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SOME RESULTS ON σ-DERIVATIONS

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ABSTRACT. Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and let $\mathcal{M}$ be a Banach $\mathcal{B}$-bimodule. Suppose that $\sigma : \mathcal{A} \to \mathcal{B}$ is a linear mapping and $d : \mathcal{A} \to \mathcal{M}$ is a σ-derivation. We prove several results about automatic continuity of σ-derivations on Banach algebras. In addition, we define a notion for m-weakly continuous linear mapping and show that, under certain conditions, d and σ are m-weakly continuous. Moreover, we prove that if $\mathcal{A}$ is commutative and $\sigma : \mathcal{A} \to \mathcal{A}$ is a continuous homomorphism such that $\sigma^2 = \sigma$ then $\sigma d\sigma(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A})) \subseteq rad(\mathcal{A})$.

1. Introduction and preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras and let $\mathcal{M}$ be a $\mathcal{B}$-bimodule. Suppose that $\sigma : \mathcal{A} \to \mathcal{B}$ is a linear mapping. A linear mapping $d : \mathcal{A} \to \mathcal{M}$ is called a σ-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$. Clearly if $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and $\sigma = id$, the identity mapping on $\mathcal{A}$, then a σ-derivation is an ordinary derivation. On the other hand, each homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ is a $\frac{\theta}{2}$-derivation. Mirzavaziri and Moslehian [5] have presented several important results of σ-derivations. Hosseini et al [3] defined generalized σ-derivation on Banach algebras and presented some results about automatic continuity of generalized σ-derivations and σ-derivations on Banach algebras. So far, numerous derivations have been defined such as σ-derivation, generalized σ-derivation, (σ, τ)-derivation and so on. In 2009, Mirzavaziri and Omidvar Tehrani [8] defined (δ, ε)-double derivation and also the automatic continuity of the former derivation on C^*-algebras was considered. Next, Hejazian et al [4] studied the automatic

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continuity of (δ, ε)-double derivations on Banach algebras. The investigation of automatic continuity of (δ, ε)-double derivations and generalized σ-derivations in detail, will result in some theorems about automatic continuity of derivations and σ-derivations. Moreover, Mirzavaziri and Moslehian ([6] and [7]) acquired some results about automatic continuity of σ-derivations. In this article the m-weakly continuity of a linear mapping is defined as follows:
The linear mapping $T : \mathcal{B} \to \mathcal{A}$ is called m-weakly continuous if the linear mapping $\varphi T : \mathcal{B} \to \mathbb{C}$ is continuous for all multiplicative linear functional φ from $\mathcal{A}$ in to $\mathbb{C}$. Suppose that $\mathcal{A}$ is unital and $d : \mathcal{A} \to \mathcal{B}$ is a σ-derivation such that $\varphi d(\mathbf{1}) \neq 0$ for all $\varphi \in \Phi_{\mathcal{B}}$, the set of all non-zero multiplicative linear functionals from $\mathcal{B}$ in to $\mathbb{C}$. If for all $\varphi \in \Phi_{\mathcal{B}}$ there exists an element $a_\varphi \in \mathcal{A}$ such that $a_\varphi \notin ker(\varphi d)$ and $\varphi d(a_\varphi^2) = (\varphi d(a_\varphi))^2$ then φd is a homomorphism. Moreover, d and σ are m-weakly continuous. In particular, if $\mathcal{A}$ is semi-simple and commutative then d and σ are continuous.
Singer and Wermer (see Corollary 2.7.20 of [2]) proved that, when $\mathcal{A}$ is a commutative Banach algebra and $D : \mathcal{A} \to \mathcal{A}$ is a continuous derivation, $D(\mathcal{A}) \subseteq rad(\mathcal{A})$, where $rad(\mathcal{A})$ is the Jacobson radical of $\mathcal{A}$. They conjectured that $D(\mathcal{A}) \subseteq rad(\mathcal{A})$ for each (possibly discontinuous) derivation D on $\mathcal{A}$. In 1988, Thomas [9] proved this conjecture. We prove that if $d : \mathcal{A} \to \mathcal{A}$ is a σ-derivation on a commutative Banach algebra $\mathcal{A}$ such that σ is a continuous homomorphism and $\sigma^2 = \sigma$ then $\sigma d \sigma(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A})) \subseteq rad(\mathcal{A})$. In particular if $d(\mathcal{A}) \subseteq \sigma d \sigma(\mathcal{A})$ then $d(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A})) \subseteq rad(\mathcal{A})$, where $Q(\mathcal{A})$ is the set of all quasi-nilpotent elements of $\mathcal{A}$.

2. Main results

Throughout this paper $\mathcal{A}$ and $\mathcal{B}$ denote two Banach algebras. Moreover, $\mathcal{M}$ denotes a Banach $\mathcal{B}$-bimodule. Furthermore, if an algebra is unital then $\mathbf{1}$ will show its unit element. Recall that if E is a subset of an algebra B, the *right annihilator* $ran(E)$ of E (resp. the *left annihilator* $lan(E)$ of E) is defined to be $\{b \in B\colon Eb = \{0\}\}$ (resp. $\{b \in B\colon bE = \{0\}\}$). The set $ann(E) := ran(E) \bigcap lan(E)$ is called the *annihilator* of E. Suppose $S \subseteq \mathcal{M}$. The right annihilator $ran(S)$ of S is defined to be $\{b \in \mathcal{B} : \ Sb = \{0\}\}$. The left annihilator of S is defined, similarly. Also, recall that if Y and Z are Banach spaces and $T\colon Y \to Z$ is a linear mapping, then the set $\{z \in Z : \exists \ \{y_n\} \ \subseteq Y \ s.t \ y_n \to 0, T(y_n) \to z\}$ is called the separating space $S(T)$ of T. By the closed graph Theorem, T is continuous if and only if $S(T) = \{0\}$. The reader is referred to [2] for more about separating spaces.

Definition 2.1. Suppose $\sigma : \mathcal{A} \to \mathcal{B}$ is a linear mapping. A linear mapping $d : \mathcal{A} \to \mathcal{M}$ is called a σ-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$.

It is clear that if $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and $\sigma = id$, the identity mapping on $\mathcal{A}$, then a σ-derivation is an ordinary derivation.

Theorem 2.2. *Suppose that $d : \mathcal{A} \to \mathcal{B}$ is a linear mapping. We define $d_1 : \mathcal{A}_1 \to \mathcal{B}_1$ by $d_1(a + \alpha) = d(a) + \alpha$ for all $a + \alpha \in \mathcal{A}_1$, whenever $\mathcal{A}_1 = \mathcal{A} \bigoplus \mathbb{C}$*

and $\mathcal{B}_1 = \mathcal{B}\bigoplus\mathbb{C}$ are the unitization of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then d_1 is a σ-derivation if and only if d is a homomorphism.

Proof. We denote the unit element of $\mathcal{A}_1$ and $\mathcal{B}_1$ by $\mathbf{1}$. Clearly $d_1(\mathbf{1}) = \mathbf{1}$. Suppose that d_1 is a σ-derivation. We have $\mathbf{1} = d_1(\mathbf{1}) = d_1(\mathbf{1})\sigma(\mathbf{1}) + \sigma(\mathbf{1})d_1(\mathbf{1})$. Therefore $\sigma(\mathbf{1}) = \frac{1}{2}$ and $d_1((a+\alpha)\mathbf{1}) = d_1(a+\alpha)\sigma(\mathbf{1}) + \sigma(a+\alpha)d_1(\mathbf{1}) = \frac{d_1(a+\alpha)}{2} + \sigma(a + \alpha)$. Hence $\sigma(a+\alpha) = \frac{d_1(a+\alpha)}{2}$ for all $a+\alpha \in \mathcal{A}_1$. Moreover, we have

$$\begin{aligned} d_1((a+\alpha)(b+\beta)) &= d_1(a+\alpha)\sigma(b+\beta) + \sigma(a+\alpha)d_1(b+\beta) \\ &= d_1(a+\alpha)\frac{d_1(b+\beta)}{2} + \frac{d_1(a+\alpha)}{2}d_1(b+\beta) \\ &= d_1(a+\alpha)d_1(b+\beta). \end{aligned}$$

It means that d_1 is a homomorphism. Hence d is a homomorphism. Conversely, assume that d is a homomorphism, i.e. $d(ab) = d(a)d(b)$ for all $a,b \in \mathcal{A}$. We have $d(ab) + \beta d(a) + \alpha d(b) + \alpha\beta = d(a)d(b) + \beta d(a) + \alpha d(b) + \alpha\beta$ for all $a + \alpha$, $b+\beta \in \mathcal{A}_1$. It means that d_1 is a homomorphism. Put $\sigma = \frac{d_1}{2}$. Then

$$\begin{aligned} d_1((a+\alpha)(b+\beta)) &= d_1(a+\alpha)d_1(b+\beta) \\ &= d_1(a+\alpha)\frac{d_1(b+\beta)}{2} + \frac{d_1(a+\alpha)}{2}d_1(b+\beta) \\ &= d_1(a+\alpha)\sigma(b+\beta) + \sigma(a+\alpha)d_1(b+\beta). \end{aligned}$$

Hence d_1 is a σ-derivation. □

Corollary 2.3. *Suppose $\mathcal{B}$ is commutative and semisimple and let $d : \mathcal{A} \to \mathcal{B}$ be a linear mapping. If $d_1 : \mathcal{A}_1 \to \mathcal{B}_1$, defined by $d_1(a+\alpha) = d(a) + \alpha$, is a σ-derivation then d and d_1 are continuous operators.*

Proof. According to Theorem 2.2, d is a homomorphism. By Theorem 2.3.3 of [2], d is continuous and so d_1 is continuous. □

Theorem 2.4. *Suppose that $\mathcal{A}$ is unital and $d : \mathcal{A} \to \mathcal{M}$ is a σ-derivation. If σ is continuous and $\|\sigma(\mathbf{1})\| < 1$ then d is continuous.*

Proof. Suppose $d(\mathbf{1}) = 0$. Then for each $a \in \mathcal{A}$, $\|d(a)\| = \|d(a)\sigma(\mathbf{1})\| \le \|d(a)\|\|\sigma(\mathbf{1})\|$. Thus $\|d(a)\|(1 - \|\sigma(\mathbf{1})\|) \le 0$. It follows that $d(a) = 0$. Since a was arbitrary, d is identically zero and hence d is continuous. Now assume that $d(\mathbf{1}) \neq 0$ and a is an arbitrary element of $\mathcal{A}$ such that $d(a) \neq 0$. We have

$$\begin{aligned} \|d(a)\| &= \|d(\mathbf{1})\sigma(a) + \sigma(\mathbf{1})d(a)\| \\ &\le \|d(\mathbf{1})\sigma(a)\| + \|\sigma(\mathbf{1})d(a)\| \\ &\le \|d(\mathbf{1})\|\|\sigma\|\|a\| + \|\sigma(\mathbf{1})\|\|d(a)\|. \end{aligned}$$

Hence $(1 - \|\sigma(\mathbf{1})\|)\|d(a)\| \le \|d(\mathbf{1})\|\|\sigma\|\|a\|$. This implies that d is continuous. □

Recall that an element a in a normed algebra $\mathcal{A}$ is called quasi-nilpotent if $lim_{n\to\infty}\|a^n\|^{\frac{1}{n}} = 0$. The set of all quasi-nilpotent elements of $\mathcal{A}$ is denoted by $Q(\mathcal{A})$.

Theorem 2.5. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital and $\mathcal{B}$ has no zero divisors and assume that $d : \mathcal{A} \to \mathcal{B}$ is a σ-derivation such that $d(\mathbf{1}) \neq 0$. If there exists a sequence $\{a_n\} \subseteq \mathcal{A}$ such that $d(a_n) \to a_0$ and $\sigma(a_n) \to a_0$, where $a_0 \neq 0$, then $d = \sigma$. Moreover, if d is continuous then $d(Q(\mathcal{A})) \subseteq Q(\mathcal{B})$.*

Proof. We have $d(a_n) = d(a_n)\sigma(\mathbf{1}) + \sigma(a_n)d(\mathbf{1})$. Thus $a_0(\sigma(\mathbf{1}) + d(\mathbf{1}) - \mathbf{1}) = 0$. Since $\mathcal{B}$ has no zero divisors and $a_0 \neq 0$, $d(\mathbf{1}) + \sigma(\mathbf{1}) = \mathbf{1}$. We have $d(\mathbf{1}) \neq \mathbf{1}$, since if $d(\mathbf{1}) = \mathbf{1}$ then $\sigma(\mathbf{1}) = 0$. Thus $d(\mathbf{1}) = d(\mathbf{1})\sigma(\mathbf{1}) + \sigma(\mathbf{1})d(\mathbf{1}) = 0$, which is a contradiction. We have $d(\mathbf{1}) = (\mathbf{1} - \sigma(\mathbf{1}))\sigma(\mathbf{1}) + \sigma(\mathbf{1})(\mathbf{1} - \sigma(\mathbf{1}))$. Therefore $(\mathbf{1} - 2\sigma(\mathbf{1}))d(\mathbf{1}) = 0$. Since $d(\mathbf{1}) \neq 0$ and $\mathcal{B}$ has no zero divisors, $\sigma(\mathbf{1}) = \frac{\mathbf{1}}{2}$. It follows that $d(\mathbf{1}) = \frac{\mathbf{1}}{2}$. Let a be an arbitrary element of $\mathcal{A}$. We have

$$d(a) = d(a)\sigma(\mathbf{1}) + \sigma(a)d(\mathbf{1}) = \frac{d(a)}{2} + \frac{\sigma(a)}{2},$$

and hence $d = \sigma$. By induction on n, we obtain

$$d(a^n) = 2^{n-1}(d(a))^n$$

therefore $(d(a))^n = \frac{d(a^n)}{2^{n-1}}$. Assume that d is continuous and $a \in Q(\mathcal{A})$. Then

$$\|(d(a))^n\|^{\frac{1}{n}} = \|\frac{d(a^n)}{2^{n-1}}\|^{\frac{1}{n}} \leq (\frac{1}{2^{n-1}})^{\frac{1}{n}}\|d\|^{\frac{1}{n}}\|a^n\|^{\frac{1}{n}} \to 0.$$

It means that $d(a) \in Q(\mathcal{B})$. □

Remark 2.6. Suppose that $\sigma : \mathcal{A} \to \mathcal{B}$ is a continuous linear mapping and $\{\sigma(ab) - \sigma(a)\sigma(b) \mid a, b \in \mathcal{A}\} \subseteq ann(\mathcal{M})$. Then $U_\sigma = \mathcal{A} \bigoplus \mathcal{M}$ is an algebra by the following action: $(a, x) \bullet (b, y) = (ab, \sigma(a)y + x\sigma(b))$ for all $a, b \in \mathcal{A}$ and $x, y \in \mathcal{M}$. Put $m = \max\{1, \|\sigma\|\}$. We define $|||a||| = m\|a\|$ $(a \in \mathcal{A})$, which is clearly a complete norm on $\mathcal{A}$. Then $|||ab||| = m\|ab\| \leq m^2\|a\|\|b\| = m\|a\|m\|b\| = |||a||||||b|||$. Let $d : \mathcal{A} \to \mathcal{M}$ be a σ-derivation. Define two norms $\|.\|_1$ and $\|.\|_2$ on U_σ by $\|(a, x)\|_1 = |||a||| + \|x\|$, $\|(a, x)\|_2 = |||a||| + \|d(a) - x\|$.

Theorem 2.7. *Suppose that U_σ,$\|.\|_1$ and $\|.\|_2$ are as in the* Remark 2.6. *Then U_σ is a Banach algebra with respect to $\|.\|_1$ and $\|.\|_2$. Furthermore, these two norms are equivalent if and only if d is continuous.*

Proof. Clearly $(U_\sigma, \|.\|_1)$ is a Banach algebra and $\|.\|_2$ is a norm on U_σ. We prove that $\|.\|_2$ is a complete algebra norm on U_σ. Suppose $\{(a_n, x_n)\}$ is a Cauchy sequence in $(U_\sigma, \|.\|_2)$. Then $\{a_n\}$ and $\{d(a_n) - x_n\}$ are Cauchy sequences in $\mathcal{A}$ and $\mathcal{M}$, respectively. Since $\mathcal{A}$ and $\mathcal{M}$ are Banach spaces, there exist $a \in \mathcal{A}$ and $x \in \mathcal{M}$ such that $a_n \to a$ in $\mathcal{A}$ and $d(a_n) - x_n \to x$ in $\mathcal{M}$. Therefore $(a_n, x_n) \to (a, d(a) - x)$ in $\|.\|_2$. Thus $(U_\sigma, \|.\|_2)$ is a Banach space. Assume that

(a,x) and (b,y) are two arbitrary elements of U_σ. We have

$$\begin{aligned}\|(a,x)\bullet(b,y)\|_2 &= \|(ab,\sigma(a)y+x\sigma(b)\|_2\\ &= \||ab\|| + \|d(ab)-\sigma(a)y-x\sigma(b)\|\\ &= \||ab\||+\|d(a)\sigma(b)+\sigma(a)d(b)-\sigma(a)y-x\sigma(b)\|\\ &\le \||a\||\ \||b\||+\|d(a)-x\|\|\sigma\|\|b\|+\|\sigma\|\|a\|\|d(b)-y\|\\ &\le \||a\||\ \||b\||+\|d(a)-x\|\ \||b\||+\||a\||\ \|d(b)-y\|\\ &\le (\||a\||+\|d(a)-x\|)(\||b\||+\|d(b)-y\|)\\ &= \|(a,x)\|_2\|(b,y)\|_2.\end{aligned}$$

Therefore $(U_\sigma,\|.\|_2)$ is a Banach algebra. Suppose d is continuous. We have

$$\begin{aligned}\|(a,x)\|_2 &= \||a\||+\|d(a)-x\|\\ &\le \||a\||+\|d(a)\|+\|x\|\\ &\le \||a\||+\|d\|\|a\|+\|x\|\\ &\le \||a\||+\|d\|\ m\|a\|+\|x\|\\ &= \||a\||+\|d\|\ \||a\||+\|x\|\\ &\le (1+\|d\|)(\||a\||+\|x\|)\\ &= (1+\|d\|)\|(a,x)\|_1\end{aligned}$$

for all $(a,x)\in U_\sigma$. Applying the open mapping Theorem, we obtain that $\|.\|_1$ and $\|.\|_2$ are equivalent. Conversely, suppose that $\|.\|_1$ and $\|.\|_2$ are equivalent. Then there exists a positive number c such that $\|(a,x)\|_2 \le c\|(a,x)\|_1$ $((a,x)\in U_\sigma)$. Thus $\|d(a)\| \le \|(a,0)\|_2 \le c\|(a,0)\|_1 = c\||a\||$. It means that d is continuous. □

Suppose that $d:\mathcal{A}\to\mathcal{M}$ is a linear mapping. We define a linear mapping $\Theta: U_\sigma \to U_\sigma$ by $\Theta(a,x)=(a,d(a)-x)\quad(a\in\mathcal{A},x\in\mathcal{M})$. It is clear that Θ is an endomorphism if and only if d is a σ-derivation.

Theorem 2.8. *Suppose that $\sigma:\mathcal{A}\to\mathcal{B}$ is a continuous linear mapping such that $\{\sigma(ab)-\sigma(a)\sigma(b)\mid a,b\in\mathcal{A}\}\subseteq ann(\mathcal{M})$ and assume that $d:\mathcal{A}\to\mathcal{M}$ is a σ-derivation. Consider U_σ and $\|.\|_2$ as in Remark 2.6. Then d is continuous if and only if $\Theta:(U_\sigma,\|.\|_2)\to(U_\sigma,\|.\|_2)$ is continuous.*

Proof. We have $\|\Theta(a,x)\|_2 = \|(a,d(a)-x)\|_2 = \||a\||+\|x\| = \|(a,x)\|_1$. Let d be continuous. By Theorem 2.7, $\|.\|_1$ and $\|.\|_2$ are equivalent. So there exists a positive number c such that $\|(a,x)\|_1 \le c\ \|(a,x)\|_2$. On the other hand, $\|\Theta(a,x)\|_2 = \|(a,x)\|_1 \le c\ \|(a,x)\|_2$. It means that Θ is continuous. Now assume that Θ is continuous. Then there exists a positive number c such that $\|\Theta(a,x)\|_2 \le c\ \|(a,x)\|_2$. This implies that $\|(a,x)\|_1 \le c\ \|(a,x)\|_2$. It follows from Theorem 2.7 that d is continuous. □

Suppose that $\mathcal{A}$ is a Banach algebra. We denote by $\Phi_{\mathcal{A}}$, the set of all non-zero multiplicative linear functionals from $\mathcal{A}$ into $\mathbb{C}$. We know that each member of $\Phi_{\mathcal{A}}$ is continuous. Since the case $\Phi_{\mathcal{A}}=\phi$ makes every thing trivial, so we will assume that $\Phi_{\mathcal{A}}$ is not equal to empty set.

Definition 2.9. Let $\mathcal{B}$ and $\mathcal{A}$ be two Banach algebras and suppose that $T : \mathcal{B} \to \mathcal{A}$ is a linear mapping. T is called *m-weakly continuous* if the linear mapping $\varphi T : \mathcal{B} \to \mathbb{C}$ is continuous for all $\varphi \in \Phi_{\mathcal{A}}$.

It is clear that if a linear mapping is continuous then it is m-weakly continuous but the converse is not true, in general. To see this, suppose that $\mathcal{A}$ is a Banach algebra. Set $\mathfrak{B} = \mathbb{C} \bigoplus \mathcal{A}$. Consider $\mathfrak{B}$ as a commutative algebra with pointwise addition and scalar multiplication and the product defined by $(\alpha, a).(\beta, b) = (\alpha\beta, \alpha b + \beta a)$ $(\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathcal{A})$. The algebra $\mathfrak{B}$ with the norm $\|(\alpha, a)\| = |\alpha| + \|a\|$ is a Banach algebra. Hence $rad(\mathfrak{B}) = Q(\mathfrak{B}) = \{0\} \bigoplus \mathcal{A}$. On the other hand, $rad(\mathfrak{B}) = \bigcap_{\varphi \in \Phi_{\mathfrak{B}}} ker(\varphi)$. Note that $\Phi_{\mathfrak{B}} \neq \phi$, since $\mathfrak{B}$ is a unital commutative Banach algebra. Assume that $T : \mathcal{A} \to \mathcal{A}$ is a discontinuous linear mapping. Define $D : \mathfrak{B} \to \mathfrak{B}$ by $D(\alpha, a) = (0, T(a))$. Clearly D is discontinuous and $D(\mathfrak{B}) \subseteq \{0\} \bigoplus \mathcal{A} = rad(\mathfrak{B}) = \bigcap_{\varphi \in \Phi_{\mathfrak{B}}} ker(\varphi)$. So $\varphi(D(\mathfrak{B})) = \{0\}$ for all $\varphi \in \Phi_{\mathfrak{B}}$ and it cause that $\varphi D : \mathfrak{B} \to \mathbb{C}$ is continuous for all $\varphi \in \Phi_{\mathfrak{B}}$. Thus D is m-weakly continuous but it is not continuous. In fact D is a discontinuous derivation on $\mathfrak{B}$. Moreover, every derivation from a commutative Banach algebra $\mathcal{A}$ into $\mathcal{A}$ is m-weakly continuous (see Theorem 4.4 of [9]).

Proposition 2.10. *Suppose that $\mathcal{A}$ is a Banach algebra. Then $\mathcal{A}$ is commutative and semi-simple if and only if $\bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker(\varphi) = \{0\}$.*

Proof. Obviously if $\mathcal{A}$ is commutative and semi-simple then $\bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker(\varphi) = \{0\}$. Conversely, suppose that $\bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker(\varphi) = \{0\}$ and a,b are two arbitrary elements of $\mathcal{A}$. Then $\varphi(ab) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a) = \varphi(ba)$ for all $\varphi \in \Phi_{\mathcal{A}}$. So $\varphi(ab - ba) = 0$. Since φ was arbitrary, we have $ab - ba \in \bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker(\varphi) = \{0\}$. Hence $\mathcal{A}$ is commutative. Since $\mathcal{A}$ is commutative and $\bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker(\varphi) = \{0\}$, $rad(\mathcal{A}) = \{0\}$. Thus $\mathcal{A}$ is semi-simple. □

Theorem 2.11. *Suppose that $\mathcal{B}$ and $\mathcal{A}$ are two Banach algebras and assume that $T : \mathcal{B} \to \mathcal{A}$ is an m-weakly continuous linear mapping. If $\bigcap_{\varphi \in \Phi_{\mathcal{A}}} ker(\varphi) = \{0\}$ then T is continuous.*

Proof. By part (ii) of Proposition 5.2.2 in [2], we have the result. □

Theorem 2.12. *Suppose that $d : \mathcal{A} \to \mathcal{B}$ is a σ-derivation such that σ is m-weakly continuous. If $\bigcap_{\varphi \in \Phi_{\mathcal{B}}} ker(\varphi) = \{0\}$ and $S(\varphi d) \neq \{0\}$ for all $\varphi \in \Phi_{\mathcal{B}}$ then σ is a homomorphism.*

Proof. Suppose that φ is an arbitrary element of $\Phi_{\mathcal{B}}$. Put $\varphi \mathrm{d} = d_1$ and $\varphi\sigma = \sigma_1$. Obviously d_1 is a σ_1-derivation. Since σ_1 is continuous, $\{\sigma_1(ab) - \sigma_1(a)\sigma_1(b) \mid a, b \in \mathcal{A}\} \subseteq ann(\mathrm{S}(d_1)) = \{0\}$ (see Lemma 2.3 of [6]). Therefore $\{\sigma(ab) - \sigma(a)\sigma(b) \mid a, b \in \mathcal{A}\} \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{B}}} ker(\varphi) = \{0\}$. So σ is a homomorphism. □

Theorem 2.13. *Suppose that $\mathcal{A}$ is unital and $d : \mathcal{A} \to \mathcal{B}$ is a σ-derivation such that $\varphi d(\mathbf{1}) \neq 0$ for all $\varphi \in \Phi_{\mathcal{B}}$. If for all $\varphi \in \Phi_{\mathcal{B}}$ there exists an element $a_\varphi \in \mathcal{A}$ such that $a_\varphi \notin ker(\varphi d)$ and $\varphi d(a_\varphi^2) = (\varphi d(a_\varphi))^2$ then φd is a homomorphism. Moreover, d and σ are m-weakly continuous.*

Proof. Suppose that φ is an arbitrary element of $\Phi_{\mathcal{B}}$. Put $\varphi d = d_1$ and $\varphi\sigma = \sigma_1$. At first we show that $ker(d_1) \subseteq ker(\sigma_1)$. Let $a \in ker(d_1)$. We have

$$\begin{aligned} 0 &= d_1(a) \\ &= d_1(a)\sigma_1(\mathbf{1}) + \sigma_1(a)d_1(\mathbf{1}) \\ &= \sigma_1(a)d_1(\mathbf{1}). \end{aligned}$$

Since $d_1(\mathbf{1}) \neq 0$, $\sigma_1(a) = 0$ and hence $a \in ker(\sigma_1)$. It means that $ker(d_1) \subseteq ker(\sigma_1)$. Therefore there exists a complex number λ_φ such that $\sigma_1 = \lambda_\varphi d_1$. By hypothesis, there exists $a_\varphi \notin ker(\varphi d)$ such that $\varphi d(a_\varphi^2) = (\varphi d(a_\varphi))^2$. We have

$$\begin{aligned} (d_1(a_\varphi))^2 &= d_1(a_\varphi^2) \\ &= d_1(a_\varphi)\sigma_1(a_\varphi) + \sigma_1(a_\varphi)d_1(a_\varphi) \\ &= d_1(a_\varphi)\lambda_\varphi d_1(a_\varphi) + \lambda_\varphi d_1(a_\varphi)d_1(a_\varphi) \\ &= 2\lambda_\varphi(d_1(a_\varphi))^2. \end{aligned}$$

Since $d_1(a_\varphi) \neq 0$, $\lambda_\varphi = \frac{1}{2}$. This implies that $\sigma_1 = \frac{d_1}{2}$. We have

$$\begin{aligned} d_1(ab) &= d_1(a)\sigma_1(b) + \sigma_1(a)d_1(b) \\ &= d_1(a)\frac{d_1(b)}{2} + \frac{d_1(a)}{2}d_1(b) \\ &= d_1(a)d_1(b) \end{aligned}$$

for all $a, b \in \mathcal{A}$. Hence $d_1 : \mathcal{A} \to \mathbb{C}$ is a complex homomorphism. We know that every complex homomorphism on a Banach algebra is continuous. Clearly σ_1 is also continuous. Since φ was arbitrary, d and σ are m-weakly continuous. □

Suppose that $a \in \mathcal{A}$ we define $L_a : \mathcal{A} \to \mathcal{A}$ by $L_a(b) = ab$ for all $b \in \mathcal{A}$. Set $L_{\mathcal{A}} = \{L_a \mid a \in \mathcal{A}\}$. It is clear that $L_{\mathcal{A}}$ is a subalgebra of $B(\mathcal{A})$, here $B(\mathcal{A})$ denotes the set of all continuous linear mapping from $\mathcal{A}$ into $\mathcal{A}$. It is well known that $a \in Q(\mathcal{A})$ if and only if $L_a \in Q(L_{\mathcal{A}})$.

Theorem 2.14. *$Q(\mathcal{A}) = lan(\mathcal{A})$ if and only if $Q(L_{\mathcal{A}}) = \{0\}$.*

Proof. Suppose that $Q(L_{\mathcal{A}}) = \{0\}$ and $a \in Q(\mathcal{A})$. So $L_a \in Q(L_{\mathcal{A}}) = \{0\}$ and hence $a \in lan(\mathcal{A})$. It means that $Q(\mathcal{A}) \subseteq lan(\mathcal{A})$. It is easy to see that $lan(\mathcal{A}) \subseteq Q(\mathcal{A})$. Thus $Q(\mathcal{A}) = lan(\mathcal{A})$. Conversely, assume that $Q(\mathcal{A}) = lan(\mathcal{A})$. Suppose that $L_a \in Q(L_{\mathcal{A}})$. So $a \in Q(\mathcal{A}) = lan(\mathcal{A})$. It follows that $ab = 0$ for all $b \in \mathcal{A}$. It means that $L_a = 0$. Hence $Q(L_{\mathcal{A}}) = \{0\}$. □

Theorem 2.15. *Suppose that $d : \mathcal{A} \to \mathcal{A}$ is a σ-derivation such that σ is an endomorphism and $\sigma^2 = \sigma$. If $\sigma d\sigma$ is a continuous mapping and $\sigma(a)\sigma d\sigma(a) = \sigma d\sigma(a)\sigma(a)$ for all $a \in \mathcal{A}$ then $\sigma d\sigma(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A})) \subseteq Q(\mathcal{A})$. In particular if $d(\mathcal{A}) \subseteq \sigma d\sigma(\mathcal{A})$ then $d(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A}))$.*

Proof. First of all, we define another action on $\mathcal{A}$ by the following form: $a \bullet b = \sigma(ab)$ for all $a, b \in \mathcal{A}$. It is clear that $\mathcal{A}$ is an algebra by this action. We denote this algebra by $\widetilde{\mathcal{A}_\sigma}$. Put $D = \sigma d\sigma$. It is clear that $\sigma D = D\sigma = D$ and D is a

σ-derivation on $\mathcal{A}$. Moreover, D is a derivation on $\widetilde{\mathcal{A}}_\sigma$. Because,

$$\begin{aligned} D(a \bullet b) &= D(\sigma(ab)) = D(\sigma(a)\sigma(b)) \\ &= D(\sigma(a))\sigma^2(b) + \sigma^2(a)D(\sigma(b)) \\ &= \sigma(D(a))\sigma(b) + \sigma(a)\sigma(D(b)) \\ &= D(a) \bullet b + a \bullet D(b) \end{aligned}$$

for all $a, b \in \widetilde{\mathcal{A}}_\sigma$. Suppose that $a \in \mathcal{A}$ is a non-zero arbitrary element. We define a linear mapping $\Delta_{L_a} : B(\widetilde{\mathcal{A}}_\sigma) \to B(\widetilde{\mathcal{A}}_\sigma)$ by $\Delta_{L_a}(T) = TL_a - L_aT$ for all $T \in B(\widetilde{\mathcal{A}}_\sigma)$. We have $\Delta_{L_a}(D)(x) = (DL_a - L_aD)(x) = D(a \bullet x) - a \bullet D(x) = L_{D(a)}(x)$ for all $x \in \widetilde{\mathcal{A}}_\sigma$. Therefore $\Delta^2_{L_a}(D) = \Delta_{L_a}(L_{D(a)}) = L_{D(a)}L_a - L_aL_{D(a)} = 0$. Hence $\Delta_{L_a}(D) \in Q(B(\widetilde{\mathcal{A}}_\sigma))$. This implies that $L_{D(a)} \in Q(L_{\widetilde{\mathcal{A}}_\sigma})$. So $D(a) \in Q(\widetilde{\mathcal{A}}_\sigma)$. Since $D\sigma = \sigma D = D$, $D(a) \in Q(\mathcal{A})$. It means that $\sigma d\sigma(\mathcal{A}) \subseteq Q(\mathcal{A})$. Since $D(\mathcal{A}) \subseteq Q(\mathcal{A})$, $\sigma D(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A}))$. Hence $\sigma d\sigma(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A}))$. Note that $\sigma(Q(\mathcal{A})) \subseteq Q(\mathcal{A})$. □

We know that if $\sigma : \mathcal{A} \to \mathcal{A}$ is an endomorphism such that $\sigma^2 = \sigma$ then we can define $\widetilde{\mathcal{A}}_\sigma - algebra$ which introduced in 2.15. We want to define a norm on $\widetilde{\mathcal{A}}_\sigma$ such that it is a Banach algebra. Suppose σ is continuous. Obviously $\|\sigma\| \geq 1$. We define $|||a||| = \|\sigma\|\|a\|$. Clearly $\widetilde{\mathcal{A}}_\sigma$ is a Banach algebra with respect to $|||.|||$.

Theorem 2.16. *Suppose that $\mathcal{A}$ is commutative and $d : \mathcal{A} \to \mathcal{A}$ is a σ-derivation such that σ is a continuous endomorphism and $\sigma^2 = \sigma$. Then $\sigma d\sigma(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A}))$ $\subseteq rad(\mathcal{A})$. In particular if $d(\mathcal{A}) \subseteq \sigma d\sigma(\mathcal{A})$ then $d(\mathcal{A}) \subseteq \sigma(Q(\mathcal{A})) \subseteq rad(\mathcal{A})$.*

Proof. Consider $\widetilde{\mathcal{A}}_\sigma$-algebra with $|||.|||$. Clearly it is a commutative Banach algebra. We know that $D = \sigma d\sigma : \widetilde{\mathcal{A}}_\sigma \to \widetilde{\mathcal{A}}_\sigma$ is a derivation. By Theorem 4.4 in [9], $D(\widetilde{\mathcal{A}}_\sigma) \subseteq rad(\widetilde{\mathcal{A}}_\sigma) = Q(\widetilde{\mathcal{A}}_\sigma)$. Since $D\sigma = \sigma D = D$, $D(\mathcal{A}) \subseteq Q(\mathcal{A})$. A similar argument to Theorem 2.15 gives the result. □

Definition 2.17. A Banach algebra $\mathcal{A}$ has the Cohen's factorization property if $\mathcal{A}^2 = \mathcal{A}$, where $\mathcal{A}^2 = \{bc \mid b, c \in \mathcal{A}\}$.

Corollary 2.18. *Suppose that $d : \mathcal{A} \to \mathcal{A}$ is a σ-derivation such that all conditions in Theorem 2.16 are hold and furthermore $d\sigma = \sigma d = d$. If $Q(L_\mathcal{A}) = \{0\}$ and $\mathcal{A}$ has the Cohen's factorization property then d is identically zero.*

Proof. By Theorem 2.16, $d(\mathcal{A}) \subseteq Q(\mathcal{A})$. Since $\mathcal{A}$ is commutative and $Q(L_\mathcal{A}) = \{0\}$, it follows from Theorem 2.14 that $Q(\mathcal{A}) = lan(\mathcal{A}) = ann(\mathcal{A})$. Suppose that a is an arbitrary element of $\mathcal{A}$. Then there exist two elements b and c in $\mathcal{A}$ such that $a = bc$. We have $d(a) = d(bc) = d(b)\sigma(c) + \sigma(b)d(c) = 0$. Since a was arbitrary, $d \equiv 0$. □

Remark 2.19. Suppose that $\mathcal{A}$ is commutative and has the Cohen's factorization property and assume that $d : \mathcal{A} \to \mathcal{A}$ is a derivation. If $Q(L_\mathcal{A}) = \{0\}$ then by Theorem 4.4 of [9], we have $d(\mathcal{A}) \subseteq Q(\mathcal{A})$. It follows from Theorem 2.14 that $d \equiv 0$.

Theorem 2.20. *Suppose $\mathcal{B}$ is commutative and $d : \mathcal{A} \to \mathcal{B}$ is a $\sigma -$ derivation such that σ is an isomorphism. Then $d(\mathcal{A}) \subseteq rad(\mathcal{B})$.*

Proof. We define a map $D : \mathcal{B} \to \mathcal{B}$ by $D(b) = d\sigma^{-1}(b)$ for all $b \in \mathcal{B}$. It is clear that D is a derivation on $\mathcal{B}$. According to Theorem 4.4 of [9], $D(\mathcal{B}) \subseteq rad(\mathcal{B})$. Hence $d(\mathcal{A}) \subseteq rad(\mathcal{B})$. □

Proposition 2.21. *Suppose that $d : \mathcal{A} \to \mathcal{A}$ is a $\sigma -$derivation such that $\sigma^2 = \sigma$ and σ is an endomorphism. If $d\sigma = \sigma d$ then $d^n(\sigma(ab)) = \sum_{k=0}^{n} \binom{n}{k}\, d^{n-k}\sigma(a)\, d^k\sigma(b)$ $(n \in \mathbb{N}$ and $a, b \in \mathcal{A})$. With the convention that $d^0 = id$, the identity operator on $\mathcal{A}$.*

Proof. We consider $\widetilde{\mathcal{A}}_\sigma - algebra$. Clearly $d : \widetilde{\mathcal{A}}_\sigma \to \widetilde{\mathcal{A}}_\sigma$ is a derivation. According to part (i) of Proposition 18.4 of [1], we have $d^n(a \bullet b) = \sum_{k=0}^{n} \binom{n}{k}\, d^{n-k}(a) \bullet d^k(b)$ for all $a, b \in \widetilde{\mathcal{A}}_\sigma$. Therefore

$$\begin{aligned} d^n(\sigma(ab)) &= \sum_{k=0}^{n} \tbinom{n}{k} \sigma(d^{n-k}(a)\ d^k(b)) \\ &= \sum_{k=0}^{n} \tbinom{n}{k} \sigma d^{n-k}(a)\ \sigma d^k(b) \\ &= \sum_{k=0}^{n} \tbinom{n}{k} d^{n-k}\sigma(a)\ d^k\sigma(b). \end{aligned}$$

□

Theorem 2.22. *Suppose that $d : \mathcal{A} \to \mathcal{A}$ is a continuous $\sigma -$ derivation such that σ is an endomorphism and $\sigma^2 = \sigma$. If $d\sigma = \sigma d$ and $d\sigma$ is continuous then $e^d\sigma$ is a continuous endomorphism and e^d is a continuous bijective mapping on $\mathcal{A}$.*

Proof. First, we define a linear mapping d_1 by the following form: $d_1^0 = \sigma$ and $d_1 = d\sigma$. Clearly $d_1^n = d^n\sigma$ for all non-negative integer n. It follows from Proposition 2.21 that $d_1^n(ab) = \sum_{k=0}^{n} \binom{n}{k} d_1^{n-k}(a) d_1^k(b)$ for all $a, b \in \mathcal{A}$. We have

$$\begin{aligned} e^{d_1} &= \sum_{n=0}^{\infty} \frac{d_1^n}{n!} = \sigma + \sum_{n=1}^{\infty} \frac{d_1^n}{n!} \\ &= \sigma + \sum_{n=1}^{\infty} \frac{(d\sigma)^n}{n!} \\ &= \sigma + \sum_{n=1}^{\infty} \frac{d^n\sigma}{n!} \\ &= (id + \sum_{n=1}^{\infty} \frac{d^n}{n!})\sigma \\ &= e^d\sigma. \end{aligned}$$

Since d_1 is a continuous derivation, Proposition 18.7 of [1] implies that $e^{d_1}(ab) = e^{d_1}(a)\ e^{d_1}(b)$. Therefore $e^d\sigma(ab) = e^d\sigma(a)e^d\sigma(b)$ for all $a, b \in \mathcal{A}$. It means that

$e^{d_1} = e^d\sigma$ is a continuous endomorphism on $\mathcal{A}$. We know that $d : \widetilde{\mathcal{A}}_\sigma \to \widetilde{\mathcal{A}}_\sigma$ is a continuous derivation. By Proposition 18.7 of [1], we obtain $e^d(a \bullet b) = e^d(a) \bullet e^d(b)$, i.e. $e^d : \widetilde{\mathcal{A}}_\sigma \to \widetilde{\mathcal{A}}_\sigma$ is a continuous automorphism. Hence e^d is a continuous bijective mapping on $\mathcal{A}$. □

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