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ON COMPACTNESS IN COMPLEX INTERPOLATION

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ABSTRACT. We show that, in complex interpolation, an operator function that is compact on one side of the interpolation scale will be compact for all proper interpolating spaces if the right hand side (Y^0, Y^1) is reduced to a single space. A corresponding result, in restricted generality, is shown if the left hand side (X^0, X^1) is reduced to a single space. These results are derived from the fact that a holomorphic operator valued function on an open subset of $\mathbb{C}$ which takes values in the compact operators on part of the boundary is in fact compact operator valued.

INTRODUCTION

Let (X^0, X^1) and (Y^0, Y^1) be interpolation pairs of Banach spaces, and let $A\colon X_\Sigma \to Y_\Sigma$ be a linear operator such that $A|_{X^j} \in \mathcal{L}(X^j, Y^j)$ $(j = 0, 1)$ and such that $A|_{X^0}\colon X^0 \to Y^0$ is compact. (For notation concerning interpolation we refer to Section 2.) For many important cases it is known that then the interpolating operator $A|_{X_t}\colon X_t \to Y_t$ resulting from complex interpolation is compact for all $t \in (0, 1)$. In particular, for the case that the pair (Y^0, Y^1) obeys a certain approximation hypothesis, this property was shown by Persson [12]. However, whether this is true in general is an open problem since the fundamental paper of Calderón [2]. For a survey on developments and the state of the art concerning this problem we refer to [3, 5, 6, 14] and the references therein.

In the particular cases where $X^0 = X^1$ or $Y^0 = Y^1$, it was shown already in [10] that the answer is positive, without further requirements on the spaces. It is the purpose of the present paper to present an analogous result for the case that the operator A is replaced by a suitable operator function $A\colon S \to \mathcal{L}(X_\Delta, Y_\Sigma)$,

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where S is the strip

$$S := \{z \in \mathbb{C};\, 0 \leqslant \operatorname{Re} z \leqslant 1\}.$$

Assume that $A(z)\colon X^0 \to Y^0$ is compact for all z in some subset of the imaginary axis, having positive measure. If $Y^0 = Y^1 =: Y$, then we show that $A(z)$ is compact as an operator in $\mathcal{L}(X_{\operatorname{Re} z}, Y)$, for all $z \in \mathring{S}$ (Theorem 2.1). Similarly, if $X^0 = X^1 := X$, the dual of X is separable, and one of the spaces Y^0, Y^1 is reflexive, then we show that $A(z)$ is compact as an operator in $\mathcal{L}(X, Y_{\operatorname{Re} z})$, for all $z \in \mathring{S}$ (Theorem 2.3).

As a preliminary result we show that holomorphic operator valued functions on open subsets of $\mathbb{C}$ which take values in the compact operators on part of the boundary are compact operator valued (Theorem 1.1).

1. On compactness for holomorphic operator functions

Theorem 1.1. *Let X, Y be Banach spaces, and let $Z \subseteq Y'$ be a separating set for Y. Let $\Omega := \big\{z \in \mathbb{C};\, |z| < 1,\ \operatorname{Im} z > 0\big\}$,*

$$T\colon \Omega \cup (-1,1) \to \mathcal{L}(X,Y)$$

bounded, $z \mapsto y'(T(z)x)$ continuous for all $y' \in Z$, $x \in X$, and $T|_\Omega$ holomorphic. Assume that $T(z)$ is a compact operator for all $z \in E$, where $E \subseteq (-1,1)$ is a set of positive measure, and that $T|_{(-1,1)}$ is strongly measurable.

Then $T(z)$ is compact for all $z \in \Omega$.

Proof. Without restriction we may assume that X is separable.

Also, we may assume that T has an extension to $\overline{\Omega}$ such that $z \mapsto y'(T(z)x)$ is continuous on $\overline{\Omega}$ for all $y' \in Z$, $x \in X$. (Replace T by $z \mapsto T(\alpha z)$, for suitable $\alpha \in (0,1)$.)

Let $D := B_{\mathbb{C}}(0,1)$ be the open unit disc in $\mathbb{C}$, let $C := \partial D$ be the unit circle, and let μ be the normalised line measure on C. There exists a homeomorphism $\kappa\colon \overline{\Omega} \to \overline{D}$, such that $\kappa\colon \Omega \to D$ is biholomorphic and such that the setup is transformed into the situation that

$$T\colon \overline{D} \to \mathcal{L}(X,Y)$$

is bounded, $z \mapsto y'(T(z)x)$ is continuous on $\overline{D}$, for all $y' \in Z$ and $x \in X$, that $T|_D$ is holomorphic, that $T(\xi)$ is a compact operator for all $\xi \in E$, where $E \subseteq C$ is a set of positive measure, and that $T|_C$ is strongly measurable.

Let $p\colon D \times C \to (0,\infty)$ be the 2-dimensional Poisson kernel,

$$p(z,\xi) = \frac{1 - |z|^2}{|z - \xi|^2}.$$

For $z \in D$, Poisson's formula implies that

$$T(z) = \int_C T(\xi) p(z,\xi)\, d\mu(\xi).$$

Here, the integral is a strong integral, and the equality is obtained by inserting $x \in X$ and applying functionals $y' \in Z$. We split the integral as $T(z) = T_1(z) + T_2(z)$, with

$$T_1(z) := \int_E T(\xi)p(z,\xi)\,d\mu(\xi),$$
$$T_2(z) := \int_{C\setminus E} T(\xi)p(z,\xi)\,d\mu(\xi) \qquad (z \in D).$$

Then $T_1(z)$ is a compact operator, by the strong convex compactness property of the compact operators; cf. [16; Theorem 1.3]. We estimate $T_2(z)$ by

$$\|T_2(z)\| \leqslant \int_{C\setminus E} \|T(\xi)\|p(z,\xi)\,d\mu(\xi)$$

(noting that $\xi \mapsto \|T(\xi)\|$ is measurable, by the separability of X). Now it follows that $\|T_2(r\xi)\| \to 0$ as $r \to 1-$, for $\xi \in E$ a.e.; see, e.g., [8; Corollary 2 of Theorem 1.2].

Let $\eta \in \mathcal{L}(X,Y)'$ be vanishing on the compact operators, and define

$$\phi(z) := \eta(T(z)) \qquad (z \in D).$$

Then ϕ is bounded and holomorphic, and the previous considerations show that $\phi(r\xi) = \eta(T_2(r\xi)) \to 0$ $(r \to 1-)$ for $\xi \in E$ a.e. This implies that $\eta(T(z)) = \phi(z) = 0$ $(z \in D)$, by [13; Theorem 17.18]. Since this is true for any η as above, the Hahn-Banach theorem implies that $T(z)$ is compact for all $z \in D$. $\square$

Remarks 1.2. (a) In the proofs of Theorems 2.1 and 2.3 we will apply Theorem 1.1 with the interior of the strip S instead of the set Ω. The transformation to this situation is achieved by a suitable biholomorphic mapping.

(b) If, in the hypothesis of Theorem 1.1, one assumes more strongly that T is operator norm continuous on $\Omega \cup (-1,1)$, then the proof is easier, as one can use the last step of the proof directly: Proceeding as above one obtains a function ϕ which is continuous on $\overline{D}$ and vanishes on the set $E \subseteq C$ of positive measure. This implies that $\phi = 0$, by [13; Theorem 17.18]. It is the main point of Theorem 1.1 to deal with a less restrictive hypothesis for T concerning the behaviour on $(-1,1)$.

(c) The hypotheses of Theorem 1.1 are somewhat reminiscent of the setting used in [2; 9.6].

(d) If $Z \subseteq Y'$ is an *almost norming* subspace (i.e.,

$$\|y\|_Z := \sup\{|y'(y)|;\, y' \in Z,\ \|y'\| \leqslant 1\} \qquad (y \in Y),$$

defines a norm $\|\cdot\|_Z$ which is equivalent to the norm on Y), then the hypothesis that $T|_{(-1,1)}$ is strongly measurable can be replaced by the requirement that $T(\cdot)x$ is essentially separably valued on $(-1,1)$, for all $x \in X$. This is a consequence of a slight strengthening of Pettis' theorem on measurability as described in [7; Section II.1, Corollary 4].

2. Compact interpolation of operator functions

In this section we assume that (X^0, X^1) and (Y^0, Y^1) are interpolation pairs of Banach spaces (cf. [1; Sect. 2.3], [11; Introduction]).

We recall the notation $X_\Delta := X^0 \cap X^1$, $X_\Sigma := X^0 + X^1$, and we assume that $\check{X}$ is a dense subspace of X_Δ. For the definition of the complex interpolation spaces we recall the space

$$\mathcal{F}_0(X) := \big\{ f \in C_0(S; X_\Sigma);\ f \text{ holomorphic on } \mathring{S}, \\ (s \mapsto f(j + \mathrm{i}s)) \in C_0(\mathbb{R}; X^j)\ (j = 0, 1) \big\},$$

with the norm given by

$$|||f||| := \sup\big\{ \|f(j + \mathrm{i}s)\|_j;\ s \in \mathbb{R},\ j = 0, 1 \big\}.$$

For $z \in S$, the interpolation space X_z is defined by

$$X_z := \big\{ f(z);\ f \in \mathcal{F}_0(X) \big\},$$

with norm

$$\|x\|_z := \inf\big\{ |||f|||;\ f \in \mathcal{F}_0(X),\ f(z) = x \big\}.$$

(Usually, the spaces X_t are only considered for $0 \leqslant t \leqslant 1$; note that evidently $X_z = X_{\operatorname{Re} z}$ for all $z \in S$.) Finally, we define

$$A_0(S) := \big\{ \phi \in C_0(S);\ \phi\big|_{\mathring{S}} \text{ holomorphic} \big\},$$

and we recall that

$$\check{\mathcal{F}}_0(X, \check{X}) := \operatorname{lin}\big\{ \phi x;\ \phi \in A_0(S),\ x \in \check{X} \big\}$$

is a dense subspace of $\mathcal{F}_0(X)$; cf. [2] or [1; Lemma 4.2.3].

In addition to the previous assumptions we assume that we are given a family $(A(z))_{z \in S}$ of linear mappings $A(z) \colon \check{X} \to Y_\Sigma$, with the following properties:

(i) For all $x \in \check{X}$ the function $A(\cdot)x \colon S \to Y_\Sigma$ is continuous, bounded, and holomorphic on $\mathring{S}$;

(ii) for $x \in \check{X}$, $j = 0, 1$, the function $\mathbb{R} \ni s \mapsto A(j + \mathrm{i}s)x \in Y^j$ is continuous, and

$$M_j := \sup\big\{ \|A(j + \mathrm{i}s)x\|_{Y^j};\ s \in \mathbb{R},\ x \in \check{X},\ \|x\|_{X^j} \leqslant 1 \big\} < \infty.$$

If these conditions are satisfied, then, for all $z \in S$, the operator $A(z)$ has a unique extension to an operator $A(z) \in \mathcal{L}(X_z, Y_z)$, and this operator satisfies $\|A(z)\|_{\mathcal{L}(X_z, Y_z)} \leqslant M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$. We refer to [4] (or [15; Theorem 2.1], [11; Theorem 2.7]) for the proof of this statement.

Theorem 2.1. *In the previous setup, assume that $Y^0 = Y^1 =: Y$, and assume additionally that there exists a subset $E \subseteq \mathbb{R}$ of positive measure such that $A(\mathrm{i}s) \in \mathcal{L}(X_0, Y)$ is a compact operator, for all $s \in E$.*

Then the operator $A(z) \in \mathcal{L}(X_z, Y)$ is compact for all $z \in \mathring{S}$.

Proof. The hypotheses imply that for $f \in \check{\mathcal{F}}(X, \check{X})$, the function $z \mapsto A(z)f(z)$ belongs to $\mathcal{F}_0(Y) := \mathcal{F}_0((Y, Y))$. Moreover, the mapping

$$f \mapsto A(\cdot)f(\cdot)$$

is continuous with respect to the norms on $\mathcal{F}_0(X)$ and $\mathcal{F}_0(Y)$, which implies that it has a unique extension to $\mathcal{F}_0(X)$. As $A(z) \in \mathcal{L}(X_z, Y)$ for all $z \in S$, this extension is given pointwise by $A(z)f(z)$ $(f \in \mathcal{F}_0(X))$. This shows that the function $z \mapsto A(z)f(z)$ belongs to $\mathcal{F}_0(Y)$, for all $f \in \mathcal{F}_0(X)$.

Now we define the operator function $T\colon S \to \mathcal{L}(\mathcal{F}_0(X), Y)$ by

$$T(z)f := A(z)f(z) \qquad (f \in \mathcal{F}_0(X),\ z \in S).$$

Because $\mathcal{F}_0(X) \ni f \mapsto f(z) \in X_z$ is a continuous linear operator, the hypothesis of the theorem implies that $T(\mathrm{i}s)$ is compact for all $s \in E$. Further, the properties mentioned above imply that T is holomorphic on $\mathring{S}$, strongly continuous on S, and $T(z)$ tends strongly to 0 as $|z| \to \infty$.

Therefore the application of Theorem 1.1 yields that $T(z)$ is compact for all $z \in \mathring{S}$. For $z \in S$, the operator $\mathcal{F}_0(X) \ni f \mapsto f(z) \in X_z$ is a quotient map (in particular, maps the unit ball of $\mathcal{F}_0(X)$ onto the unit ball of X_z), and therefore the definition of $T(z)$ implies that $A(z) \in \mathcal{L}(X_z, Y)$ is compact for all $z \in \mathring{S}$. $\square$

Remarks 2.2. (a) For the case that the pair (Y^0, Y^1) obeys an approximation hypothesis, Calderón [2; 10.4] presented a result corresponding to Theorem 2.1. We just mention another noteworthy difference between the hypotheses in the results: In [2] the function A is required to be operator norm continuous (in a certain sense) on S.

(b) One might ask whether, analogously to the case of a single operator, one could also treat the case that $X^0 = X^1$ whereas the interpolation pair (Y^0, Y^1) is not reduced to a single space.

This would be highly desirable. Indeed, a glance at the proof of Theorem 2.1 shows that this would mean that one could also treat the general case where both interpolation pairs are non-trivial. In particular, this would solve the long-standing open problem mentioned above, concerning a single operator. We did not see how to apply Theorem 1.1 to this case, in the general setup. In Theorem 2.3 below we present a result of this kind, under more restrictive hypotheses.

Theorem 2.3. *In the setup of the beginning of this section, assume that $X^0 = X^1 =: X$. Further assume that one of the spaces Y^0, Y^1 is reflexive, that there exists a subset $E \subseteq \mathbb{R}$ of positive measure such that $A(\mathrm{i}s) \in \mathcal{L}(X, Y^0)$ is a compact operator for all $s \in E$, and that the function $\mathbb{R} \ni s \mapsto A(\mathrm{i}s)' \in \mathcal{L}(Y^{0\prime}, X')$ is strongly measurable.*

Then the operator $A(z) \in \mathcal{L}(X, Y_z)$ is compact for all $z \in \mathring{S}$.

Proof. It is easy to see that, without restriction, one may assume that Y_Δ is dense in Y^0 and Y^1, and thus $Y_j = Y^j$ $(j = 0, 1)$. ¿From [2; 9.5, 12.1 and 12.2] we then know that the space Y_z is reflexive and that $(Y')_z = (Y_z)'$ (with the obvious notation), for all $z \in \mathring{S}$.

For all $x \in X$, $y' \in Y_0' \cap Y_1' = (Y')_\Delta = (Y_\Sigma)'$, we obtain that the function

$$z \mapsto \langle x, A(z)'y' \rangle = \langle A(z)x, y' \rangle$$

is bounded and continuous on S and holomorphic on $\mathring{S}$. Therefore, for all $f \in \check{\mathcal{F}}_0(Y', (Y')_\Delta)$, the function

$$z \mapsto \langle x, A(z)'f(z) \rangle$$

belongs $C_0(S)$, is holomorphic on $\mathring{S}$, and we have the estimate

$$|\langle x, A(z)'f(z) \rangle| \leqslant \max\{M_0, M_1\} \|x\| \|\!| f |\!\|.$$

The denseness of $\check{\mathcal{F}}_0(Y', (Y')_\Delta)$ in $\mathcal{F}_0(Y')$ implies that the last properties carry over to all $f \in \mathcal{F}_0(Y')$. As in the proof of Theorem 2.1 we now define $T\colon S \to \mathcal{L}(\mathcal{F}_0(Y'), X')$ by

$$T(z)f := A(z)'f(z) \qquad (f \in \mathcal{F}_0(Y'),\ z \in S).$$

Then we can apply Theorem 1.1 to the function T. Indeed, $X \subseteq X''$ is a norming subspace for X', the continuity and holomorphy assumptions hold (for the holomorphy we refer to [9; Chapter III, Remark 1.38]), $T(\mathrm{i}s)\colon f \mapsto A(\mathrm{i}s)'f(\mathrm{i}s)$ is compact because $f \mapsto f(\mathrm{i}s)$ is continuous and the dual operator $A(\mathrm{i}s)' \in \mathcal{L}(Y_0', X')$ of the compact operator $A(\mathrm{i}s) \in \mathcal{L}(X, Y_0)$ is compact for all $s \in E$, and the measurability of $\mathbb{R} \ni s \mapsto A(\mathrm{i}s)'f(\mathrm{i}s)$, for $f \in \mathcal{F}_0(Y')$, follows from the strong measurability of $\mathbb{R} \ni s \mapsto A(\mathrm{i}s)'$ and the continuity of $\mathbb{R} \ni s \mapsto f(\mathrm{i}s)$.

The application of Theorem 1.1 then yields that $T(z)$ is compact for all $z \in \mathring{S}$, and as in the proof of Theorem 2.1 we conclude that $A(z)' \in \mathcal{L}((Y')_z, X') = \mathcal{L}((Y_z)', X')$ is compact. By Schauder's theorem we obtain that $A(z) \in \mathcal{L}(X, Y_z)$ is a compact operator. $\square$

Remark 2.4. In Theorem 2.3, the assumption of strong measurability follows from the remaining hypotheses if the dual X' of X is separable. This is a consequence of Remark 1.2(d).

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