Ann. Funct. Anal. 3 (2012), no. 1, 19–31
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

p-POWER QUASICONCAVITY OF A REARRANGEMENT INVARIANT FUNCTION SPACE

CHONGSUNG LEE[1*] AND KYUGEUN CHO[2]

Communicated by C. P. Niculescu

ABSTRACT. We define the p-power quasiconcave function and show relationships between p-power quasiconcave fundamental function and r.i. spaces like Lorentz space and Marcinkiewicz space.

1. INTRODUCTION

Let E be a rearrangement invariant function space (or r.i. space, in short), which consists of measurable functions defined on a measure space (Ω, Σ, μ). The general theory on r.i. space can be found in [4]. In this paper, we focus on r.i. space defined on a positive real line with Legesque measure since our objectivity is to investigate relationships between the fundamental function $\varphi_E = \|\chi_{[0,t]}\|_E$ and some geometric properties of a space E. We say a positive function $\varphi(t)$ on the positive real line is called quasiconcave if $\varphi(t)$ is positive and nondecreasing and $\varphi(t)/t$ is nonincreasing. In [6], it has been revealed that the necessary and sufficient condition of a positive function $\varphi(t)$ is a fundamental function of r.i. space $\varphi(t)$ that is quasiconcave and $\varphi(0) = 0$. The followings are the known results of a positive concave function $\varphi(t)$ on $[0, \infty)$ and their proof can be found in [3] and [5].

Property 1.1. A positive function $\varphi(t)$ is equivalent to a positive concave function if and only if

$$\varphi(t) \leq C \max\left(1, \frac{t}{s}\right) \varphi(s) \qquad \text{for all } s, t > 0.$$

Date: Received: 7 October 2011; Accepted: 30 December 2011.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46B03; Secondary 46B04, 46B42.
Key words and phrases. p-power concavity, upper p-estimate, rearrangement invariant space.

In particular, when $C = 1$, there exists a concave function $\widetilde{\varphi}(t)$ such that

$$\frac{1}{2}\widetilde{\varphi}(t) \leq \varphi(t) \leq \widetilde{\varphi}(t).$$

Indeed, we can find a concave function which is equivalent to a given quasiconcave function.

Property 1.2. If $\varphi(t)$ is positive and everywhere finite on $(0, \infty)$, which satisfies $\varphi(t_1 \cdot t_2) \leq \varphi(t_1) \cdot \varphi(t_2)$, then we get

$$\lim_{t\to\infty} \frac{\log \varphi(t)}{\log t} = \inf_{1<t} \frac{\log \varphi(t)}{\log t} = \overline{\alpha}$$

and

$$\lim_{t\to 0} \frac{\log \varphi(t)}{\log t} = \inf_{t<1} \frac{\log \varphi(t)}{\log t} = \underline{\alpha}$$

Furthermore, we have $-\infty < \underline{\alpha} \leq \overline{\alpha} < \infty$.

Definition 1.3. For a given positive, everywhere finite function $\varphi(t)$ on $(0, \infty)$, we define the function $D_\varphi(s)$, which is called the dilation function of $\varphi(t)$, by

$$D_\varphi(s) = \sup_{0<t<\infty} \frac{\varphi(st)}{\varphi(t)}.$$

When $\varphi(t)$ is quasiconcave, it is easy to show that $D_\varphi(s)$ is everywhere finite and satisfies the submultiplicativity condition of Property 1.2. Therefore, we can define the following two indices, which were introduced by Zippin for the fundamental function $\varphi_E(t)$ of r.i. space E (see [8]).

Definition 1.4. Let $\varphi(t)$ be a positive quasiconcave function. We then define two indices, $\overline{r}(\varphi)$ and $\underline{r}(\varphi)$, which will be called the upper and lower indices of $\varphi(t)$, by

$$\overline{r}(\varphi) = \lim_{t\to\infty} \frac{\log D_\varphi(t)}{\log t} = \inf_{1<t} \frac{\log D_\varphi(t)}{\log t}$$

and

$$\underline{r}(\varphi) = \lim_{t\to 0} \frac{\log D_\varphi(t)}{\log t} = \sup_{0<t<1} \frac{\log D_\varphi(t)}{\log t}.$$

The following simple facts are also mentioned in [8] for the case of a fundamental function $\varphi_E(t)$ and [3].

Property 1.5. Let $\varphi(t)$ be a quasiconcave function. We then have

i) $0 \leq \underline{r}(\varphi) \leq \overline{r}(\varphi) \leq 1$
ii) $\underline{r}(\varphi) + \overline{r}(\varphi) = 1$
iii) If $\Psi(t)$ is equivalent to $\varphi(t)$, then $\overline{r}(\Psi) = \overline{r}(\varphi)$ and $\underline{r}(\Psi) = \underline{r}(\varphi)$.

Property 1.6. Let $\varphi(t)$ be nondecreasing and $D_\varphi(s_1) \leq s_1$ for some $s_1 > 1$. Then there exists a concave function $\Psi(t)$ which is equivalent to $\varphi(t)$.

2. p-Power quasiconcave function

Definition 2.1. Let $\varphi(t)$ be a positive quasiconcave function on $[0,\infty)$. We say that $\varphi(t)$ is p-power quasiconcave with a constant C if it satisfies

$$\sum_{i=1}^{n} \varphi^p(a_i) \leq C\varphi^p\left(\sum_{i=1}^{n} a_i\right), \qquad \text{for all } a_i \text{ in } [0,\infty). \tag{2.1}$$

In particular, if $\varphi(t)$ is concave and satisfies (2.1), we say that $\varphi(t)$ is p-power concave with a constant C.

It is clear that the concave function $x^{1/p}$ is p-power concave. In the following, we consider some conditions, which are useful in showing the existence of nontrivial p-power quasiconcave functions.

Lemma 2.2. i) *If $\varphi(t)$ is a p-power quasiconcave function with a constant C, then $\varphi(t)$ is also q-power quasiconcave with a constant $C^{q/p}$ when $p \leq q$.*

ii) *Let $\varphi(t)$ satisfy (2.1) and $\Psi(t)$ be a positive nondecreasing function. Then $\varphi(t)\Psi(t)$ also satisfies (2.1).*

iii) *Let $\varphi(t)$ be a positive p-power quasiconcave function on $[0,\infty)$. If $\Psi(t)$ is equivalent to $\varphi(t)$, then $\Psi(t)$ also satisfies (2.1).*

Proof. i) It is clear since $\|\cdot\|_{l_p} \geq \|\cdot\|_{l_q}$ for $p \leq q$. Indeed, we have

$$\left\{\sum_{i=1}^{n} \varphi^q(a_i)\right\}^{1/q} \leq \left\{\sum_{i=1}^{n} \varphi^p(a_i)\right\}^{1/p} \leq C^{1/p}\varphi\left(\sum_{i=1}^{n} a_i\right).$$

Hence,

$$\sum_{i=1}^{n} \varphi^q(a_i) \leq C^{q/p}\varphi^q\left(\sum_{i=1}^{n} a_i\right).$$

ii)

$$\begin{aligned}\sum_{i=1}^{n} \varphi^p(a_i)\Psi^p(a_i) &\leq \left(\sum_{i=1}^{n} \varphi^p(a_i)\right)\Psi^p\left(\sum_{i=1}^{n} a_i\right)\\ &\leq C\varphi^p\left(\sum_{i=1}^{n} a_i\right)\Psi^p\left(\sum_{i=1}^{n} a_i\right)\end{aligned}$$

iii) Suppose that $C_1\varphi(t) \leq \Psi(t) \leq C_2\varphi(t)$. Then,

$$\begin{aligned}\sum_{i=1}^{n} \Psi^p(a_i) &\leq \sum_{i=1}^{n} C_2^p\varphi^p(a_i) \leq C_2^p \cdot C\varphi^p\left(\sum_{i=1}^{n} a_i\right)\\ &\leq C\cdot\left(\frac{C_2}{C_1}\right)^p\Psi^p\left(\sum_{i=1}^{n} a_i\right).\end{aligned}$$

□

Theorem 2.3. *Let $\varphi(t)$ be a positive quasiconcave function with $\overline{r}(\varphi) < 1$ and $w(t)$ be a quasiconcave function. If $\Psi(t) = \varphi(t)w^{\alpha}(t)$ for $0 \le \alpha < 1 - \overline{r}(\varphi)$, then $\Psi(t)$ is quasiconcave. Furthermore, if $\varphi(t)$ is p-power quasiconcave, there exists a concave function $\Phi(t)$ which is also p-power quasiconcave and equivalent to $\Psi(t)$.*

Proof. By Property 1.6, it is enough to show that $\Psi(t)$ is nondecreasing and $D_{\Psi}(s_1) \le s_1$, for some $s_1 > 1$. Since $\varphi(t)$ and $w(t)$ are quasiconcave and α is nonnegative, $\Psi(t)$ is nondecreasing. In order to show that $D_{\Psi}(s_1) \le s_1$, for some $s_1 > 1$, take ϵ such that $\alpha + \epsilon \le 1 - \overline{r}(\varphi)$. With this ϵ, choose s_1 sufficiently large such that $D_{\varphi}(s_1) \le s_1^{\overline{r}(\varphi)+\epsilon}$ from the definition of upper index of φ. Then, for all $t > 0$,

$$\begin{aligned}
\frac{\Psi(s_1 t)}{\Psi(t)} &= \frac{\varphi(s_1 t)\{w(s_1 t)\}^{\alpha}}{\varphi(t)\{w(t)\}^{\alpha}} \\
&= \frac{\varphi(s_1 t)\{w(s_1 t)/s_1 t\}^{\alpha}(s_1 t)^{\alpha}}{\varphi(t)\{w(t)/t\}^{\alpha} t^{\alpha}} \\
&\le \frac{\varphi(s_1 t)}{\varphi(t)} s_1^{\alpha}, \\
&\quad \text{since } w(t)/t \text{ is nonincreasing and } \alpha \text{ is nonnegative,} \\
&\le D_{\varphi}(s_1) s_1^{\alpha} \\
&\le s_1^{\alpha + \overline{r}(\varphi) + \epsilon} \le s_1
\end{aligned}$$

since $\overline{r}(\varphi) + \alpha + \epsilon \le 1$ and $s_1 \ge 1$. This implies that $D_{\Psi}(s_1) \le s_1$ for some $s_1 > 1$. Hence, $\Psi(t)$ is quasiconcave and p-power quasiconcave by Lemma 2. 2. ii. The existence of a concave function which is equivalent to $\Psi(t)$ is also obtained by Property 1.1 ☐

The above theorem tells us that for a given p-power quasiconcave function $\varphi(t)$, we can construct another p-power quasiconcave function which is nonequivalent to $\varphi(t)$. The next example is a p-power concave function which is not equivalent to $x^{1/p}$. The following concave function has $1/p$ as its lower and upper indices. Thus, the converse of Property 1.5-iii) is not true.

Example 2.4. Let $p > 1$ be fixed. Define

$$\varphi(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 1 + \log t & 1 \le t \end{cases}$$

Define $\Psi(t) = t^{1/p}\varphi(t)$. We now compute the dilation function $D_{\Psi}(s)$ of $\Psi(t)$. When $s > 1$, a simple calculation shows that

$$\frac{\Psi(st)}{\Psi(t)} = \begin{cases} s^{1/p} & 0 < t \le 1/s \\ s^{1/p}(1 + \log st) & 1/s < t \le 1 \\ s^{1/p}(1 + \log st)/(1 + \log t) & 1 < t. \end{cases}$$

Therefore,

$$\begin{aligned} D_\Psi &= \sup_{0<t} \{\Psi(st)/\Psi(t)\} \\ &= \sup_{1/s<t\le 1} s^{1/p}(1+\log st) = s^{1/p}(1+\log s). \end{aligned}$$

From this, we get

$$\begin{aligned} \overline{r}(\Psi) &= \lim_{s\to\infty} \left[\log\left\{s^{1/p}(1+\log s)\right\}/\log s\right] \\ &= 1/p + \lim_{s\to\infty} \log\{1+\log s\}/\log s = 1/p \end{aligned}$$

Since $1/p$ is strictly less than 1 and $\Psi(t)$ is nondecreasing, we take α such that $\overline{r}(\Psi) = \frac{1}{p} < \alpha \le 1$. We then have $D_\Psi(s) \le s^\alpha \le s$ for $s > 1$ hence there is a concave function $\overline{\Psi}(t)$ by Property 1.6. When $s \le 1$, we get

$$\frac{\Psi(st)}{\Psi(t)} = \begin{cases} s^{1/p} & t \le 1 \\ s^{1/p}(1+\log t) & 1 < t \le 1/s \\ s^{1/p}(1+\log st)/(1+\log t) & 1/s < t. \end{cases}$$

Thus, by computation,

$$\begin{aligned} D_\Psi(s) &= \sup_{0<t} \Psi(st)/\Psi(t) \\ &= \sup_{0<t\le 1} \Psi(st)/\Psi(t) = s^{1/p}. \end{aligned}$$

Therefore, $\underline{r}(\Psi) = 1/p$. Finally, $\overline{\Psi}(t)$ is not equivalent to $t^{1/p}$ since $\Psi(t)$ is not equivalent to $t^{1/p}$. Also, $\overline{\Psi}(t)$ is p-power concave by Lemma 2.2.

Theorem 2.5. *Let $\varphi(t)$ be a p-power quasiconcave function with a constant C and a lower index $\underline{r}(\varphi)$. Then, $1/\underline{r}(\varphi) \le p$.*

Proof. By taking t for each a_i in (2.1), we get

$$D_\varphi(1/n) \le C^{1/p}(1/n)^{1/p}.$$

Dividing both sides by $\log(1/n) < 0$, we get

$$\frac{D_\varphi(1/n)}{\log(1/n)} \ge \frac{(1/p)\log C}{\log(1/n)} + \frac{(1/p)\log(1/n)}{\log(1/n)}.$$

Letting n go to ∞, we have $\underline{r}(\psi) \ge 1/p$. □

From this result, we know that if $\varphi(t)$ is p-power quasiconcave, then we have $p \ge 1$ by Property 1.5. We now consider conditions under which a given quasiconcave function $\varphi(t)$ is p-power quasiconcave.

Lemma 2.6. *Let $\varphi(t)$ be a positive quasiconcave function with $\underline{r}(\varphi) > 0$. Then there exists a constant $1 \le C$ and a positive quasiconcave differentiable function $\Psi(t)$ with $\Psi(0) = 0$ such that*

$$\varphi(t) \le \Psi(t) \le C\varphi(t)$$

and

$$\frac{1}{C}\frac{\Psi(t)}{t} \le \frac{d\Psi}{dt}(t) \le \frac{\Psi(t)}{t}.$$

Proof. Define $\Psi(t) = \int_0^t \frac{\varphi(x)}{x}dx$ and let $\widetilde{\varphi}(x) = \frac{x}{\varphi(x)}$.

$\Psi(t)$ is nondecreasing and we show that $\frac{\Psi(t)}{t}$ is nonincreasing. We have, for $t_1 \le t_2$,

$$\begin{aligned} \frac{\Psi(t_1)}{t_1} &= \frac{1}{t_1}\int_0^{t_1} \frac{\varphi(x)}{x}dx \\ &\ge \frac{1}{t_1}\int_0^{t_1} \frac{\varphi(t_1)}{t_1}dx = \frac{\varphi(t_1)}{t_1}. \end{aligned}$$

Thus,

$$\begin{aligned} \Psi(t_2) &= \int_0^{t_2} \frac{\varphi(x)}{x}dx \\ &= \int_0^{t_1} \frac{\varphi(x)}{x}dx + \int_{t_1}^{t_2} \frac{\varphi(x)}{x}dx \\ &\le \Psi(t_1) + \frac{\varphi(t_1)}{t_1}(t_2 - t_1) \\ &\le \Psi(t_1) + \frac{\Psi(t_1)}{t_1}(t_2 - t_1) \\ &= \frac{t_2}{t_1}\Psi(t_1). \end{aligned}$$

Therefore, we have shown that $\Psi(t)$ is qusiconcave. We now show that $\Psi(t)$ is equivalent to $\varphi(t)$. Since $\underline{r}(\varphi) > 0$, we know that $\overline{r}(\widetilde{\varphi}) < 1$ by Property 1.5. We take α such that $\overline{r}(\widetilde{\varphi}) < \alpha < 1$. By definition of $\overline{r}(\widetilde{\varphi})$, there exists $M > 1$ such that

$$D_{\widetilde{\varphi}}(s) \le s^{\alpha} \qquad \text{for} \quad s > M.$$

Now, let t be fixed and let $\beta = tM$. If x is in $(0, t)$ we have $M < \frac{\beta}{x}$. Thus we get

$$\frac{\widetilde{\varphi}(\beta)}{\widetilde{\varphi}(x)} \le D_{\widetilde{\varphi}}\left(\frac{\beta}{x}\right) \le \left(\frac{\beta}{x}\right)^{\alpha}$$

and

$$\frac{\widetilde{\varphi}(t)}{\widetilde{\varphi}(\beta)} \le 1.$$

We then have

$$\begin{aligned}\Psi(t) &= \int_0^t \frac{1}{\widetilde{\varphi}(x)}dx = \frac{1}{\widetilde{\varphi}(t)} \cdot \frac{\widetilde{\varphi}(t)}{\widetilde{\varphi}(\beta)} \int_0^t \frac{\widetilde{\varphi}(\beta)}{\widetilde{\varphi}(x)}dx \\ &\leq \frac{1}{\widetilde{\varphi}(t)} \int_0^{\frac{\beta}{M}} \left(\frac{\beta}{x}\right)^{\alpha} dx \\ &= \frac{\beta^{\alpha}}{\widetilde{\varphi}(t)} \frac{1}{1-\alpha} \left(\frac{\beta}{M}\right)^{1-\alpha} \\ &= \frac{\beta}{1-\alpha} \cdot \frac{1}{M^{1-\alpha}} \cdot \frac{1}{\widetilde{\varphi}(t)} \\ &= \frac{M^{\alpha}}{1-\alpha} \frac{t}{\widetilde{\varphi}(t)} = \frac{M^{\alpha}}{1-\alpha}\varphi(t).\end{aligned}$$

Now take $C = \frac{M^{\alpha}}{1-\alpha}$ then we have the right inequality.
For the left inequality, the nonincreasing property of $\frac{\varphi(t)}{t}$ gives

$$\varphi(t) = \frac{t}{\widetilde{\varphi}(t)} = \int_0^t \frac{dx}{\widetilde{\varphi}(t)} \leq \int_0^t \frac{dx}{\widetilde{\varphi}(x)} = \Psi(t).$$

By definition of $\Psi(t)$, we have $\frac{d\Psi(t)}{dt} = \frac{\varphi(t)}{t}$. Thus,

$$\frac{1}{C}\Psi(t) \leq \varphi(t) = t \cdot \frac{d\Psi(t)}{dt} \leq \Psi(t)$$

and we get the result by dividing the above inequality by t. □

Theorem 2.7. *Let $\varphi(t)$ be a quasiconcave function with $\underline{r}(\varphi) > 0$. Then there exists a finite p such that $\varphi(t)$ becomes a p-power concave function.*

Proof. By Lemma 2.2, it is enough to show that $\Psi(t) = \int_0^t \frac{\varphi(x)}{x}dx$ satisfies (2.1) since $\Psi(t)$ is equivalent to $\varphi(t)$ by Lemma 2.6. First, we want to show that $\frac{\Psi^p(t)}{t}$ is nondecreasing for some finite p. Note that $\Psi(t)$ is differentiable. We then have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\Psi^p(t)}{t}\right) &= \frac{p\Psi^{p-1}(t)\Psi'(t) \cdot t - \Psi^p(t)}{t^2} \\ &= \frac{\Psi^{p-1}(t)}{t^2}\left\{p\Psi'(t) \cdot t - \Psi(t)\right\} \\ &\geq \frac{\Psi^{p-1}(t)}{t^2}\left\{\frac{p\Psi(t)}{C} - \Psi(t)\right\} \qquad \text{by Lemma 2.6} \\ &= \frac{\Psi^p(t)}{t^2}\left(\frac{p}{C} - 1\right).\end{aligned}$$

Hence, for $1 \leq C < p < \infty$, $\frac{d}{dt}\left(\frac{\Psi^p(t)}{t}\right) \geq 0$ and so $\frac{\Psi^p(t)}{t}$ is nondecreasing. We now show that $\Psi(t)$ satisfies (2.1).

$$\begin{aligned}\Psi^p\left(\sum_{i=1}^n a_i\right) &= \sum_{i=1}^n a_i\left\{\Psi^p\left(\sum_{i=1}^n a_i\right)\Big/\sum_{i=1}^n a_i\right\}\\ &\geq \sum_{i=1}^n a_i\left\{\Psi^p(a_i)/a_i\right\}\\ &= \sum_{i=1}^n \Psi^p(a_i).\end{aligned}$$

□

Theorem 2.8. *Let $\varphi(t)$ be the fundamental function of r.i. space E.*

i) Suppose that $\varphi(t)$ is p_1-power quasiconcave. If E satisfies an upper q-estimate, then $q \leq p_1$.

ii) Suppose that $\widetilde{\varphi}(t) = \frac{t}{\varphi(t)}$ is equivalent to $\Psi(t)$ which is p_2-power quasiconcave. If E satisfies a lower q estimate, then $q \geq \widetilde{p}_2$ where $1/p_2 + 1/\widetilde{p}_2 = 1$.

Proof. i) Since $\varphi(t)$ is p_1-power quasiconcave and the fundamental function of r.i.-space, there exist constant C_1 and C_2 such that

$$\frac{1}{C_1}n^{1/p_1} \leq \varphi(n) \leq C_2 n^{1/q}$$

for all integers n. Thus we have $q \leq p_1$.

ii) Since $\widetilde{\varphi}(t)$ is equivalent to the p_2-power concave function, there exists C_3 such that

$$\widetilde{\varphi}(\frac{1}{n}) = \left(\frac{1}{n}\right)\varphi\left(\frac{1}{n}\right) \leq C_3\left(\frac{1}{n}\right)^{1/p_2}.$$

By the lower q estimate property, we have C_4 such that

$$\varphi(\frac{1}{n}) \leq C_4\left(\frac{1}{n}\right)^{1/q}.$$

Thus,

$$\frac{1}{C_3}\left(\frac{1}{n}\right)^{1-(1/p_2)} \leq \varphi\left(\frac{1}{n}\right) \leq C_4\left(\frac{1}{n}\right)^{1/q}$$

for all integers. Thus, we have $1/q \leq 1-(1/p_2) = 1/\widetilde{p}_2$.

□

3. Main result

We now apply the p-power quasiconcave property to some r.i. space like Lorentz space and Marcinkiewicz space. Although there are several versions of these spaces, we take Sharpley's version with minor modifications [7]. Let f be a real valued function defined on $[0,\infty)$ with Lebegue measure μ. The distribution function of f, denoted by $\lambda_f(t)$, is defined by $\mu(\{x : |f(x)| > t\})$. We define $f^*(t)$

is the non-increasing, right continuous function which is equimeasurable with f and $f^{**}(t) = \frac{1}{t}\int_0^t f^*(x)dx$.

For an explicit formula of $f^*(t)$, we have $f^*(t) = \inf\{y \geq 0 : \lambda_f(y) \leq t\}$ [1].

Definition 3.1. Let $\varphi(t)$ be a quasiconcave function on $[0,\infty)$ with $0 < \underline{r}(\psi) \leq \overline{r}(\psi) < 1$ and $1 \leq q < \infty$.

The Lorentz space $\Lambda_{\psi,q}$ is the set of all measurable functions f such that f^* exists and

$$\|f\|_{\Lambda_{\psi,q}} = \left\{\int_0^\infty (f^{**}(t)\psi(t))^q \frac{dt}{t}\right\}^{1/q} < \infty.$$

The Marcinkiewicz space M_ψ is the set of all measurable functions f such that

$$\|f\|_{M_\psi} = \sup_{t>0} \psi(t) f^{**}(t) < \infty.$$

By definition, we can easily show that $\|f\|_{\Lambda_{\varphi,q}}$ and $\|f\|_{M_\varphi}$ are equivalent to $\|f\|_{\Lambda_{\psi,q}}$ and $\|f\|_{M_\psi}$ respectively when φ and ψ are equivalent. It is well known that $\int_0^t f^*(x)dx = \sup_{\mu(E)=t}\int_A |f|dx$ (See page 64 in [3]). Thus we have an alternate form of $\|f\|_{M_\varphi}$:

$$\|f\|_{M_\psi} = \sup_{\mu(E)>0} \frac{\psi(\mu(E))}{\mu(E)} \int_A |f| d\mu.$$

For the space $\Lambda_{\psi,q}$, we have also useful form of its norm

$$\|f\|^*_{\Lambda_{\psi,q}} = \left\{\int_0^\infty [f^*(t)\psi(t)]^q \frac{dt}{t}\right\}^{1/q},$$

which is equivalent to $\|f\|_{\Lambda_{\psi,q}}$ (see [7, Theorem 2.3]).

We now introduce some functional which is convenient to compute the norm in $\Lambda_{\Psi,q}$. We modify the proof in Sharpley's version of Lorentz space (see [3, 7]).

Lemma 3.2. *i) Let f be an element in $\Lambda_{\psi,q}$. Then, the functional*

$$\|f\|^0_{\Lambda_{\psi,q}} = \left\{\int_0^\infty ((f^*(t))^q \, d\psi^q\right\}^{1/q}$$

is equivalent to $\|f\|_{\Lambda_{\psi,q}}$.
ii) The fundamental function $\varphi_{\Lambda_{\psi,q}}$ in $\Lambda_{\psi,q}$ is equivalent to $\psi(t)$.
iii) If $f \in \Lambda_{\psi,q}$, we have

$$\|f\|^0_{\Lambda_{\psi,q}} = \left\{q\int_0^\infty y^{q-1}\left[\psi^q(\lambda_f(y)\right] dy\right\}^{1/q}.$$

Proof. i) By Lemma2.6, $\psi(t)$ has an equivalent quasiconcave differentiable function $\Psi(t) = \int_0^t \frac{\psi}{x}dx$ and we can renorm $\Lambda_{\psi,q}$ by replacing $\psi(t)$ with $\Psi(t)$. Thus we may assume, for some constant C,

$$\frac{1}{C}\frac{\psi(t)}{t} \leq \frac{d\psi}{dt}(t) \leq \frac{\psi(t)}{t}.$$

Then, by the improper Stieltjes integral, we have

$$\begin{aligned}\left\{\|f\|^0_{\Lambda_{\psi,q}}\right\}^q &= \psi(+0)(f^*(+0))^q + \int_{+0}^{\infty}(f^*(t))^q d\psi^q, \\ &\quad \text{since } \ 0<\underline{r}(\psi),\ \text{ it is easy to get } \lim_{s\to 0}\psi(s)=0, \\ &= \int_0^\infty (f^*(t))^q q\psi^{q-1}(t)\frac{d\psi(t)}{dt}t\frac{dt}{t}.\end{aligned}$$

Since $\frac{q}{C}\psi^q(t) \le q\psi^{q-1}\cdot\frac{d\psi(t)}{dt}\cdot t \le q\psi^q(t)$, we have

$$\frac{q}{C}\|f\|^*_{\Lambda_{\psi,q}} \le \|f\|^0_{\Lambda_{\psi,q}} \le q\|f\|^*_{\Lambda_{\psi,q}}.$$

Thus $\|f\|^0_{\psi_{\Lambda,q}}$ is equivalent to $\|f\|_{\Lambda_{\psi,q}}$.

ii) Since $\|\chi_{[0,t]}\|^0_{\Lambda_{\psi,q}} = \left\{\int_0^t d\psi^q\right\}^{1/q} = \psi(t)$, the fundamental function $\varphi_{\Lambda_{\psi,q}}$ is equivalent to $\psi(t)$.

iii) Since simple functions are dense in $\Lambda_{\psi,q}$ (see [2, 7]), we show the result for a simple function $f=\sum_{i=1}^n a_i\chi_{A_i}$, where $\{A_i\}$ are pairwise disjoint measurable sets. Without loss of generality, we may assume $a_1>\cdots>a_n$. Define $d_i=\sum_{j=1}^i \mu(A_i)$ and $d_0=0$. Then, we have

$$\lambda_f(y) = \begin{cases} d_i & a_{i+1}\le y<a_i \\ 0 & a_1\le y\end{cases}$$

Hence,

$$\begin{aligned}\|f\|^0_{\Lambda_{\psi,q}} &= \left\{\int_0^\infty (f^*(t))^q d\psi^q\right\}^{1/q} \\ &= \left\{\sum_{i=1}^n a_i^q\left[\psi^q(d_i)-\psi^q(d_{i-1})0\right]\right\}^{1/q}\end{aligned}$$

On the other hand,

$$\begin{aligned}&\left(q\int_0^\infty y^{q-1}\left\{\psi[\lambda_f(y)]\right\}^q dy\right)^{1/q} \\ =\ &\left\{\psi^q(d_n)a_n^q+\psi^q(d_{n-1})(a_{n-1}^q-a_n^q)+\cdots+\psi^q(d_1)(a_1^q-a_2^q)\right\}^{1/q} \\ =\ &\left(\sum_{i=1}^n a_1^q\left[\psi^q(d_i)-\psi^q(d_{i-1})\right]\right)^{1/q} \\ =\ &\|f\|^0_{\Lambda_{\psi,q}}\end{aligned}$$

□

Theorem 3.3. *If $\psi(t)$ is p-power quasiconcave with constant C, the space $\Lambda_{\psi,q}$ satisfies a lower-p-estimate for $1\le q<p$.*

Proof. Since a lower estimate is metric invariant, we use the equivalent functional $\|f\|^0_{\Lambda_{\psi,q}}$ by Lemma 3.2. Let $\{f_i\}$ be elements in $\Lambda_{\psi,q}$ with pairwise disjoint support. Note that $\sum_{i=1}^n \lambda_{f_i}(y) = \lambda \sum_{i=1}^n f_i(y)$.

By Lemma 3.2, we have

$$\begin{aligned}
\left\{\sum_{i=1}^n \left(\|f_i\|^0_{\Lambda_{\psi,q}}\right)^p\right\}^{1/p} &= \left(\sum_{i=1}^n \left\{\int_0^\infty \psi^q(\lambda_{f_i}(y))dy^q\right\}^{p/q}\right)^{1/p} \\
&= \left(\sum_{i=1}^n \left\{\int_0^\infty \left[\psi^p(\lambda_{f_i(y)}\right]^{q/p} dy^q\right\}^{p/q}\right)^{1/p} \\
&\le \left(\int_0^\infty \left\{\sum_{i=1}^n \psi^p(\lambda_{f_i}(y)\right\}^{q/p} dy^q\right)^{1/q} \\
&\quad \text{since } \sum_{i=1}^n \|f_i\|_{L_r} \le \left\|\sum_{i=1}^n f_i\right\|_{L_r} \text{ when } r<1 \\
&\le \left(\int_0^\infty \left\{C\psi^p\left[\sum_{i=1}^n \lambda_{f_i}(y)\right]\right\}^{q/p} dy^q\right)^{1/q} \\
&\quad \text{since } \psi \text{ is } p\text{-power quasiconcave} \\
&= C^{1/p}\left(\int_0^\infty \left\{\psi\left[\sum_{i=1}^n \lambda_{f_i}(y)\right]\right\}^q dy^q\right)^{1/q} \\
&= C^{1/p}\left(\int_0^\infty \left\{\psi\left(\lambda_{\sum_{i=1}^n f_i}(y)\right)\right\}^q dy^q\right)^{1/q} \\
&= C^{1/p}\left\|\sum_{i=1}^n f_i\right\|^0_{\Lambda_{\psi,q}}
\end{aligned}$$

□

Theorem 3.4. *Let $\psi(t)$ be quasiconcave such that $\widetilde{\psi}(t) = \frac{t}{\psi(t)}$ is q-power quasiconcave with constant C. Then M_ψ satisfies an upper p-estimate where $\frac{1}{p}+\frac{1}{q}=1$.*

Proof. Let $\{f_i\}_{i=1}^n$ be a set of measurable functions with disjoint supports $\{E_i\}_{i=1}^n$ respectively. Let E be any measurable set in $[0,\infty)$. Define $F_i = A\bigcap E_i$. Since each E_i's are disjoint, we know $\sum_{i=1}^n \mu(F_i) \le \mu(E)$. Thus

$$\begin{aligned}
\frac{\psi(\mu(E))}{\mu(E)} \int_E \sum_{i=1}^m f_i &= \frac{\psi(\mu(E))}{\mu(E)} \sum_{i=1}^m \int_E f_i \\
&\le \frac{\psi(\mu(E))}{\mu(E)} \sum \frac{\mu(F_i)}{\psi(\mu(F_i))} \|f_i\|_{M_\psi} \\
&\le \frac{\psi(\mu(E))}{\mu(E)} \left\{ \sum_{i=1}^n \left(\frac{\mu(F_i)}{\psi(\mu(F_i))} \right)^q \right\}^{1/q} \left\{ \sum_{i=1}^m \|f_i\|_{M_\psi}^p \right\}^{1/p} \\
&\le \frac{\psi(\mu(E))}{\mu(E)} \left\{ \sum_{i=1}^n \widetilde{\psi} \left(\sum_{i=1}^m \mu(F_i) \right)^q \right\}^{1/q} \left\{ \sum_{i=1}^m \|f_i\|_{M_\psi}^p \right\}^{1/p} \\
&= \frac{\psi(\mu(E))}{\mu(E)} \left\{ \sum_{i=1}^n \widetilde{\psi} \left(\sum_{i=1}^m \mu(F_i) \right) \right\} \left\{ \sum_{i=1}^n \|f_i\|_{M_\psi}^p \right\}^{1/p} \\
&= C \frac{\psi(\mu(E))}{\mu(E)} \frac{\sum_{i=1}^n \mu(F_i)}{\psi\left(\sum_{i=1}^n \mu(F_i)\right)} \left\{ \sum_{i=1}^n \|f_i\|_{M_\psi}^p \right\}^{1/p} \\
&\le C \left\{ \sum_{i=1}^n \|f_i\|_{M_\psi}^p \right\}^{1/p},
\end{aligned}$$

since $\frac{\psi(t)}{t}$ is nonincreasing and $\sum_{i=1}^n \mu(F_i) \le \mu(E)$.

We thus have

$$\left\| \sum_{i=1}^n f_i \right\|_{M_{psi}} \le C \left\{ \sum_{i=1}^n \|f_i\|_{M_\psi}^p \right\}^{1/p}.$$

□

The following are easily obtained from the above theorems.

Corollary 3.5. *Weak L_p satisfies an upper p-estimate.*

Corollary 3.6. *Let $\psi(t)$ be a quasiconcave with $\overline{r}(\psi) < 1$. Then there exists p such that $1 < p < \infty$ and M_ψ satisfies an upper p-estimate.*

Acknowledgement. This work was supported by the Inha University Research Grant.

References

1. M. Cwikel, *The dual of weak L_p*, Ann. Inst. Fourier, Grenoble **25** (1975), 81–126
2. R.A. Hunt, *On $L_{p,q}$ spaces*, Enseignement Math. **12** (1966), 249–276.
3. S. Krein, J. Petunim and E. Semenov, *Interpolation of linear operators*, English transl., Transl. Math. Mono. vol 54, AMS Providence, R.I. 1980

4. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II, Function Spaces*, Ergeb. der Math. u. ihr. Grenzgeb, 97, Berlin, Springer, 1979
5. J. Peetree, *Concave majorants of positive functions*, Acta. Math. Acad. Sci. Hunger **21** (1970), 327–333
6. E.M. Semenov, *Embedding theorems for Banach spaces of measurable functions* (Russian), Dokl. Akad. Nauk SSSR **156** (1964), 1292–1295.
7. R. Sharpley, *Spaces $\Lambda_\alpha(X)$ and interpolation*, J. Functional Analysis **11** (1972), 479–513
8. M. Zippin, *Interpolation of operators of weak type between rearrangement invariant function spaces*, J. Funct. Anal. **7** (1971), 267–284

[1] Department of Mathematics Education, Inha University, Inchon 402-751, Korea.

E-mail address: cslee@inha.ac.kr

[2] Bangmok College of General Education, Myong Ji University, Yong-In 449-728, Korea.

E-mail address: kgjo@mju.ac.kr