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FIXED POINTS OF (ψ,ϕ)-WEAK CONTRACTIONS IN CONE METRIC SPACES

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ABSTRACT. In this paper we have established the fixed point theorem of self maps for (ψ,ϕ)-weak contractions in cone metric spaces. Also our result is supported by an example.

1. INTRODUCTION AND PRELIMINARIES

In 1997 Alber and Guerre-Delabriere [4] introduced the notion of the weak contraction. They proved the existence of fixed points for single-valued maps satisfying weak contractive condition on Hilbert spaces. Rhoades [23] showed that most results of [4] are still true for any Banach space. The weak contraction was defined as follows

Definition 1.1. A mapping $T : X \to X$, where (X, d) is a metric space, is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$$

where $x, y \in X$ and $\phi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

In fact Banach contraction is a special case of weak contraction by taking $\phi(t) = (1 - k)t$ for $0 \leq k < 1$. In this connection Rhoades [23] proved the following very interesting fixed point theorem.

Theorem 1.2. [23] *Let (X, d) be a complete metric space, and let A be a ϕ-weak contraction on X. If $\phi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing*

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function with $\phi(t) > 0$ for all $t \in (0, \infty)$ and $\phi(0) = 0$, then A has a unique fixed point.

We seen some common fixed point theorems in [9, 25, 26, 27, 5, 21] and number of hybrid contractive mapping results in [4, 24, 14, 28]. Recently Dutta and Chaudhary [12] generalized weak contraction by using concept of alternating distance and proved existence and uniqueness of the fixed points.

Huang and Zhang [19] generalized the notion of metric spaces by replacing the real numbers by ordered Banach space and define cone metric spaces. They have proved the Banach contraction mapping theorem and some other fixed point theorems of contractive type mappings in cone metric spaces. Subsequently, Rezapour and Hamlbarani[29], Ilic and Rakocević [15, 16] studied fixed point theorems for contractive type mappings in cone metric spaces. In this paper we proved some fixed point theorems for expansion mappings in complete cone metric spaces.

Let E be a real Banach space and P a subset of E. P is called a cone if and only if:
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii)$ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$; we shall write $x \ll y$ if $y - x \in intP$, where int$P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|. \tag{1.1}$$

The least positive number K satisfying the above is called the normal constant of P, see [19]. In [29] the authors showed that there are no normal cones with normal constant $M < 1$ and for each $k > 1$ there are cones with normal constant $M > k$. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

The cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Lemma 1.3. [29] *Every regular cone is normal.*

In the following we always suppose that E is a real Banach space, P is a cone in E with int$P \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 1.4. Let X be a non-empty set and $d : X \times X \to E$ a mapping such that
$(d_1)0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
$(d_2)d(x, y) = d(y, x)$ for all $x, y \in X$,
$(d_3)d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space [19].

Example 1.5. Let $E = R^2, P = \{(x, y) \in E : x, y \geq 0\}, X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space [19].

Definition 1.6. (See [19]) Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then
$(i)\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
$(ii)\{x_n\}_{n\geq 1}$ is said to be a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
$(iii)(X, d)$ is called a complete cone metric space if every Cauchy sequence in X is convergent.

Lemma 1.7. [16]. *If P is a normal cone in E, then*
(i) If $0 \leq x \leq y$ and $a > 0$, where a is real number, then $0 \leq ax \leq ay$.
(ii) If $0 \leq x_n \leq y_n$, for $n \in N$ and $\lim_n x_n = x, lim_n y_n = y$, then $0 \leq x \leq y$.

Lemma 1.8. [18]. *If E be a real Banach space with cone P in E, then for $a, b, c \in E$*
(i) If $a \leq b$ and $b \ll c$, then $a \ll c$.
(ii) If $a \ll b$ and $b \ll c$, then $a \ll c$.

Definition 1.9. [17]. Let $(Y, \leq)$ be a partially ordered set. Then a function $F : Y \to Y$ is said to be monotone increasing if it preserves ordering, i.e., given $x, y \in Y$, $x \leq y$ implies that $F(x) \leq F(y)$.

Let $f, g : X \to X$ be mappings with $f(X) \subset g(X)$. Let $x_0 \in X$ be arbitrary. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$. This can be done since $f(X) \subset g(X)$. Continuing this process, having chosen $x_n \in X$, we choose $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$ for all $n \in N$. $(f(x_n))$ is called an f-g-sequence with initial point x_0.

Definition 1.10. [13] Let $f, g : X \to X$ be mappings. If $y = f(z) = g(z)$ for some $z \in X$, then z is called a coincidence point of f and g, and y is called a point of coincidence of f and g.

Definition 1.11. [13] The mappings $f, g : X \to X$ are weakly compatible if, for every $x \in X$, holds: $f(g(x)) = g(f(x))$ whenever $f(x) = g(x)$.

Lemma 1.12. [3] *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Recently Choudhury and Metiya [11] established following result,

Theorem 1.13. *Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in intP$, for $x, y \in X$ with $x \neq y$. Let $T : X \to X$ be a mapping satisfying the inequality*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \text{ for } x, y \in X$$

where $\phi : intP \cup \{0\} \to intP \cup \{0\}$ *is a continuous and monotone increasing function with*
(i)$\phi(t) = 0$ *if and only if* $t = 0$*;*
(ii)$\phi(t) \ll t$*, for* $t \in intP$*;*
(iii) either $\phi(t) \leq d(x, y)$ *or* $d(x, y) \leq \phi(t)$*, for* $t \in intP \cup \{0\}$ *and* $x, y \in X$*. Then* T *has a unique fixed point in* X*.*

In this paper we generalize above theorem, for this, we need following definition

Definition 1.14. Let $\psi, \phi : IntP \cup \{0\} \to IntP \cup \{0\}$ be two continuous and monotone increasing functions satisfying
(a) $\psi(t) = \phi(t) = 0$ if and only if $t = 0$,
(b) $t - \psi(t) \in P \cup \{0\}, \phi(t) \ll t$, for $t \in intP$.

2. Main results

Theorem 2.1. *Let* (X, d) *be a complete cone metric space with regular cone* P *such that* $d(x, y) \in intP$*, for* $x, y \in X$ *with* $x \neq y$*. Let* $T : X \to X$ *be a mapping satisfying the inequality*

$$\psi((Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) \text{ for } x, y \in X$$

where ψ, ϕ *are defined in Definition* 1.11 *and* ψ *satisfies*
(i)$\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$ *for* $t_1, t_2 \in IntP$*;*
(ii) either $\psi(t), \phi(t) \leq d(x, y)$ *or* $d(x, y) \leq \psi(t), \phi(t)$*, for* $t \in intP \cup \{0\}$ *and* $x, y \in X$*. Then* T *has a unique fixed point in* X*.*

Proof. Let $x_0 \in X$, then $Tx_0 = x_1$, in this way we obtain a sequence $\{x_n\}$ such that $Tx_n = x_{n+1}$ for all $n \geq 0$. If for some $x_n = x_{n+1}$, then x_n is fixed point of T. Now we assume that $x_n \neq x_{n+1}$ for $n \in N$. By the given condition we have,

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \\ &\leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)) \\ &< \psi(d(x_{n-1}, x_n)). \end{aligned}$$

Since ψ is monotone increasing, we deduce that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$$

It follows that the sequence $d(x_n, x_{n+1})$ is monotone decreasing. Since cone P is regular and $0 \leq d(x_n, x_{n+1})$, for all $n \in N$, there exists $r \in P$ such that

$$d(x_n, x_{n+1}) \to r \quad \text{as } n \to \infty$$

Since ϕ, ψ are continuous and

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n))$$

we have by taking $n \to \infty$

$$\psi(r) \leq \psi(r) - \phi(r)$$

which is a contradiction unless $r = 0$. Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Let $c \in E$ with $0 \ll c$ be arbitrary. Since $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, there exists $m \in N$ such that

$$\psi(d(x_m, x_{m+1})) \ll \phi(\phi(c/2)).$$

Let $B(x_m, c) = \{x \in X : \psi(d(x_m, x)) \ll c\}$. Clearly, $x_m \in B(x_m, c)$. Therefore, $B(x_m, c)$ is nonempty. Now we will show that $Tx \in B(x_m, c)$, for $x \in B(x_m, c)$. Let $x \in B(x_m, c)$. By property (ii) of ψ, we have the following two possible cases.
Case (i): $\phi(d(x, x_m)) \ll \phi(c/2), \psi(d(x, x_m)) \ll \phi(c/2)$ or
Case (ii): $\phi(c/2) \leq \phi(d(x, x_m)), \phi(c/2) \leq \psi(d(x, x_m))$. Here we have,

Case (i):

$$\begin{aligned}
\psi(d(Tx, x_m)) &\leq \psi(d(Tx, Tx_m) + d(x_m Tx_m)) \\
&\leq \psi(d(x, x_m)) - \phi(d(x, x_m)) + \psi(d(x_m Tx_m)) \\
&\leq \psi(d(x, x_m)) + \psi(d(x_m, x_{m+1})) \\
&\leq \phi(c/2) + \phi(\phi(c/2)) \\
&\leq \phi(c/2) + \phi(c/2) \\
&\ll c/2 + c/2 \\
&= c.
\end{aligned}$$

Case (ii):

$$\begin{aligned}
\psi(d(Tx, x_m)) &\leq \psi(d(Tx, Tx_m) + d(x_m Tx_m)) \\
&\leq \psi(d(x, x_m)) - \phi(d(x, x_m)) + \psi(d(x_m Tx_m)) \\
&\leq \psi(d(x, x_m) - \phi(c/2) + \phi(\phi(c/2)) \\
&(\because \phi(x, x_m) \geq \phi(c/2), \psi(d(x, Tx_m)) \leq \phi(\phi(c/2))) \\
&\leq \psi(d(x, x_m) \\
&\ll c.
\end{aligned}$$

In any case $Tx \in B(x_m, c)$ for $x \in B(x_m, c)$. Therefore, T is a self mapping of $B(x_m, c)$. Since $x_m \in B(x_m, c)$ and $Tx_{n-1} = x_n, n \geq 1$, it follows that $x_n \in B(x_m, c)$, for all $n \geq m$. Again, c is arbitrary. This establish that $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Now,

$$\begin{aligned}
\psi(d(x_n, Tx)) &= \psi(d(Tx_{n-1}, Tx)) \\
&\leq \psi(d(x_{n-1}, x)) - \phi(d(x_{n-1}, x)).
\end{aligned}$$

Taking $n \to \infty$, we have,

$$\psi(d(x, Tx)) \leq 0.$$

But $\psi(d(x, Tx)) \geq 0$. This implies that $d(x, Tx) = 0$ and $x = Tx$. That is x is a fixed point of T.

If $y \in X$, with $y \neq x$, is a fixed point of T. Then $\phi(d(x, y)) \in intP$ and so

$$\begin{aligned}
\psi(d(x, y)) &= \psi(Tx, Ty) \\
&\leq \psi(d(x, y)) - \phi((x, y)) \\
&< \psi(d(x, y)),
\end{aligned}$$

which is a contradiction and hence $d(x, y) = 0$, i.e. $x = y$. □

Let (X, d) be a cone metric space and let $f, g : X \to X$ be two mappings. For every $x, y \in X$ let

$$M_{f,g}(x, y) = \{d(g(x), g(y)), d(f(x), g(x)), d(f(y), g(y))\}.$$

Definition 2.2. [8] Let P be an order cone. A nondecreasing function $\phi : P \to P$ is called a ϕ-map if
(i) $\phi(0) = 0$ and $0 < \phi(\omega) < \omega$ for all $\omega \in P \setminus \{0\}$,
(ii) $\omega \in IntP$ implies $\omega - \phi(\omega) \in IntP$,
(iii) $\lim_{n\to\infty} \phi^n(\omega) = 0$ for every $\omega \in P \setminus \{0\}$.

Definition 2.3. Let $f, g : X \to X$ be a pair of mappings is said to be a weakly ϕ-pair, if

$$d(f(x), f(y)) \leq \phi(z),$$

for some $z \in M_{f,g}(x, y)$, for all $x, y \in X$.

Di Bari and Vetro [8] proved following theorem

Theorem 2.4. *Let (X, d) be a cone metric space, let P be an order cone and let $f, g : X \to X$ be a weakly ϕ-pair. Assume that f and g are weakly compatible with $f(X) \subset g(X)$. If $f(X)$ or $g(X)$ is a complete subspace of X, then the mappings f and g have a unique common fixed point in X. Moreover for any $x_0 \in X$, every f-g-sequence $(f(x_n))$ with initial point x_0 converges to the common fixed point of f and g.*

We have generalized the weakly ϕ-pair by defining weakly (ψ, ϕ)-pair as follows

Definition 2.5. Let $f, g : X \to X$ be said to be weakly (ψ, ϕ)-pair if

$$\psi(d(fx, fy)) \leq \psi(z) - \phi(z) \tag{2.1}$$

for some $z \in M_{f,g}(x, y)$, for all $x, y \in X$, where $\psi : P \to P$ and $\phi : intP \cup \{0\} \to intP \cup \{0\}$ are continuous functions with the following properties:
(i) ψ is strongly monotonic increasing,
(ii) $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$,
(iii) $\phi(t) \ll t$, for $t \in intP$ and
(iv) either $\phi(t) \leq d(x, y)$ or $d(x, y) \leq t$, for $t \in intP \cup \{0\}$ and $x, y \in X$

Choudhury and Metiya [10] proved following result

Lemma 2.6. *Let (X, d) be a cone metric space with regular cone P such that $d(x, y) \in intP$, for $x, y \in X$ with $x \neq y$. Let $\phi : intP \cup \{0\} \to intP \cup \{0\}$ be a function with the following properties:*
(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii)$\phi(t) \ll t$, for $t \in intP$ and
(iii) either $\phi(t) \leq d(x, y)$ or $d(x, y) \leq \phi(t)$, for $t \in intP \cup \{0\}$ and $x, y \in X$. Let $\{x_n\}$ be a sequence in X for which $\{d(x_n, x_{n+1})\}$ is monotonic decreasing. Then $\{d(x_n, x_{n+1}\}$ is convergent to either $r = 0$ or $r \in intP$.

Theorem 2.7. *Let (X,d) be a cone metric space with regular cone P such that $d(x,y)\in intP$, for all $x,y\in X$ with $x\neq y$. Let $f,g:X\to X$ be weakly (ψ,ϕ)-pair. If $f(X)\subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof. Let $x_0\in X$ and construct $(f(x_n))$ be a f-g-sequence with initial point x_0. If $f(x_n)=f(x_{n-1})$ for some $n\in N$, then $f(x_m)=f(x_n)$ for all $m\in N$ with $m>n$ and so $(f(x_n))$ is a Cauchy sequence. Therefore we consider that $f(x_n)\neq f(x_{n-1})$ for all $n\in N$.

We have for all $n\geq 0$,

$$\psi(d(fx_{n+1},fx_{n+2}))\leq\psi(z)-\phi(z)$$

where $z\in M_{f,g}(x_{n+1},x_{n+2})=\{d(fx_n,fx_{n+1}),d(fx_{n+1},fx_n),d(fx_{n+2},fx_{n+1})\}$. If $z=d(fx_{n+1},fx_{n+2})$, then we have

$$\psi(d(fx_{n+1},fx_{n+2}))\leq\psi(d(fx_{n+1},fx_{n+2}))-\phi(d(fx_{n+1},fx_{n+2})).\qquad(2.2)$$

Using a property of ψ and ϕ, the inequality (2.2) hold if and only if $d(fx_{n+1},f_{n+2})=0$ and $fx_{n+1}=fx_{n+2}$, a contradiction. Now if $z=d(fx_n,fx_{n+1})$, then,

$$\psi(d(fx_{n+1},fx_{n+2}))\leq\psi(d(fx_n,fx_{n+1}))-\phi(d(fx_n,fx_{n+1})).\qquad(2.3)$$

Using a property of ϕ, we have for all $n\geq 0$,

$$\psi(d(fx_{n+1},fx_{n+2}))\leq\psi(d(fx_n,fx_{n+1})),$$

which implies that

$$d(fx_{n+1},fx_{n+2})\leq d(fx_n,fx_{n+1}),$$

since ψ is strongly monotone increasing. Therefore, $\{d(fx_n,fx_{n+1})\}$ is a monotone decreasing sequence. Hence by Lemma 2.6, there exists an $r\in P$ with either $r=0$ or $r\in intP$, such that

$$d(fx_n,fx_{n+1})\to r\ as\ n\to\infty.\qquad(2.4)$$

Letting limit as $n\to\infty$ in (2.3), using (2.4) and the continuities of ψ and ϕ,

$$\psi(r)\leq\psi(r)-\phi(r)$$

which is a contradiction unless $r=0$. So we must have,

$$d(fx_n,fx_{n+1})\to 0\text{ as }n\to\infty.\qquad(2.5)$$

Now we claim that $\{fx_n\}$ is a Cauchy sequence. If $\{fx_n\}$ is not a Cauchy sequence, then there exists a $c\in E$ with $0\ll c$, such that $\forall n_0\in N$, $\exists n,m\in N$ with $n>m\geq n_0$ such that $d(fx_m,fx_n)<\not\leq\phi(c)$. Hence by a property of ϕ, $\phi(c)\leq d(fx_m,fx_n)$. Therefore, there exist sequences $\{m(k)\}$ and $\{n(k)\}$ in N such that for all positive integers k,

$$n(k)>m(k)>k\text{ and }d(fx_{m(k)},fx_{n(k)})\geq\phi(c).$$

Assuming that $n(k)$ is the smallest such positive integer, we get

$$d(fx_{m(k)},fx_{n(k)})\geq\phi(c)$$

and

$$d(fx_{m(k)}, fx_{n(k)-1} \ll \phi(c).$$

Now,

$$\phi(c) \leq d(fx_{m(k)}, fx_{n(k)}) \leq d(fx_{m(k)}, fx_{n(k)-1}) + d(fx_{n(k)-1}, fx_{n(k)})$$

that is,

$$\phi(c) \leq d(fx_{m(k)}, fx_{n(k)}) \leq \phi(c) + d(fx_{n(k)-1}, fx_{n(k)}).$$

Letting $k \to \infty$ in the above inequality, using inequality (2.5), we have

$$\lim_{k\to\infty} d(fx_{m(k)}, fx_{n(k)}) = \phi(c). \tag{2.6}$$

Again,

$$d(fx_{m(k)}, fx_{n(k)}) \leq d(fx_{m(k)}, fx_{m(k)+1}) + d(fx_{m(k)+1}, fx_{n(k)+1}) + d(fx_{n(k)+1}, fx_{n(k)})$$

and

$$d(fx_{m(k)+1}, fx_{n(k)+1}) \leq d(fx_{m(k)+1}, fx_{m(k)}) + d(fx_{m(k)}, fx_{n(k)}) + d(fx_{n(k)}, fx_{n(k)+1})$$

Letting $k \to \infty$ in the above inequalities, using (2.5) and (2.6), we have

$$\lim_{k\to\infty} d(fx_{m(k)+1}, fx_{n(k)+1}) = \phi(c). \tag{2.7}$$

Putting $x = x_{m(k)+1}$ and $y = x_{n(k)+1}$ in (2.1), we have

$$d(fx_{m(k)+1}, fx_{n(k)+1}) \leq \psi(z) - \phi(z)$$

where

$$\begin{aligned} z \in M_{f,g}(x,y) &= \{d(gx_{m(k)+1}, gx_{n(k)+1}), d(fx_{m(k)+1}, gx_{m(k)+1}), \\ &\quad d(fx_{n(k)+1}, gx_{n(k)+1})\} \\ &= \{d(fx_{m(k)}, fx_{n(k)}), d(fx_{m(k)+1}, fx_{m(k)}), \\ &\quad d(fx_{n(k)+1}, fx_{n(k)})\}. \end{aligned}$$

Case.1 If $z = d(fx_{m(k)}, fx_{n(k)})$ and letting $k \to \infty$ the above inequality, using (2.6), (2.7) and the continuities of and ϕ, we have

$$\psi(\phi(c)) \leq \psi(\phi(c)) - \phi(\phi(c)).$$

Ii only true for $\phi(c) = 0$. This implies $c = 0$, a contradiction to $0 \ll c$.

Case.2 If $z = d(fx_{m(k)+1}, fx_{m(k)})$ and letting $k \to \infty$ the above inequality, using (2.6), (2.7) and the continuities of ϕ and ψ, we have

$$\psi(\phi(c)) \leq \psi(0) - \phi(\phi(0)) = 0.$$

This implies that $\psi(\phi(c)) = 0 \Rightarrow \phi(c) = 0 \Rightarrow c = 0$. It is again a contradiction.

Case.3 Similarly in case.2 we get a contradiction.

Therefore $\{fx_n\}$ be a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists a $q \in g(X)$ such that $\{fx_n\} \to q$ as $n \to \infty$. Since $q \in g(X)$, we can find $p \in X$ such that $gp = q$. Now, putting $x = x_{n+1}$ and $y = p$ in (2.1), we have

$$\psi(d(fx_{n+1}, fp)) \leq \psi(z) - \phi(z),$$

where $z \in M_{f,g}(x_{n+1}, p) = \{d(fx_n, gp), d(fx_{n+1}, fx_n), d(fp, gp)\}$. Now

Case.1 If $z = d(fx_n, gp)$,then

$$\psi(d(fx_{n+1}, fp)) \le \psi(d(fx_n, gp)) - \phi(d(fx_n, gp)),$$

Letting limit $n \to \infty$, we have

$$\psi(d(q, fp)) \le \psi(d(q, q)) - \phi(d(q, q)),$$

i.e. $\psi(d(q, fp)) \le 0$. By definition of ψ, $\psi(d(q, fp)) \ge 0$, so we have $\psi(d(q, fp)) = 0$ implies $fp = q = gp$.

Case.2 If $z = d(fx_{n+1}, fx_n)$, then

$$\psi(d(fx_{n+1}, fp)) \le \psi(d(fx_{n+1}, fx_n)) - \phi(d(fx_{n+1}, fx_n)),$$

Letting limit $n \to \infty$, we have

$$\psi(d(q, fp)) \le \psi(d(q, q)) - \phi(d(q, q)),$$

i.e. $\psi(d(q, fp)) \le 0$. By definition of ψ, $\psi(d(q, fp)) \ge 0$, so we have $\psi(d(q, fp)) = 0$ implies $fp = q$.

Case.3 If $z = d(fp, gp)$, then

$$\psi(d(fx_{n+1}, fp)) \le \psi(d(fp, gp)) - \phi(d(fp, gp)),$$

Letting limit $n \to \infty$, we have

$$\psi(d(q, fp)) \le \psi(fp, q)) - \phi(fp, q)).$$

This is contradiction if $(d(fp, q)) \ne 0$. Hence $d(fp, q) = 0$ and $fp = q$. Therefor we have

$$q = fp = gp.$$

Hence p is a coincidence point and q is a point of coincidence of f and g.

We next show that the point of coincidence is unique. For this, assume that there exists a point r in X such that $z_1 = fr = gr$. Then, from (2.1),

$$\psi(d(fp, fr)) \le \psi(z) - \phi(z) \tag{2.8}$$

where $z \in \{M_{f,g}(p, r) = \{d(gp, gr), d(fp, gp), d(fr, gr)\}$.

Case1. If $z = d(gp, gr)$, then from (2.8)

$$\psi(d(q, z_1)) \le \psi(d(q, z_1)) - \phi(d(q, z_1)),$$

it is only true for $d(q, z_1) = 0$. Hence $q = z_1$.

Case2. If $z = d(fp, gp)$, then from (2.8)

$$\psi(d(q, z_1)) \le \psi(d(q, q)) - \phi(d(q, q)) = 0,$$

i.e. $d(q, z_1) \le 0$. But $d(q, z_1) \ge 0$. Hence $q = z_1$.

Case3. If $z = d(fr, gr)$, then from (2.8)

$$\psi(d(q, z_1)) \le \psi(d(z_1, z_1)) - \phi(d(z_1, z_1)) = 0,$$

i.e. $d(q, z_1) \le 0$, but $d(q, z_1) \ge 0$. Hence $d = z_1$.

Therefore, q is the unique point of coincidence of f and g. Now, if f and g are weakly compatible, then by Lemma 1.12, z is the unique common fixed point of f and g. Hence the roof is completed. □

Example 2.8. Let $X = [0,1] \cup \{2,3,\cdots\}$, $E = \mathbb{R}^2$ with usual norm, be a real Banach space, $P = \{(x,y) \in E : x,y \geq 0\}$ be a regular cone and the partial ordering $\leq$ with respect to the cone P, be the usual partial ordering in E. We define $d : X \times X \to E$ as

$$d(x,y) = \begin{cases} (|x-y|,|x-y|), & \text{if } x,y \in [0,1], x \neq y \\ (x+y,x+y), & \text{if at least one of } x \text{ or } y \notin [0,1] \text{ and } x \neq y, \\ (0,0), & \text{if } x = y. \end{cases}$$

for $x,y \in X$. Then (X,d) is a complete cone metric space with $d(x,y) \in intP$, for $x,y \in X$ with $x \neq y$. Define $\psi, \phi : intP \cup \{0\} \to intP \cup \{0\}$ as

$$\psi(t_1,t_2) = \begin{cases} (t_1,t_2), & \text{if } t_1,t_2 \in [0,1], \\ (t_1^2,t_1^2) & \text{for otherwise.} \end{cases}$$

$$\phi(t_1,t_2) = \begin{cases} (\frac{1}{2}t_1^2, \frac{1}{2}t_2^2), & \text{if } t_1,t_2 \in [0,1], \\ (\frac{1}{2}, \frac{1}{2}) & \text{for otherwise.} \end{cases}$$

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} x - \frac{1}{2}x^2, & \text{if } x \in [0,1], \\ x - 1, & \text{if } x \in \{2,3\cdots\}. \end{cases}$$

Without loss of generality, we assume that $x \geq y$ and discuss the following cases.

Case 1. For $x,y \in [0,1]$. Then

$$\begin{aligned} \psi(d(Tx,Ty)) &= \Big((x - \frac{1}{2}x^2) - (y - \frac{1}{2}y^2), (x - \frac{1}{2}x^2) - (y - \frac{1}{2}y^2)\Big) \\ &= \Big((x-y) - \frac{1}{2}(x-y)(x+y), (x-y) - \frac{1}{2}(x-y)(x+y)\Big) \\ &\leq \Big((x-y) - \frac{1}{2}(x-y)^2, (x-y) - \frac{1}{2}(x-y)^2\Big) \\ &= \Big((x-y),(x-y)\Big) - \frac{1}{2}\Big((x-y)^2,(x-y)^2\Big) \\ &= \psi(d(x,y)) - \phi(d(x,y)) \end{aligned}$$

Case 2. For $x \in \{3,4,\cdots\}$. Then, If $y \in [0,1]$

$$\begin{aligned} d(Tx,Ty) &= d(x-1, y - \frac{1}{2}y^2) \\ &= \Big(x - 1 + y - \frac{1}{2}y^2, x - 1 + y - \frac{1}{2}y^2\Big) \\ &\leq \Big(x + y - 1, x + y - 1\Big). \end{aligned}$$

If $y \in \{2, 3 \cdots\}$

$$\begin{aligned} d(Tx, Ty) &= d(x-1, y-1) \\ &= \Big(x+y-2, x+y-2\Big) \\ &< \big(x+y-1, x+y-1\big). \end{aligned}$$

Therefore

$$\begin{aligned} \psi(d(Tx, Ty)) &\le \Big((x+y-1)^2, (x+y-1)^2\Big) \\ &< \Big((x+y-1)(x+y-1), (x+y-1)(x+y-1)\Big) \\ &< \Big((x+y)^2 - 1, (x+y)^2 - 1\Big) \\ &< \Big((x+y)^2 - 1/2, (x+y)^2 - 1/2\Big) \\ &= \Big((x+y)^2, (x+y)^2\Big) - \Big(1/2, 1/2\Big) \\ &= \psi(d(x,y)) - \phi(d(x,y)). \end{aligned}$$

Case 3. For $x = 2$ and $y \in [0, 1]$. Then, $Tx = 1$, and

$$d(Tx, Ty) = \Big(1 - (y - \frac{1}{2}y^2), 1 - (y - \frac{1}{2}y^2)\Big) \le (1, 1).$$

So, we have

$$\psi(d(Tx, Ty)) \le \psi(1, 1) = (1, 1).$$

Again $d(x, y) = (2+y, 2+y)$. So,

$$\begin{aligned} \psi(d(x,y)) - \phi(d(x,y)) &= \Big((2+y)^2, (2+y)^2\Big) - \phi(d(x,y)) \\ &= \Big((2+y)^2, (2+y)^2\Big) - (\frac{1}{2}, \frac{1}{2}) \\ &= (\frac{7}{2} + 4y + y^2, \frac{7}{2} + 4y + y^2\Big) \\ &> (1, 1) \\ &= \psi(d(Tx, Ty)). \end{aligned}$$

Now it fulfills the requirement of Theorem 2.1 and 0 is the unique fixed point of T.

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References

1. C.T. Aage and J.N. Salunke, *On common fixed points for contractive type mappings in cone metric spaces*, Bull. Math. Anal. Appl. **1** (2009), no. 3, 10–15.
2. M. Abbas and B.E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett. **22** (2009), 511–515.
3. M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric space*, J. Math. Anal. Appl. **341** (2008), 416–420.

4. Ya.I. Alber and S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces*, In: I. Gohberg, Yu. Lyubich (Eds.), New Results in Operator Theory, in: Advances and Appl., vol. 98, Birkhuser, Basel, 1997, 7-22.
5. I. Altun, M. Abbas and H. Simsek, *A fixed point theorem on cone metric spaces with new type contractivity*, Banach J. Math. Anal. **5** (2011), no. 2, 15–24.
6. M. Arshad, A. Azam and I. Beg, *Common fixed points of two maps in cone metric spaces*, Rend. Circ. Mat. Palermo **57** (2008), 433–441.
7. C. Di. Bari and P. Vetro, *ϕ-pairs and common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo **57** (2008), 279–285.
8. C. Di. Bari and P. Vetro, *Weakly ϕ-pairs and common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo **58** (2009), 125–132.
9. I. Beg and M. Abbas, *Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition*, Fixed Point Theory Appl. **2006**, Art. ID 74503, 7 pp.
10. B.S. Choudhury and N. Metiya, *The point of coincidence and common fixed point for a pair of mappings in cone metric spaces*, Comput. Math. Appl. **60** (2010), no. 6, 1686–1695.
11. B.S. Choudhury and N. Metiya, *Fixed points of weak contractions in cone metric spaces*, Nonlinear Analysis (2009), doi:10.1016/j.na.2009.08.040.
12. P.N. Dutta and B.S. Choudhury, *A generalisation metric spaces*, Fixed Point Theory App. **2008**, Article ID 406368, 8 pages.
13. G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9** (1986) 771–779.
14. N. Hussain and G. Jungck, *Common fixed point and invariant approximation results for noncommuting generalized (f,g)-nonexpansive maps*, J. Math. Anal. Appl. **321** (2006), 851–861.
15. D. Ilić and V. Rakocević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** (2008), 876–882.
16. D. Ilić, V. Rakolević, *Quasi-contraction on a cone metric space*, Appl. Math. Lett. **22** (2009), 728–731.
17. J. Jachymski, *Order-theoretic aspects of metric fixed point theory*, In: W.A. Kirk, B. Sims (Eds.), Handbook of Metric Fixed Point Theory, Kluwer Academic Publishers, Boston, 2001.
18. G. Jungck, S. Radenović, S. Radojević and V. Rakočević, *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory Appl. **2009**, Art. ID 643840, 13 pp.
19. H. Long-Guang and Z. Xian, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1468–1476.
20. A. Najati and A. Rahimi, *A fixed point approach to the stability of a generalized cauchy functional equation*, Banach J. Math. Anal. **2** (2008), no. 1, 105–112.
21. S. Radenović and Z. Kadelburg, *Quasi-contractions on symmetric and cone symmetric spaces*, Banach J. Math. Anal. **5** (2011), no. 1, 38–50.
22. S. Reich, *Some fixed point problems*, Atti. Accad. Naz. Lincei **57** (1974), 194–198.
23. B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. **47** (2001), 2683–2693.
24. N. Shahzad, Invariant approximations, *Generalized I-contractions, and R-subweakly commuting maps*, Fixed Point Theory Appl. **1** (2005), 79–86.
25. Y. Song, *Coincidence points for noncommuting f-weakly contractive mappings*, Int. J. Comput. Appl. Math. **2** (2007), no. 1, 51–57.
26. Y. Song, *Common fixed points and invariant approximations for generalized (f,g)-nonexpansive mappings*, Commun. Math. Anal. **2** (2007), 17–26.
27. Y. Song and S. Xu, *A note on common fixed-points for Banach operator pairs*, Int. J. Contemp. Math. Sci. **2** (2007), 1163–1166.

bibitemgha Gh. Abbaspour Tabadkan and M. Ramezanpour, *A fixed point approach to the stability of ϕ-morphisms on hilbert C^*-modules,* Ann. Funct. Anal. **1** (2010), no. 1, 44–50
28. Qingnian Zhanga and Yisheng Songb, *Fixed point theory for generalized ϕ-weak contractions*, Appl.Math. Lett. **22** (2009), 75–78.
29. Sh. Rezapour and R. Hamlbarani, *Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **345** (2008), 719–724.
30. P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo **56** (2007), 464–468.
31. D. Wardowski, *Endpoint and fixed points of set-valued contractions in cone metric spaces*, Nonlinear Anal. **71** (2009), 512–516.

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