frac{r}{p+r-1} = \frac{\alpha r}{1+\alpha r - \alpha}$, we have Proposition 2.3. □

As in the discussion as above, Theorem 2.2 can be proved by showing Proposition 2.3. Finally we cite its proof. Since it is equivalent to the Furuta inequality, we have an alternative proof of it. It is done by the usual induction, whose technical point is a multiplicative property of the index $\frac{\alpha r}{(1-\alpha)+\alpha r}$ of $\natural$ as appeared below.

Proof of Proposition 2.3. For convenience, we show that if $A^{-1} \natural_\alpha B \le I$, then

$$A^{-r} \natural_{\frac{\alpha r}{(1-\alpha)+\alpha r}} B \le I \quad \text{for} \quad r \ge 1. \tag{2.2}$$

Now the assumption says that

$$C^\alpha = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^\alpha \le A.$$

For any $\epsilon \in (0,1]$, we have $C^{\alpha\epsilon} \le A^\epsilon$ by the Löwner-Heinz inequality and so

$$\begin{aligned} A^{-(1+\epsilon)} \,\natural_{\frac{\alpha(1+\epsilon)}{(1-\alpha)+\alpha(1+\epsilon)}}\, B &= A^{-\frac{1}{2}}(A^{-\epsilon} \,\natural_{\frac{\alpha(1+\epsilon)}{1+\alpha\epsilon}}\, A^{\frac{1}{2}}BA^{\frac{1}{2}})A^{-\frac{1}{2}} \\ \le \ & A^{-\frac{1}{2}}(C^{-\alpha\epsilon} \,\natural_{\frac{\alpha(1+\epsilon)}{1+\alpha\epsilon}}\, C)A^{-\frac{1}{2}} = A^{-\frac{1}{2}}C^{\alpha}A^{-\frac{1}{2}} = A^{-1} \,\natural_\alpha\, B \le I. \end{aligned}$$

Hence we proved the conclusion (2.2) for $1 \le r \le 2$. So we next assume that (2.2) holds for $1 \le r \le 2^n$. Then the discussion of the first half ensures that

$$(A^{-r})^{r_1} \,\natural_{\frac{\alpha_1 r_1}{(1-\alpha_1)+\alpha_1 r_1}}\, B \le I \quad \text{for } 1 \le r_1 \le 2, \text{ where } \alpha_1 = \frac{\alpha r}{(1-\alpha)+\alpha r}.$$

Thus the multiplicative property of the index

$$\frac{\alpha_1 r_1}{(1-\alpha_1)+\alpha_1 r_1} = \frac{\alpha r r_1}{(1-\alpha)+\alpha r r_1}$$

shows that (2.2) holds for all $r \ge 1$. □

We here consider an expression of (AH)-type for satellite of (FI): Suppose that $A^{-1} \,\natural_\alpha\, B \le I$ and put $\alpha = \frac{1}{p}$. It is equivalent to $C = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{p}} \le A$. So (SF) says that

$$A^{-r} \,\natural_{\frac{1+r}{p+r}}\, C^p \le C, \text{ or } \quad A^{-(r+1)} \,\natural_{\frac{1+r}{p+r}}\, B \le A^{-\frac{1}{2}}CA^{-\frac{1}{2}} = A^{-1} \,\natural_{\frac{1}{p}}\, B.$$

Namely (SF) has an (AH)-type representation as follows:

Theorem 2.7. *Let A and B be positive invertible operators. Then*

$$A \,\natural_\alpha\, B \le I \ \Rightarrow\ A^r \,\natural_{\frac{\alpha r}{\alpha r+1-\alpha}}\, B \le A \,\natural_\alpha\, B \ (\le I) \ \text{ for } \ r \ge 1.$$

3. Grand Furuta Inequality

To compare with (AH) and (FI), (AH) is arranged as a Furuta type operator inequality. As in the proof of (AH), its assumption is that

$$B_1 = C^\alpha = (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha \le A^{-1} = A_1.$$

Replacing $p = \alpha^{-1}$, it is reformulated that

$$A \ge B > 0 \implies A^r \ge (A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^rA^{\frac{r}{2}})^{\frac{1}{p}} \qquad (\dagger)$$

for $r \ge 1$ and $p \ge 1$.

Moreover, to make a simultaneous extension of both (FI) and (AH), Furuta added variables as in the case of (FI). Actually he paid his attention to $A^{-\frac{1}{2}}$ in (†), presicely, he replaced it to $A^{-\frac{t}{2}}$ ($t \in [0,1]$). Consequently he established so-called grand Furuta inequality, simply (GFI). It is sometimes said to be generalized Furuta inequality. We refer [17, 18, 10, 13, 26, 28].

Theorem 3.1 (Grand Furuta inequality (GFI))**.** *If $A \ge B > 0$ and $t \in [0,1]$, then*

$$[A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \le A^{1-t+r}$$

holds for $r \ge t$ and $p, s \ge 1$.

It is easily seen that

$$\begin{aligned}&\text{(GFI) for } t=1,\ r=s \iff \text{(AH)}\\&\text{(GFI) for } t=0,\ (s=1) \iff \text{(FI)}.\end{aligned}$$

Proof of (GFI). We prove it by the induction on s. For this, we first prove it for $1 \le s \le 2$: Since $(X^*C^2X)^s = X^*C(CXX^*C)^{s-1}CX$ for arbitrary X, $C \ge 0$ and $0 \le s-1 \le 1$, (LH) implies that

$$A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2} = A^{\frac{r-t}{2}}B^{\frac{p}{2}}(B^{\frac{p}{2}}A^{-t}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}A^{\frac{r-t}{2}}$$

$$\le A^{\frac{r-t}{2}}B^{\frac{p}{2}}(B^{\frac{p}{2}}B^{-t}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}A^{\frac{r-t}{2}} = A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}}.$$

Furthermore it follows from (LH) and (FI) that

$$\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}} \le \{A^{\frac{r-t}{2}}B^{(p-t)s+t}A^{\frac{r-t}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \le A^{1-t+r}$$

by noting that $(p-t)s+t+(r-t) = (p-t)s+r$. Hence (GFI) is proved for $1 \le s \le 2$.

Next, under the assumption (GFI) holds for some $s \ge 1$, we now prove that (GFI) holds for $s+1$. Since (GFI) holds for s, we take $r=t$ in it. Thus we have

$$A \ge \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{\frac{1}{(p-t)s+t}}.$$

Put $C = \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{\frac{1}{(p-t)s+t}}$, that is, $A \ge C$. By using that $s \ge 1$ if and only if $1 \le \frac{s+1}{s} \le 2$ and that (GFI) for $1 \le s \le 2$ has been proved, we obtain that

$$\begin{aligned}A^{1-t+r} &\ge \{A^{r/2}(A^{-t/2}C^{(p-t)s+t}A^{-t/2})^{\frac{s+1}{s}}A^{r/2}\}^{\frac{1-t+r}{\{(p-t)s+t-t\}(\frac{s+1}{s})+r}}\\&= \{A^{r/2}(A^{-t/2}C^{(p-t)s+t}A^{-t/2})^{\frac{s+1}{s}}A^{r/2}\}^{\frac{1-t+r}{(p-t)(s+1)+r}}\\&= \{A^{r/2}(A^{-t/2}\{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}A^{-t/2})^{\frac{s+1}{s}}A^{r/2}\}^{\frac{1-t+r}{(p-t)(s+1)+r}}\\&= \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s+1}A^{r/2}\}^{\frac{1-t+r}{(p-t)(s+1)+r}}.\end{aligned}$$

This means that (GFI) holds for $s+1$, and so the proof is complete. □

Next we point out that (GFI) for $t=1$ includes both Ando–Hiai and Furuta inequalities.

Since Ando–Hiai inequality is just (GFI; $t=1$) for $r=s$, it suffices to check that Furuta inequality is contained in (GFI; $t=1$). As a matter of fact, it is just (GFI; $t=1$) for $s=1$.

Theorem 3.2. *Furuta inequality* (FI) *is equivalent to* (GFI) *for* $t=s=1$.

Proof. We write down (GFI; $t=1$) for $s=1$: If $A \ge B > 0$, then

$$[A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})A^{\frac{r}{2}}]^{\frac{r}{p-1+r}} \le A^r$$

for $p, r \ge 1$, or equivalently,

$$A^{-(r-1)} \natural_{\frac{r}{p-1+r}} B^p \le A$$

for $p, r \geq 1$. Replacing $r-1$ by r_1, (GFI; $t=1$) for $s=1$ is rephrased as follows: If $A \geq B > 0$, then

$$A^{-r_1} \sharp_{\frac{1+r_1}{p+r_1}} B^p \leq A$$

for $p \geq 1$ and $r_1 \geq 0$, which is nothing but Furuta inequality. □

Furthermore Theorem 2.2, generalized Ando–Hiai inequality, is understood as the case $t=1$ in (GFI):

Theorem 3.3. (GFI; $t=1$) *is equivalent to Theorem* 2.2 (GAH).

Proof. (GFI; $t=1$) is written as

$$A \geq B > 0 \ \Rightarrow\ [A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^sA^{\frac{r}{2}}]^{\frac{r}{(p-1)s+r}} \leq A^r \quad (p, r, s \geq 1).$$

We here put

$$\alpha = \frac{1}{p}, \quad B_1 = A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}}.$$

Then we have

$$A \geq B > 0 \iff A^{-1} \sharp_{\frac{1}{p}} A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}} \leq 1 \iff A^{-1} \sharp_{\alpha} B_1 \leq 1$$

and for each $p, r, s \geq 1$

$$\begin{aligned} &[A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^sA^{\frac{r}{2}}]^{\frac{r}{(p-1)s+r}} \leq A^r \\ \iff\ & A^{-r} \sharp_{\frac{r}{(p-1)s+r}} (A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^s \leq 1 \\ \iff\ & A^{-r} \sharp_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} B_1^s \leq 1. \end{aligned}$$

This shows the statement of Theorem 2.2 (GAH). □

Next we consider some variants of (GFI), which are useful in the discussion of Kantorovich type inequalities, see [7].

Theorem 3.4. *If $A \geq B \geq 0$, then*

$$A^{\frac{(p+t)s+r}{q}} \geq (A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for all $p,\ t,\ s,\ r \geq 0$ and $q \geq 1$ with $(p+t+r)q \geq (p+t)s+r$ and $(1+t+r)q \geq (p+t)s+r$.

Proof. First of all, we may assume $p > 0$. Now Furuta inequality says that

$$A_1 = A^{\frac{p+t}{q_1}} \geq B_1 = (A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^{\frac{1}{q_1}}$$

holds for $t \geq 0$, where $q_1 = \max\{1, \frac{p+t}{1+t}\}$. Applying Furuta inequality again, we have

$$A_1^{\frac{p_1+r_1}{q}} \geq (A_1^{\frac{r_1}{2}}B_1^{p_1}A_1^{\frac{r_1}{2}})^{\frac{1}{q}},$$

that is,

$$A^{\frac{(p+t)(p_1+r_1)}{qq_1}} \geq (A^{\frac{(p+t)r_1}{2q_1}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^{\frac{p_1}{q_1}}A^{\frac{(p+t)r_1}{2q_1}})^{\frac{1}{q}},$$

for all $p_1,\ r_1 \geq 0$ and $q \geq 1$ with $(1+r_1)q \geq p_1 + r_1$. So we take $p_1 = sq_1$ and $r_1 = \frac{rq_1}{p+t}$. Since $(1+r_1)q \geq p_1 + r_1$ is equivalent to the condition that $(p+t+r)q \geq (p+t)s + r$ and $(1+t+r)q \geq (p+t)s + r$, the statement is proved. □

In the remainder, we reconsider (GFI). For this, we cite it by the use of operator means. For convenience, we use the notation $\natural_s$ for the binary operation

$$A \natural_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}} \quad \text{for} \quad s \notin [0,1],$$

whose formula is the same as $\sharp_s$.

Grand Furuta inequality (GFI).

$$A \geq B > 0, t \in [0,1] \Rightarrow A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A \ (r \geq t;\ p, s \geq 1)$$

This mean theoretic expression of (GFI) induces the following improvement of it.

Satellite of Grand Furuta inequality (SGF).

$$A \geq B > 0, t \in [0,1] \Rightarrow A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq B \ (r \geq t;\ p, s \geq 1)$$

We here clarify that the case $t = 1$ is essential in (GFI), in which (SGF) is quite meaningful. As a matter of fact, we prove that (SGF; t=1) implies (SGF) for every $t \in [0,1]$.

For readers' convenience, we prove (SGF). For this, the following lemma is needed, which is a variational expression of (LH):

Lemma 3.5. *If $A \geq B > 0$, $t \in [0,1]$ and $1 \leq s \leq 2$, then*

$$A^t \natural_s C \leq B^t \natural_s C$$

holds for arbitrary $C > 0$, in particular,

$$A^t \natural_s B^p \leq B^{(p-t)s+t}$$

holds for $p \geq 1$.

Proof. Since $A^{-t} \leq B^{-t}$ by (LH), we have

$$A^t \natural_s C = C(C^{-1} \#_{s-1} A^{-t})C \leq C(C^{-1} \#_{s-1} B^{-t})C = B^t \natural_s C.$$

Similarly we have

$$A^t \natural_s B^p = B^p(B^{-p} \#_{s-1} A^{-t})B^p \leq B^p(B^{-p} \#_{s-1} B^{-t})B^p = B^{(p-t)s+t}.$$

□

We here give a short comment on the first statement in the above lemma: Suppose that if $A \geq B > 0$ and $t \in [0,1]$, then

$$A^t \natural_s C \leq B^t \natural_s C$$

holds for arbitrary $C > 0$ and $1 \le s \le 2$. Then taking $C = B^t$ and $s = 2$, we have

$$A^t \natural_2 B^t \le B^t \natural_2 B^t = B^t,$$

so that $B^t A^{-t} B^t \le B^t$, or $A^t \ge B^t$. That is, it is equivalent to (LH).

More generally, we know the following fact:

Lemma 3.6. *If $A \ge B > 0$ and $t \in [0,1]$, then*

$$(A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \le B \le A$$

holds for $p, s \ge 1$.

Proof. We fix $p \ge 1$ and $t \in [0,1]$. It follows from Lemma 3.5 and (LH) that

$$(\dagger) \qquad A \ge B > 0 \ \Rightarrow \ B_1 = (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \le B \le A$$

for $s \in [1,2]$. So we assume that (†) holds for some $s \ge 1$, and prove that

$$B_2 = (A^t \natural_{2s} B^p)^{\frac{1}{2(p-t)s+t}} \le B_1 \le B.$$

Actually we apply (†) to $B_1 \le A$. Then we have

$$(A^t \natural_2 B_1^{p_1})^{\frac{1}{2(p_1-t)+t}} \le B_1 \le B, \ \text{where } p_1 = (p-t)s + t,$$

and moreover

$$(A^t \natural_2 B_1^{p_1})^{\frac{1}{2(p_1-t)+t}} = [A^t \natural_2 (A^t \natural_s B^p)]^{\frac{1}{(p-t)2s+t}} = (A^t \natural_{2s} B^p)^{\frac{1}{(p-t)2s+t}} = B_2,$$

which completes the proof. □

Under this preparation, we can easily prove (SGF) by virtue of (SF) in Theorem 1.5.

Proof of (SGF). For given p, t, s, we use the same notation as above; $p_1 = (p-t)s + t$ and $B_1 = (A^t \natural_s B^p)^{\frac{1}{p_1}}$. Then Lemma 3.6 implies that $B_1 \le B \le A$. Hence it follows from (SF) for $B_1 \le A$ and $r_1 = r - t$ that

$$A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) = A^{r_1} \#_{\frac{1+r_1}{p_1+r_1}} B_1^{p_1} \le B_1 \le B.$$

□

It is shown that (SGF; $t = 1$) is essential among (SGF; $t \in [0,1]$), in which (LH) completely works. That is,

Theorem 3.7. (SGF; $t = 1$) *implies* (SGF; t) *for* $t \in [0,1]$.

Proof. Suppose that for $A \ge B > 0$,

$$A^{-r+1} \#_{\frac{r}{(p-1)s+r}} (A \natural_s B^p) \le B$$

holds for $r \ge 1$.

We fix arbitrary $t \in (0,1)$. As $A^t \geq B^t$ by (LH), we have

$$(A^t)^{-\frac{r}{t}+1} \#_{\frac{\frac{r}{t}}{(\frac{p}{t}-1)s+\frac{r}{t}}} (A^t \natural_s B^p) \leq B^t$$

for $r \geq t$. It is arranged as

$$A^{-r+t} \#_{\frac{r}{(p-t)s+r}} (A^t \natural_s B^p) \leq B^t,$$

or equivalently,

$$(A^t \natural_s B^p) \#_{\frac{(p-t)s}{(p-t)s+r}} A^{-r+t} \leq B^t.$$

Therefore it follows from Lemma 3.5 that for $s \in [1,2]$

$$\begin{aligned} A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) &= (A^t \natural_s B^p) \#_{\frac{(p-t)s-(1-t)}{(p-t)s+r}} A^{-r+t} \\ &= (A^t \natural_s B^p) \#_{\frac{(p-t)s-(1-t)}{(p-t)s}} \{(A^t \natural_s B^p) \#_{\frac{(p-t)s}{(p-t)s+r}} A^{-r+t}\} \\ &\leq (A^t \natural_s B^p) \#_{\frac{(p-t)s-(1-t)}{(p-t)s}} B^t \\ &= B^t \#_{\frac{1-t}{(p-t)s}} (A^t \natural_s B^p) \\ &\leq B^t \#_{\frac{1-t}{p-t}} B^{(p-t)s+t} = B. \end{aligned}$$

Namely we have

$$(**) \qquad A \geq B > 0 \Rightarrow A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq B$$

for $1 \leq s \leq 2,\ r \geq t$ and $p \geq 1$.

Next we assume that $(**)$ holds for some $s \geq 1$. Then taking $r = t$, we have

$$B \geq (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}}.$$

Put $C = (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}}$, that is, $(A \geq) B \geq C$. By $(**)$ for $\frac{s+1}{s} \in [1,2]$ and $p_1 = (p-t)s+t$, we obtain

$$\begin{aligned} C &\geq A^{-r+t} \#_{\frac{1-t+r}{((p-t)s+t-t)(\frac{s+1}{s})+r}} (A^t \natural_{\frac{s+1}{s}} C^{(p-t)s+t}) \\ &= A^{-r+t} \#_{\frac{1-t+r}{(p-t)(s+1)+r}} (A^t \natural_{\frac{s+1}{s}} (A^t \natural_s B^p)) \\ &= A^{-r+t} \#_{\frac{1-t+r}{(p-t)(s+1)+r}} (A^t \natural_{s+1} B^p). \end{aligned}$$

Hence we have

$$A^{-r+t} \#_{\frac{1-t+r}{(p-t)(s+1)+r}} (A^t \natural_{s+1} B^p) \leq C \leq B.$$

□

Remark 3.8. (GFI; t=1) implies a variant of (GFI) that

$$\begin{aligned} &A \geq B > 0,\ t \in [0,1] \\ &\quad \Rightarrow\ A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^t \#_{\frac{1-t}{p-t}} B^p \quad (r \geq t;\ p, s \geq 1) \end{aligned}$$

We here note: (1) The case $t = 0$ and $s = 1$ is just

$$(SF) \qquad A \geq B > 0\ \Rightarrow\ A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq B \quad (p \geq 1,\ r \geq 0).$$

(2) The case $t = 1$ and $r = s$ is Ando–Hiai inequality;

$$(AH) \qquad X \#_\alpha Y \le 1 \;\Rightarrow\; X^r \#_\alpha Y^r \le 1 \;\; (r \ge 1).$$

(Repalce $X = A^{-1},\ Y = A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}}$ and $\alpha = \frac{1}{p}$.)

However, it easily follows from (SGF) because

$$A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \le B = B^t \#_{\frac{1-t}{p-t}} B^p \le A^t \#_{\frac{1-t}{p-t}} B^p$$

under the same condition as in the above.

4. Chaotic order

We first remark that $\log x$ is operator monotone, i.e., $A \ge B > 0$ implies $\log A \ge \log B$ by (LH) and $\frac{X^p - 1}{p} \to \log X$ for $X > 0$. By this fact, we can introduce the chaotic order as $\log A \ge \log B$ among positive invertible operators, which is weaker than the usual order $A \ge B$. In this section, we consider Furuta inequality under the chaotic order. We refer [2, 6, 16, 27, 8, 9, 29].

Theorem 4.1. *The following assertions are mutually equivalent for $A,\ B > 0$:*

(i) $A \gg B$, *i.e.,* $\log A \ge \log B$,

(ii) $A^p \ge (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ *for* $p \ge 0$,

(iii) $A^r \ge (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for* $p,\ r \ge 0$.

Proof. We prove the implications: (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii): First we note that $(1 + \frac{\log X}{n})^n \longrightarrow X$ for $X > 0$. Since

$$A_n = 1 + \frac{\log A}{n} \ge B_n = 1 + \frac{\log B}{n} > 0$$

for sufficiently large n, Furuta inequality ensures that for given $p,\ r > 0$

$$A_n{}^{1+nr} \ge (A_n{}^{\frac{nr}{2}} B_n{}^{np} A_n{}^{\frac{nr}{2}})^{\frac{1+nr}{n(p+r)}},$$

or equivalently

$$A_n{}^{n(\frac{1}{n}+r)} \ge (A_n{}^{n\frac{r}{2}} B_n{}^{np} A_n{}^{n\frac{r}{2}})^{\frac{1}{n(p+r)} + \frac{r}{p+r}}.$$

Taking $n \to \infty$, we have the desired inequality (iii).

(iii) $\Rightarrow$ (ii) is trivial by setting $r = p$.

(ii) $\Rightarrow$ (i): Note that $\frac{X^p - 1}{p} \to \log X$ for $X > 0$. The assumption (ii) implies that

$$\frac{A^p - 1}{p} \ge \frac{(A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}} - 1}{p} = \frac{A^{\frac{p}{2}} B^p A^{\frac{p}{2}} - 1}{p((A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}} + 1)} = \frac{A^{\frac{p}{2}} (B^p - 1) A^{\frac{p}{2}} + A^p - 1}{p((A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}} + 1)}.$$

Taking $p \to +0$, we have

$$\log A \ge \frac{\log B + \log A}{2}, \text{ that is, } \log A \ge \log B.$$

So the proof is complete. $\square$

Remark 4.2. The order preserving operator inequality (i) $\Rightarrow$ (iii) in above is called chaotic Furuta inequality, simply (CFI). We here note that (iii) $\Rightarrow$ (i) is directly proved as follows:

Take the logarithm on both side of (iii), that is,

$$r \log A \geq \frac{r}{p+r} \log A^{\frac{r}{2}} B^p A^{\frac{r}{2}}$$

for $p,\ r \geq 0$. Therefore we have

$$\log A \geq \frac{1}{p+r} \log A^{\frac{r}{2}} B^p A^{\frac{r}{2}}.$$

So we put $r = 0$ in above. Namely it implies that

$$\log A \geq \frac{1}{p} \log B^p = \log B.$$

As in chaotic Furuta inequality, Theorem 3.4 has the following chaotic order version:

Theorem 4.3. *If* $\log A \geq \log B$ *for* $A,\ B > 0$*, then*

$$A^{\frac{(p+t)s+r}{q}} \geq (A^{\frac{r}{2}}(A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s A^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for all $p,\ t,\ s,\ r \geq 0$ *and* $q \geq 1$ *with* $(t+r)q \geq (p+t)s + r$.

Proof. As in the proof of chaotic Furuta inequality (i) $\Rightarrow$ (iii), we have

$$A_n = 1 + \frac{\log A}{n} \geq B_n = 1 + \frac{\log B}{n} > 0$$

for sufficiently large n. Thus Theorem implies that

$$A_n^{\frac{(p_1+t_1)s+r_1}{q}} \geq (A_n^{\frac{r_1}{2}}(A_n^{\frac{t_1}{2}} B_n^{p_1} A_n^{\frac{t_1}{2}})^s A_n^{\frac{r_1}{2}})^{\frac{1}{q}}$$

holds for all $p_1,\ t_1,\ s,\ r_1 \geq 0$ and $q \geq 1$ with $(t_1+r_1)q \geq (p_1+t_1)s + r_1$. Putting $p_1 = np$, $t_1 = nt$ and $r_1 = nr$, we have

$$A_n^{\frac{n((p+t)s+r}{q}} \geq (A_n^{\frac{nr}{2}}(A_n^{\frac{nt}{2}} B_n^{np} A_n^{\frac{nt}{2}})^s A_n^{\frac{nr}{2}})^{\frac{1}{q}}$$

for all $p,\ t,\ s,\ r \geq 0$ and $q \geq 1$ with $(t+r)q \geq (p+t)s+r$. Finally, since $A_n{}^n \longrightarrow A$ and $B_n{}^n \longrightarrow B$, we have the desired inequality by tending $n \to \infty$. $\square$

The chaotic Furuta inequality (CFI), Theorem 4.1 (iii), is expressed in terms of weighted geometric mean as well as Furuta inequality (FI) as follows:

$$A \geq B > 0 \implies A^{-r} \ \sharp_{\frac{r}{p+r}} \ B^p \leq I \tag{CFI}$$

holds for $p \geq 0$ and $r \geq 0$.

For the sake of convenience, we cite (AH): For $\alpha \in (0,1)$

$$A \ \sharp_\alpha \ B \leq I \implies A^r \ \sharp_\alpha \ B^r \leq I \tag{AH}$$

holds for $r \geq 1$.

Theorem 4.4. *The operator inequalities* (FI), (CFI) *and* (AH) *are mutually equivalent:*

Proof. Suppose that (CFI) holds. Then we prove (FI), so that we assume $A \geq B > 0$. We have

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p = B^p \natural_{\frac{p-1}{p+r}} A^{-r} = B^p \natural_{\frac{p-1}{p}} (B^p \natural_{\frac{p}{p+r}} A^{-r})$$
$$= B^p \natural_{\frac{p-1}{p}} (A^{-r} \natural_{\frac{r}{p+r}} B^p) \leq B^p \natural_{\frac{p-1}{p}} I = B \leq A,$$

which means that (FI) is shown.

Next we suppose that (FI) holds. Then we prove (AH), so that we assume $A \natural_\alpha B \leq I$ and $r \geq 0$. Then, putting $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and $p = \frac{1}{\alpha} > 1$, we have

$$B_1 = (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha = C^{\frac{1}{p}} \leq A^{-1} = A_1.$$

Applying (FI) to $A_1 \geq B_1$, it follows that for $p \geq 1$,

$$A_1^{-r} \natural_{\frac{1+r}{p+r}} B_1^p \leq B_1 \leq A_1.$$

Summing up the above discussion, for each $p > 1$,

$$A \natural_{\frac{1}{p}} B \leq I \;\Rightarrow\; A^r \natural_{\frac{1+r}{p+r}} A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq A^{-1}, \text{ or } A^{r+1} \natural_{\frac{1+r}{p+r}} B \leq I \text{ for } r \geq 0.$$

Note that

$$B \natural_{\frac{p-1}{p+r}} A^{r+1} = A^{r+1} \natural_{\frac{1+r}{p+r}} B \leq I$$

holds. That is, we can assume this and so apply it for $q = \frac{p+r}{p-1} \geq 1$. Hence it implies that

$$I \geq B^{r+1} \natural_{\frac{1+r}{q+r}} A^{r+1}.$$

Since $1 - \frac{1+r}{p_1+r} = \frac{1}{p}$,

$$I \geq B^{r+1} \natural_{\frac{1+r}{q+r}} A^{r+1} = A^{r+1} \natural_{\frac{1}{p}} B^{r+1}.$$

Namely we obtain (AH).

Finally we prove (AH) $\Rightarrow$ (CFI). So we assume that $A \geq B > 0$ and $p, r > 1$ because it holds for $0 \leq p, r \leq 1$ by (LH). For given $p, r > 1$, we put $\alpha = \frac{r}{p+r}$ and $r_1 = \frac{r}{p}$. Then we have

$$A^{-r_1} \natural_{\frac{r_1}{1+r_1}} B \leq A^{-r_1} \natural_{\frac{r_1}{1+r_1}} A = I.$$

We here apply (AH) to this and so we have

$$I \geq A^{-r_1 p} \natural_{\frac{r_1 p}{p+r_1 p}} B^p = A^{-r} \natural_{\frac{r}{p+r}} B^p,$$

as desired. □

We here present an interesting characterization of chaotic order.

Theorem 4.5. *The following assertions are equivalent for* $A,\ B > 0$*:*
(i) $\log A \geq \log B$,
(ii) *For each* $\delta > 0$ *there exists an* $\alpha = \alpha_\delta > 0$ *such that* $(e^\delta A)^\alpha > B^\alpha$.

The proof of Theorem 4.5 is not written, but its essence is shown as follows:

Theorem 4.6. *If $\log A > \log B$ for $A,\ B > 0$, then there exists an $\alpha > 0$ such that $A^\alpha > B^\alpha$.*

Proof. Since $\log A - \log B \geq 2s > 0$ for some $s > 0$, there exists an $\alpha > 0$ such that

$$\|\frac{x^h - 1}{h} - \log x\|_I < s$$

for $0 < h \leq \alpha$, where I is a bouded interval including the spectra of A and B. Hence we have

$$0 \leq \frac{A^\alpha - 1}{\alpha} - \log A \leq s, \quad 0 \leq \frac{B^\alpha - 1}{\alpha} - \log B \leq s,$$

so that

$$\begin{aligned}\frac{A^\alpha - B^\alpha}{\alpha} &= \left(\frac{A^\alpha - 1}{\alpha} - \log A\right) + \log A - \log B - \left(\frac{B^\alpha - 1}{\alpha} - \log B\right) \\ &\geq \log A - \log B - \left(\frac{B^\alpha - 1}{\alpha} - \log B\right) \\ &\geq \log A - \log B - \|\frac{B^\alpha - 1}{\alpha} - \log B\|_I \\ &\geq 2s - s = s,\end{aligned}$$

that is, $A^\alpha - B^\alpha \geq \alpha s > 0$ is shown. □

Related to this, there raises the problem: Does $\log A \geq \log B$ imply that there exists an $\alpha > 0$ such that $A^\alpha \geq B^\alpha$?

Example 4.7. Take A and B as follows:

$$A = U\begin{pmatrix} e^4 & 0 \\ 0 & e^{-1}\end{pmatrix}U; \ U = \frac{1}{\sqrt{5}}\begin{pmatrix}\sqrt{3} & \sqrt{2} \\ \sqrt{2} & -\sqrt{3}\end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2}\end{pmatrix}.$$

Then we have

$$\log A = \begin{pmatrix}\sqrt{2} & \sqrt{6} \\ \sqrt{6} & 1\end{pmatrix} \text{ and } \log B = \begin{pmatrix} 0 & 0 \\ 0 & -2\end{pmatrix},$$

so that $\log A \geq \log B$ is easily checked.

On the other hand, putting $x = e^\alpha$ for $\alpha > 0$,

$$\det(A^\alpha - B^\alpha) = -x^{-3}(x+1)(x-1)^4(2x^2 + x + 2) < 0$$

for all $x > 1$. Hence $A^\alpha \geq B^\alpha$ does not hold for any $\alpha > 0$.

Concluding this section, we mention some operator inequalities related to (CFI).

Theorem 4.8. *Let A and B be positive invertible operators. Then the following statements are mutually equivalent:*

(1) $\log A \leq \log B$,

(2) $A^{-r} \natural_{\frac{r}{p+r}} B^p \geq 1$ *for* $p,\ r \geq 0$.

(3) $A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \geq B^\delta$ *for* $p,\ r \geq 0$ *and* $0 \leq \delta \leq p$.

(4) *The operator function* $f(p) = A^{-r} \natural_{\frac{r}{p+r}} B^p$ *is increasing on* p.

Proof. (1) ⇔ (2): It follows from (i) ⇔ (iii) in Theorem 4.1.

(2) ⇒ (3): By using Lemma 1.4, we have

$$\begin{aligned} A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p &= B^p \natural_{\frac{p-\delta}{p+r}} A^{-r} = B^p \natural_{\frac{p-\delta}{p}} (B^p \natural_{\frac{p}{p+r}} A^{-r}) \\ =& B^p \natural_{\frac{p-\delta}{p}} (A^{-r} \natural_{\frac{r}{p+r}} B^p) \le B^p \natural_{\frac{p-\delta}{p}} I = B^\delta. \end{aligned}$$

(3) ⇒ (2): It is trivial by putting $\delta = 0$.

(3) ⇒ (4): By using Lemma 1.4 (iv), we have

$$\begin{aligned} f(p+\epsilon) &= A^{-r} \natural_{\frac{r}{p+\epsilon+r}} B^{p+\epsilon} \\ &= A^{-r} \natural_{\frac{r}{p+r}} (A^{-r} \natural_{\frac{p+r}{p+\epsilon+r}} B^{p+\epsilon}) \\ &\ge A^{-r} \natural_{\frac{r}{p+r}} B^p = f(p). \end{aligned}$$

(4) ⇒ (2): It is obtained by $f(p) \ge f(0) = 1$. □

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