Annals of Functional Analysis
Ann. Funct. Anal. 3 (2012), no. 1, 49–66
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

ADVANCES IN ALMOST CONVERGENCE

CHAO YOU

Communicated by G. Androulakis

ABSTRACT. In this paper, we first give the concept of properly distributed sequence, and prove that it is almost convergent with F-limit expressed as a formal integral. Basing on these, we review the work of Feng and Li, which is shown to be a special case of our generalized theory. Then we generalize Banach limit to Banach limit functional, which is the minimum requirement to characterize strong almost convergence for bounded sequences in normed vector space. With this machinery, we show that Hajduković's almost convergence and quasi-almost convergence are both equivalent to our strong almost convergence.

1. INTRODUCTION AND PRELIMINARIES

Let l^∞ be the Banach space of bounded sequences of real numbers $x := \{x(n)\}_{n=1}^\infty$ with supremum norm $\|x\|_\infty := \sup_n |x(n)|$. As an application of Hahn-Banach theorem, a *Banach limit* L is a bounded linear functional on l^∞, which satisfies the following properties:

(i) If $x = \{x(n)\}_{n=1}^\infty \in l^\infty$ and $x(n) \geq 0$, then $L(x) \geq 0$;

(ii) If $x = \{x(n)\}_{n=1}^\infty \in l^\infty$ and $Tx = \{x(2), x(3), \ldots\}$, where T is the *left-shift operator*, then $L(x) = L(Tx)$;

(iii) $\|L\| = 1$;

(iv) If $x = \{x(n)\}_{n=1}^\infty \in c$, where c is the Banach subspace of l^∞ consisting of convergent sequences, then $L(x) = \lim_{n\to\infty} x(n)$.

Since the Hahn-Banach norm-preserving extension is not unique, there must be many Banach limits in the dual space of l^∞, and usually different Banach limits have different values at the same element in l^∞. However, there indeed exist sequences whose values of all Banach limits are the same. Condition (iv) is a

Date: Received: 28 October 2011; Accepted: 8 January 2012.
2010 *Mathematics Subject Classification.* Primary 46B45; Secondary 11K36.
Key words and phrases. Banach limit functional, properly distributed sequence, strong almost convergence.

trivial example. Besides that, there also exist nonconvergent sequences satisfying this property, for such examples please see [1] and [2]. In [2], G. G. Lorentz called a sequence $x = \{x(n)\}_{n=1}^{\infty}$ *almost convergent*, if all Banach limits of x, $L(x)$, are the same, and this unique Banach limit is called *F-limit* of x. In his paper, Lorentz proved the following criterion for almost convergent sequences:

Theorem 1.1 ([2])**.** *A sequence $x = \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ is almost convergent with F-limit $L(x)$ if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=i}^{i+n-1} x(t) = L(x)$$

uniformly in i.

There is no doubt that Lorentz's theorem is a landmark in Banach limit theory, which in theory points out all the almost convergent sequences. Since then, convergence and summability of sequences and applications have become an active research field with fruitful results [3, 4, 5]. Recently, basing on Lorentz [2] and Sucheston [6], B. Q. Feng and J. L. Li gave another way [1] to find the value of Banach limits of x, where x is an element of the space of almost convergent sequences with some properties. In the first part of this paper, we will make a remark on the concept of essential subsequence (Definition 2, [1]), then cite Theorem 4([1]) to develop our theory, and at last use our theory to review two main results in [1], in the bid to include [1] into our framework and show that we have genuinely done a work of generalization in theory.

Similar to Theorem 1.1, recently D. Hajduković[7] and S. Shaw et al[8] generalized the concept of almost convergence to bounded sequences in normed vector space and bounded continuous vector-valued functions, respectively.

Suppose $(V, \|\cdot\|_V)$ is a complex normed vector space. Let $l^{\infty}(V)$ be the normed vector space of bounded V-valued sequences $x := \{x_n\}_{n=1}^{\infty}$ with supremum norm $\|x\|_{\infty} := \sup_n \|x_n\|_V$. In particular, $c(V)$ is the subspace of $l^{\infty}(V)$, which consists of convergent V-valued sequences. For any $v \in V$, let $\widetilde{v} := \{v, v, \ldots\}$ denote the sequence with constant entry v, clearly $\widetilde{v} \in c(V)$.

Definition 1.2 ([7])**.** Suppose $x = \{x_n\}_{n=1}^{\infty} \in l^{\infty}(V)$ and $v \in V$. $\{x_n\}_{n=1}^{\infty}$ is called *almost convergent* to v if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} = v$$

uniformly in j.

Let $C_b([0,\infty), V)$ be the normed vector space of bounded V-valued continuous functions f with supremum norm $\|f\| := \sup_{t\in[0,\infty)} \|f(t)\|_V$.

Definition 1.3 ([8])**.** Suppose $f \in C_b([0,\infty), V)$ and $v \in V$. $f(t)$ is called *almost convergent* to v when $t \to \infty$ if

$$\lim_{t\to\infty} \frac{1}{t} \int_a^{a+t} f(s)ds = v$$

uniformly in a.

In [7], Hajduković also gave the concept of *quasi-almost convergence* in terms of some kind of linear functionals, which are similar to Banach limit in the real sequence case. First, Hajduković defined a semi-norm q on $l^\infty(V)$ as follows:

For $x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$,

$$q(x) = \overline{\lim_{n\to\infty}} \left(\sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+jn} \right\|_V \right).$$

And then, he showed that there exists *the* family Π of nontrivial linear functionals L defined on $l^\infty(V)$ such that for all $x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$, the following assertions are valid:

(i) $L(Tx) = L(x)$;

(ii) $|L(x)| \le q(x)$;

(iii) $\forall L \in \Pi$, $L(x - \widetilde{v}) = 0$ if and only if $q(x - \widetilde{v}) = 0$.

Definition 1.4 ([7])**.** A sequence $x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$ is called *quasi-almost convergent* to $v \in V$ if $\forall L \in \Pi$, $L(x - \widetilde{v}) = 0$.

Similar to the definition of almost convergence, Hajduković gave the following equivalent characterization of quasi-almost convergence:

Theorem 1.5 ([7])**.** *Suppose $x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$ and $v \in V$. $\{x_n\}_{n=1}^\infty$ is quasi-almost convergent to v if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+jn} = v$$

uniformly in j.

From this theorem, it seems that quasi-almost convergence is weaker than almost convergence. However, in the second part of this paper, we will show that actually they are equivalent! In Section 3, we will first define the concept of *Banach limit functional*, which is a generalization of Banach limit in bounded real sequence case but much simpler, even than Hajduković's linear functionals Π. To show the existence and sufficiency of Banach limit functionals, we provide a natural construction of Banach limit functionals induced from $B_1(V^*)$. Then we will give an equivalent characterization of Banach limit functional, which shows that some items in traditional or Hajduković's definition of Banach limit are equivalent or one could imply another, so it is unnecessary to place them together in the definition.

Then, in terms of Banach limit functionals, we define the concept of *strong almost convergence*, and show that it is equivalent to almost convergence in [7], then it is immediate that Hajduković's quasi-almost convergence is equivalent to almost convergence too. We also show that our almost convergence is stronger than that of J. Kurtz's[9], so that's why we call it strong almost convergence. Some basic properties of strong almost convergence are also discussed. In particular, we show that though strong almost convergence is weaker than norm convergence, corresponding completenesses with respect to the two convergences are the same.

Finally, we would like to point out that all definitions and results here could be applied to bounded continuous functions exactly word by word from summation to integration. Thus, to save space, we avoid to state them again.

2. Distribution and almost convergence of bounded sequences

Let us first give a brief introduction to the main results of [1], making the notations and terminologies available.

Definition 2.1 (Definition 1, [1]). A real number a is said to be a sub-limit of the sequence $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$, if there exists a subsequence $\{x(n_k)\}_{k=1}^{\infty}$ of x with limit a. The set of all sub-limits of x is denoted by $S(x)$ and the set of all limit points of $S(x)$ is denoted by $S'(x)$.

Definition 2.2 (Definition 3, [1]). Let $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$, and let $\{x(n_k)\}_{k=1}^{\infty}$ be a subsequence of x. Define

$$w^u(\{x(n_k)\}) = \limsup_{n\to\infty}\left(\sup_i \frac{A(\{k : i \le n_k \le i+n-1\})}{n}\right)$$

and

$$w_l(\{x(n_k)\}) = \liminf_{n\to\infty}\left(\inf_i \frac{A(\{k : i \le n_k \le i+n-1\})}{n}\right),$$

where $A(E)$ is the cardinality of the set E. $w^u(\{x(n_k)\})$ and $w_l(\{x(n_k)\})$ are called the upper and lower weights of the subsequence $\{x(n_k)\}_{k=1}^{\infty}$, respectively. If $w^u(\{x(n_k)\}) = w_l(\{x(n_k)\})$, then the subsequence $\{x(n_k)\}_{k=1}^{\infty}$ is said to be weightable and the weight of $\{x(n_k)\}_{k=1}^{\infty}$ is denoted by $w(\{x(n_k)\})$, and $w(\{x(n_k)\}) = w^u(\{x(n_k)\}) = w_l(\{x(n_k)\})$.

Remark 2.3. It should be emphasized that our Definition 2.2 is slightly different from Definition 3([1]), with $\lim_{n\to\infty}$ there replaced by $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ for $w^u(\cdot)$ and $w_l(\cdot)$, respectively. Such expression is more accurate, since there is no reason to guarantee the existence of $\lim_{n\to\infty}$.

Definition 2.4 (Definition 2, [1]). Suppose $a \in S(x)$ for some $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$. A subsequence $\{x(n_k)\}_{k=1}^{\infty}$ of x is called an essential subsequence of a if it converges to a, and for any subsequence $\{x(m_t)\}_{t=1}^{\infty}$ of x with limit a, except finite entries, all its entries are entries of $\{x(n_k)\}_{k=1}^{\infty}$.

Theorem 2.5 (Theorem 1, [1]). *Let $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$. Suppose $a \in S(x)$. Let $\{x(n_k)\}_{k=1}^{\infty}$ and $\{x(m_t)\}_{t=1}^{\infty}$ be two essential subsequences of a. Then $w^u(\{x(n_k)\}) = w^u(\{x(m_t)\})$ and $w_l(\{x(n_k)\}) = w_l(\{x(m_t)\})$.*

Theorem 2.5 points out that, for $a \in S(x)$, all essential subsequences of a have the same upper weight and lower weight, respectively. They are called the *upper* and *lower weights* of a in the sequence x, and denoted by $w^u(a)$ and $w_l(a)$, respectively. The *weight* of a in the sequence x is denoted by $w(a)$, if $w^u(a) = w_l(a)$.

We remark that not every sub-limit $a \in S(x)$ has an essential subsequence. The following proposition shows that this happens only when a is an isolated sub-limit of x. This is an important erratum to [1], and consideration on this problem directly leads to our present work.

Proposition 2.6. [1] *Let $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and suppose $a \in S(x)$. a has an essential subsequence if and only if a is an isolated sub-limit of x.*

Proof. If a is an isolated sub-limit of x, then there exists $\varepsilon_0 > 0$ such that $(a-\varepsilon_0, a+\varepsilon_0) \bigcap S(x) = \{a\}$. Let $\{x(n_k)\}$ denote all the terms of x that lying in $(a-\varepsilon_0, a+\varepsilon_0)$, we will show that $\{x(n_k)\}$ is the desired essential subsequence of x. Since $\{x(n_k)\}$ is infinite and bounded, it must have at least one convergent subsequence or sub-limit. But a is an isolated sub-limit, hence $\{x(n_k)\}$ has just one sub-limit, i.e., a. That's to say $\{x(n_k)\}$ is convergent to a. For any subsequence $\{x(m_t)\}$ of x that converging to a, from the definition of $\{x(n_k)\}$ and convergence of $\{x(m_t)\}$ to a, all of the terms of this subsequence under consideration, except finite number of them, must be in $\{x(n_k)\}$. So $\{x(n_k)\}$ is an essential subsequence of a.

Conversely, suppose that a has an essential subsequence $\{x(n_k)\}$. Assume a is not an isolated sub-limit of sequence x, then there exist a sequence of sub-limits $\{a_n\}$ that converges to a. We know, for each a_n from $\{a_n\}$, there is a subsequence $\{x_n^i\}$ that converges to a_n when $i \to \infty$. Without loss of generality, we can assume $0 < d_n = |a - a_n| < 1/n$. Then, for each n, we can find y_n from $\{x_n^i\}$ such that y_n doesn't lie in $\{x(n_k)\}$ and $|y_n - a_n| < 1/n$. Actually, this construction is possible. Since a and a_n are distinct with distance d_n, then we can find positive integer N_1 and N_2 such that, when $k > N_1$, $i > N_2$, it holds that $|x(n_k) - a| < d_n/3$ and $|x_n^i - a_n| < d_n/3$, respectively. It is easy to see such y_n can be found and satisfying $|y_n - a| < d_n < 1/n$. Here we have constructed a subsequence $\{y_n\}$ converging to a, but not lying in the essential subsequence $\{x(n_k)\}$, which leads to a contradiction. □

Remark 2.7. Since in [1] they just considered sequences with isolated sub-limits, or a little complex case with only one limit point, this ambiguous treatment of essential subsequences wouldn't lead to serious mistakes.

The following theorem is the most important result of [1], which will be cited and reviewed later.

Theorem 2.8 (Theorem 4, [1])**.** *Suppose $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and $S(x) = \{a_1, a_2, \dots, a_m\}$ is a finite set, where $a_i \neq a_j$ if $i \neq j$. Then*

$$\sum_{0<a_j\in S(x)} a_j w_l(a_j) + \sum_{0>a_j\in S(x)} a_j w^u(a_j) \le L(x) \le \sum_{0<a_j\in S(x)} a_j w^u(a_j) + \sum_{0>a_j\in S(x)} a_j w_l(a_j).$$

[1]Special thanks goes to Prof. J. L. Li for discussion with him on this proposition. In fact, it was him that first pointed out this proposition and provided a proof for the sufficient condition.

If $w(a_j)$ exists for each j, then x is almost convergent and for any Banach limit L, $L(x) = \sum_{j=1}^{m} a_j w(a_j)$.

This form of $L(x) = \sum_{j=1}^{m} a_j w(a_j)$ is much like the *integration sum* in measure and integration theory, so we ask the question whether the unique Banach limit value of almost convergent sequence could be expressed as an integral form? Previous work shows this is related to the distribution of values appearing in the sequence. In [10], the concept of *uniform distribution of sequences* was introduced as following: Suppose $x \in l^\infty$ is a $[0,1]$-valued sequence, i.e. $0 \leq x(n) \leq 1$ for each $n \in \mathbb{N}$. x is called *uniformly distributed* if for any $[a,b) \subseteq [0,1]$,

$$\lim_{N\to\infty} \frac{A(\{n \in \mathbb{N} : x(n) \in [a,b), n \leq N\})}{N} = b - a.$$

Now we want to generalize the concept of distribution to cover both the uniform and ununiform cases.

Definition 2.9. A sequence $x := \{x(n)\}_{n=1}^{\infty} \in l^\infty$ is called properly distributed if for any Borel subset S of $[-\|x\|_\infty, \|x\|_\infty]$ it holds that

$$\begin{aligned} w(x,S) &= \liminf_{n\to\infty} \frac{A(\{k : x(k+i) \in S, k = 0,1,\ldots,n-1\})}{n} \\ &= \limsup_{n\to\infty} \frac{A(\{k : x(k+i) \in S, k = 0,1,\ldots,n-1\})}{n} \end{aligned}$$

exists uniformly in $i \in \mathbb{N}$ and $w(x,S)$ is called the weight of x with respect to S.

If we treat a properly distributed sequence x as a function defined on $\mathbb{N}$, x is analogous to the measurable function in real analysis, with $w(x,S)$ corresponding to some measure $\mu(\{n : x(n) \in S\})$ over $\mathbb{N}$. Though $w(x,S)$ indeed has some similar behavior as a measure, like nonnegativity and finite additivity, $w(x,S)$ is not a measure in general setting, for it fails to satisfy countable additivity. Here is an illustrating example:

Example 2.10. Let $s_1 = \{\underbrace{1,\ldots,1}_{n-times}, \underbrace{0,0,\ldots}_{otherwise}\}$, which is obviously properly distributed. If there exists a measure μ over $\mathbb{N}$ such that $\mu(\{n : x(n) \in S\}) = w(x,S)$ for any properly distributed sequence $x \in l^\infty$ and Borel subset S, then $\mu(\{1,2,\ldots,n\}) = w(x,[1-\varepsilon,1+\varepsilon)) = 0$, where ε is a sufficiently small positive number. Similarly, it can further be implied that for any finite subset E of $\mathbb{N}$ it always holds $\mu(E) = 0$. Since μ is countably additive and $\mathbb{N}$ is the union of pairwise disjoint finite subsets, it follows that $\mu(\mathbb{N}) = 0$. However, if we set $s_2 = \{1,\ldots,1,\ldots\}$, then s_2 is properly distributed and $\mu(\mathbb{N}) = w(s_2,[1-\varepsilon,1+\varepsilon)) = 1$, which leads to a contradiction. Thus, such measure μ over $\mathbb{N}$ doesn't exist.

From Example 2.10, you may have already realized that s_1 and s_2 represent a simple but useful class of properly distributed sequences. Hence, we naturally give the following definition of *simply distributed sequences*, which would play the similar role as "simple functions" in real analysis.

Definition 2.11. A sequence $s := \{s(n)\}_{n=1}^{\infty} \in l^{\infty}$ is called simply distributed if s is finitely-valued with range $\{a_1, \ldots, a_m\}$ and it holds that

$$\begin{aligned} w(s, a_j) &= \liminf_{n\to\infty} \frac{A(\{k : s(k+i) = a_j, k =, 0, 1, \ldots, n-1\})}{n} \\ &= \limsup_{n\to\infty} \frac{A(\{k : s(k+i) = a_j, k = 0, 1, \ldots, n-1\})}{n} \end{aligned}$$

exists uniformly in $i \in \mathbb{N}$ for $j = 1, \ldots, m$ and $w(s, a_j)$ is called the weight of s with respect to a_j.

Though we cannot bring our work into the framework of measure and integration(In fact, we really tried to do so at the beginning of our research.), we still find much common feature between them, which suggests us to generalize the measure-integration procedure in real analysis to obtain a *formal* integral to express the unique Banach limit of almost convergent sequence. This would partially answer the open question of [1].

Theorem 2.12. *If $s \in l^{\infty}$ is a simply distributed sequence with finite range $\{a_1, \ldots, a_m\}$, then it is almost convergent with the unique Banach limit $L(s) = \sum_{j=1}^{m} a_j w(s, a_j)$.*

Proof. Let $S(s)$ denote the set of all sub-limits of s. Since s is finitely-valued, we have $S(s) \subseteq \{a_1, \ldots, a_m\}$ is finite. Moreover, if $a_j \notin S(s)$, then a_j must appear finite times in s with $w(s, a_j) = 0$. Hence, by Theorem 4 of [1], it implies that s is almost convergent and for any Banach limit L, $L(s) = \sum_{j=1}^{m} a_j w(s, a_j)$. □

From Theorem 2.12, we can see that for any simply distributed sequence s, its unique Banach limit could be expressed as formal integral $L(s) = \sum_{j=1}^{m} a_j w(s, a_j)$. Then it naturally arises the question whether it is still true for general properly distributed sequences. To this end, we'd like to generalize the procedure of integration in real analysis. Firstly, let us approximate properly distributed sequences by simply distributed sequences.

Lemma 2.13. *For any properly distributed element $x \in l^{\infty}$, there is a sequence of simply distributed elements $\{s_k\}_{k=1}^{\infty} \subseteq l^{\infty}$ such that $\lim_{k\to\infty} s_k = x$ under the norm $\|\cdot\|_{\infty}$ in l^{∞}.*

Proof. For $k \in \mathbb{N}$, there is a partition

$$T_k : -\|x\|_{\infty} = a_0 < \ldots < a_{m_k} = \|x\|_{\infty}$$

of $[-\|x\|_{\infty}, \|x\|_{\infty}]$ such that $\|T_k\| < 1/k$. Define

$$s_k(n) = \begin{cases} a_0, & \text{if } a_0 \le x(n) < a_1, \\ \cdots & \cdots, \\ a_{m_k-1}, & \text{if } a_{m_k-1} \le x(n) < a_{m_k}. \end{cases} \qquad n = 1, 2, 3, \ldots$$

Since x is properly distributed, it follows easily that each s_k is simply distributed. According to the construction above, it is obvious that $\|s_k - x\|_{\infty} < 1/k$. Thus $\lim_{k\to\infty} s_k = x$. □

Theorem 2.14. *If $x \in l^\infty$ is any properly distributed sequence, then x is almost convergent. And if $\{s_k\}_{k=1}^\infty$ is any sequence of simply distributed sequences convergent to x under the $\|\cdot\|_\infty$ norm, for any Banach limit L, it always holds that $\lim_{k\to\infty} L(s_k) = L(x)$.*

Proof. For any Banach limit L, since L is a bounded linear functional on l^∞ and $\lim_{k\to\infty} s_k = x$, it follows that $\lim_{k\to\infty} L(s_k) = L(x)$. By Theorem 2.12, the value of each $L(s_k)$ is independent of L, thus so is $L(x)$. We conclude that x is almost convergent and the unique Banach limit is $\lim_{k\to\infty} L(s_k)$. □

Now we want to use the new theory to review the work of [1], which will be shown to be a special case in our framework.

Lemma 2.15. *Let $x := \{x(n)\}_{n=1}^\infty \in l^\infty$. Suppose a is an isolated sub-limit of x, and there exists $\varepsilon_0 > 0$ such that $(a-\varepsilon_0, a+\varepsilon_0) \bigcap S(x) = \{a\}$. Then for any $0 < \varepsilon \le \varepsilon_0$, $w(x, [a-\varepsilon, a+\varepsilon))$ exists if and only if $w(a)$ does. Moreover, if they both exist, they are equal.*

Proof. Like Proposition 2.6, for any $0 < \varepsilon \le \varepsilon_0$, let $\{x(n_k)\}$ denote all the terms of x that lying in $[a-\varepsilon, a+\varepsilon)$. Then, similarly, it is easy to show that $\{x(n_k)\}$ is an essential subsequence of a. And, for any $n, i \in \mathbb{N}$, we have

$$\begin{aligned}&\frac{A(\{j : a-\varepsilon \le x(i+j) < a+\varepsilon, j = 0, 1, \dots, n-1\})}{n}\\ =&\frac{A(\{k : i \le n_k \le i+n-1\})}{n}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\limsup_{n\to\infty} \frac{A(\{j : a-\varepsilon \le x(i+j) < a+\varepsilon, j = 0, 1, \dots, n-1\})}{n}\\ =&\limsup_{n\to\infty} \frac{A(\{k : i \le n_k \le i+n-1\})}{n},\end{aligned}$$

and

$$\begin{aligned}&\liminf_{n\to\infty} \frac{A(\{j : a-\varepsilon \le x(i+j) < a+\varepsilon, j = 0, 1, \dots, n-1\})}{n}\\ =&\liminf_{n\to\infty} \frac{A(\{k : i \le n_k \le i+n-1\})}{n}.\end{aligned}$$

Now it is clear that $w(x, [a-\varepsilon, a+\varepsilon))$ exists if and only if $w(a)$ does. And, if they both exist, they are equal. □

Now it's time to include Theorem 4([1]) into our framework.

Theorem 2.16. *Suppose $x := \{x(n)\}_{n=1}^\infty \in l^\infty$ and $S(x) = \{a_1, a_2, \dots, a_m\}$ is a finite set, where $a_i \ne a_j$ if $i \ne j$. If $w(a_j)$ exists for each j, then x is properly distributed.*

Proof. For any interval $[c, d)$, if $[c, d) \bigcap \{a_1, a_2, \dots, a_m\} = \emptyset$, there would be at most finite terms in $[c, d)$, so

$$\begin{aligned} w(x, [c, d)) &= \liminf_{n \to \infty} \frac{A(\{k : x(k+i) \in [c, d), k = 0, 1, \dots, n-1\})}{n} \\ &= \limsup_{n \to \infty} \frac{A(\{k : x(k+i) \in [c, d), k = 0, 1, \dots, n-1\})}{n} \\ &= 0 \end{aligned}$$

exists uniformly in $i \in \mathbb{N}$. Otherwise, there are some a_js in $[c, d)$. Without loss of generality, we can assume only a_j lying $[c, d)$. In fact, if there are more than one such a_j, we can decompose $[c, d)$ into disjoint subintervals such that each contains only one a_j. From Lemma 2.15, since $w(a_j)$ exists, we also have $w(x, [c, d))$ exists, and $w(x, [c, d)) = w(a_j)$. Thus we have proved that x is properly distributed. □

Moreover, we can reobtain the unique Banach limit of x above, using the approximation method by simply distributed sequences.

Corollary 2.17. *Suppose $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and $S(x) = \{a_1, a_2, \dots, a_m\}$ is a finite set, where $a_i \neq a_j$ if $i \neq j$. If $w(a_j)$ exists for each j, then x is almost convergent, with the unique Banach limit $L(x) = \sum_{j=1}^{m} a_j w(a_j)$ for any Banach limit L.*

Proof. For any sufficiently big $k \in \mathbb{N}$, define

$$s_k(n) = \begin{cases} a_j, & \text{if } a_j - 1/k \leq x(n) < a_j + 1/k, \\ x(n), & \text{otherwise.} \end{cases} \quad j = 1, \dots, m; n \in \mathbb{N}.$$

It is easy to see that each s_k is a simply distributed sequence with only $w(s_k, a_j) \neq 0$, and $L(s_k) = \sum_{j=1}^{m} a_j w(s_k, [a_j - 1/k, a_j + 1/k)) = \sum_{j=1}^{m} a_j w(a_j)$. From the construction of $\{s_k\}_{k=1}^{\infty}$, $\lim_{k \to \infty} s_k = x$ under the $\|\cdot\|_{\infty}$ norm. Then it follows that $L(x) = \lim_{k \to \infty} L(s_k) = \sum_{j=1}^{m} a_j w(a_j)$. □

In Theorem 5 and 6 of [1], sequences whose sub-limit sets have limit points are considered. In order to keep the form $L(x) = \sum_{a \in S(x)} a w(a)$, the authors made a great effort to give a complex definition for the weight of limit points of $S(x)$. Now, from our viewpoint of distribution, it is very easy to understand those complex formulae. Let us take Theorem 5 [1] for example, Theorem 6 [1] is treated in a similar way locally at each limit point of $S(x)$.

Theorem 2.18. *Suppose $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and $S(x)$ is infinite but countable and has a unique limit point p, that is $S'(x) = \{p\}$. If, furthermore, $w(a)$ exists for all $a \in S(x)$ and $a \neq p$, then x is properly distributed, and for any Banach limit L, $L(x) = \sum_{a \in S(x)} a w(a)$, where $w(p) = 1 - \sum_{p \neq a \in S(x)} w(a)$.*

Proof. For any sufficiently big $k \in \mathbb{N}$, define

$$s_k(n) = \begin{cases} p, & \text{if } p - 1/k \leq x(n) \leq p + 1/k, \\ a_j, & \text{if } a_j - 1/k \leq x(n) < a_j + 1/k, \text{ and } a_j \notin [p - 1/k, p + 1/k), \\ x(n), & \text{otherwise.} \end{cases}$$

Since there are only finite $a_j \notin [p-1/k, p+1/k)$, each s_k is properly distributed and $\lim_{k\to\infty} s_k = x$. Moreover, from Lemma 2.15, we have

$$L(s_k) = \sum_{a_j\notin[p-1/k,p+1/k)} a_j w(a_j) + p(1 - \sum_{a_j\notin[p-1/k,p+1/k)} w(a_j)).$$

Let $k \to \infty$, it follows that $L(x) = \lim_{k\to\infty} L(s_k) = \sum_{a\in S(x)} aw(a)$, where $w(p) = 1 - \sum_{p\neq a\in S(x)} w(a)$. □

3. Banach limit functional and strong almost convergence of bounded sequences in normed vector space

Similar to Banach limit, we will give its counterpart for bounded sequences in normed vector space as follows:

Definition 3.1. A bounded linear functional L on $l^\infty(V)$ is called a *Banach limit functional* if it satisfies the following two conditions:
(i) $\|L\| \leq 1$;
(ii) $\forall x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$ and $Tx = \{x_2, x_3, \ldots\}$, then $L(Tx) = L(x)$.

To see the existence and sufficiency of Banach limit functionals, let us begin with the following lemma, which is similar to that in Sucheston's paper[6].

Lemma 3.2. $\forall x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$,

$$\lim_{n\to\infty}\left(\sup_j \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V\right)$$

exists.

Proof. Set

$$c_n = \sup_j \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V.$$

We need to show that $\lim_{n\to\infty} c_n$ exists. For each m, n, one has

$$\begin{aligned}\sup_j\left\|\sum_{i=0}^{m+n-1} x_{i+j}\right\|_V &\leq \sup_j\left(\left\|\sum_{i=0}^{m-1} x_{i+j}\right\|_V + \left\|\sum_{i=m}^{m+n-1} x_{i+j}\right\|_V\right)\\ &\leq \sup_j\left\|\sum_{i=0}^{m-1} x_{i+j}\right\|_V + \sup_j\left\|\sum_{i=m}^{m+n-1} x_{i+j}\right\|_V\\ &\leq \sup_j\left\|\sum_{i=0}^{m-1} x_{i+j}\right\|_V + \sup_j\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V,\end{aligned}$$

i.e., $(m+n)c_{m+n} \leq mc_m + nc_n$. Thus

$$(r+km)c_{r+km} \leq rc_r + kmc_{km} \leq rc_r + kmc_m.$$

Dividing by $r + km$ and letting $k \to \infty$ with r, m fixed, we obtain

$$\limsup_{k\to\infty} c_{r+km} \leq c_m.$$

Since this holds for $r = 0, 1, \ldots, m-1$, $\limsup_{n\to\infty} c_n \le c_m$ for each m, and hence $\limsup_{n\to\infty} c_n \le \liminf_{m\to\infty} c_m$, which implies that $\lim_{n\to\infty} c_n$ exists. □

Definition 3.3. For any $x \in l^\infty(V)$, define

$$p(x) = \lim_{n\to\infty}\left(\sup_j \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V\right).$$

From Lemma 3.2, it is easy to see that p is a well-defined seminorm on $l^\infty(V)$.

Lemma 3.4. *If $x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$ such that*

$$\lim_{n\to\infty} \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V = m$$

uniformly in j, then

$$\lim_{n\to\infty}\left(\sup_j \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V\right) = \lim_{n\to\infty} \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V = m.$$

Proof. $\forall \varepsilon > 0$, since

$$\lim_{n\to\infty} \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V = m$$

uniformly in j, there exists $N \in \mathbb{N}$ such that for any $j \in \mathbb{N}$ if $n > N$, then

$$\left|\frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V - m\right| < \varepsilon,$$

i.e.,

$$m - \varepsilon < \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V < m + \varepsilon.$$

Hence

$$m - \varepsilon < \sup_j \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V \le m + \varepsilon.$$

Since ε is arbitrary, it follows that

$$\lim_{n\to\infty}\left(\sup_j \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V\right) = m.$$

□

Lemma 3.5. *If $x = \{x_n\}_{n=1}^\infty \in c(V)$ such that $\lim_{n\to\infty} x_n = 0$, then*

$$\lim_{n\to\infty} \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V = 0$$

uniformly in j.

Proof. $\forall \varepsilon > 0$, since $\lim_{n\to\infty} x_n = 0$, there exists $N_1 \in \mathbb{N}$ such that $\|x_n\|_V < \varepsilon/2$ if $n > N_1$. Choose N_2 such that $(\|x_1\|_V + \|x_2\|_V + \cdots + \|x_{N_1}\|_V)/N_2 < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$. Let $n > N$, for any $j \in \mathbb{N}$, if $j > N_1$, then

$$\frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V \le \frac{\sum_{i=0}^{n-1}\|x_{i+j}\|_V}{n} < \frac{n\varepsilon/2}{n} = \varepsilon/2;$$

if $j \le N_1$,

$$\frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V \le \frac{\sum_{i=0}^{N_1-j}\|x_{i+j}\|_V + \sum_{i=N_1-j+1}^{n-1}\|x_{i+j}\|_V}{n} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V = 0$$

uniformly in j. □

Corollary 3.6. *If $x = \{x_n\}_{n=1}^\infty \in c(V)$ with $\lim_{n\to\infty} x_n = v \in V$, then*

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V = \|v\|_V$$

uniformly in j.

Proof. Since $\lim_{n\to\infty} x_n = v$, i.e., $\lim_{n\to\infty}(x_n - v) = 0$, it follows from Lemma 3.5 that

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j} - nv\right\|_V = 0$$

uniformly in j. Since

$$\left|\frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V - \|v\|_V\right| = \frac{1}{n}\left|\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V - \|nv\|_V\right| \le \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j} - nv\right\|_V,$$

it follows that

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V = \|v\|_V$$

uniformly in j. □

Definition 3.7. Suppose that $f \in V^*$ and $\|f\| \le 1$, define the *induced bounded linear functional* L_f on $c(V)$ as following: for any $x = \{x_n\} \in c(V)$ with $\lim_{n\to\infty} x_n = v \in V$, $L_f(x) = f(v)$.

Proposition 3.8. *For any $x = \{x_n\}_{n=1}^\infty \in c(V)$ with $\lim_{n\to\infty} x_n = v \in V$, $|L_f(x)| \le p(x)$.*

Proof. From Lemma 3.4 and Corollary 3.6, we have

$$|L_f(x)| = |f(v)| \le \|v\|_V = \lim_{n\to\infty}\frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V = \lim_{n\to\infty}\left(\sup_j \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V\right) = p(x).$$

□

Corollary 3.9. $\|L_f\| \leq 1$.

Proof. $\forall x = \{x_n\}_{n=1}^{\infty} \in c(V)$, from Proposition 3.8, $|L_f(x)| \leq p(x) \leq \|x\|_\infty$. So $\|L_f\| \leq 1$. □

From Hahn-Banach Theorem, we know that there must exist a norm-preserving extension $\overline{L}_f$ of L_f on whole $l^\infty(V)$ such that

$$|\overline{L}_f(x)| \leq p(x),$$

$\forall x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V)$. Now we will show that such $\overline{L}_f$ is an example of Banach limit functional as defined in Definition 3.1.

Theorem 3.10. *If $L \in l^\infty(V)^*$ and $x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V)$ such that $|L(x)| \leq p(x)$, then $L(Tx) = L(x)$.*

Proof. Define sequence $y := \{y_n\}_{n=1}^{\infty}$ as $y_n := x_{n+1} - x_n$, i.e., $y = Tx - x$. Since x is bounded, y is also bounded, i.e., $y \in l^\infty(V)$. Then we have

$$\begin{aligned} p(y) &= \lim_{n\to\infty} \left(\sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} (x_{i+j+1} - x_{i+j}) \right\|_V \right) \\ &= \lim_{n\to\infty} \left(\sup_j \frac{1}{n} \|x_{n+j} - x_j\|_V \right) \\ &\leq \lim_{n\to\infty} \frac{2\|x\|_\infty}{n} = 0. \end{aligned}$$

Since $|L(y)| \leq p(y) = 0$, i.e., $L(y) = 0$, we have

$$L(y) = L(Tx - x) = L(Tx) - L(x) = 0,$$

i.e., $L(Tx) = L(x)$. □

So far, we have shown that $\overline{L}_f$ is indeed a Banach limit functional. Since that f is an arbitrary choice from $B_1(V^*)$ and Hahn-Banach norm-preserving extension is not unique, we can see that $l^\infty(V)$ has sufficiently many Banach limit functionals. Let us denote all the Banach limit functionals of $l^\infty(V)$ by $\mathfrak{L}(V)$.

Remark 3.11. Our definition of Banach limit functional here has greatly improved and simplified corresponding definition in D. Hajduković's paper [7]. First of all, you will find that we don't confine V to be only real normed vector space. Actually, since there is no longer positive element in normed vector space, we don't need real scalars. And we also improve the definition of $p(x)$ from $\overline{\lim}_{n\to\infty} \left(\sup_j \frac{1}{n} \left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V\right)$ to $\lim_{n\to\infty} \left(\sup_j \frac{1}{n} \left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V\right)$, which is more accurate. Moreover, due to the following theorem, we will show that suppose $L \in l^\infty(V)^*$ and $\|L\| \leq 1$, for any $x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V)$, $L(Tx) = L(x) \iff |L(x)| \leq p(x)$. Hence we exclude the condition $|L(x)| \leq p(x)$ from Definition 3.1.

Theorem 3.12. *Suppose $L \in l^\infty(V)^*$, the following two statements are equivalent:*

(i)L is a Banach limit functional;

(ii)$|L(x)| \leq p(x)$, $\forall x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V)$.

Proof. (ii)$\Longrightarrow$(i) is exactly Theorem 3.10.

For (i)$\Longrightarrow$(ii), let $c_n = \{c_{n,j}\}_{j=1}^{\infty} \in l^{\infty}(V)$, where $c_{n,j} = \frac{1}{n}\sum_{i=0}^{n-1} x_{i+j}$, i.e., $c_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$. Then for any Banach limit functional L, we have

$$\sup_j \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V = \|c_n\|_\infty \geq |L(c_n)| = |L(\frac{1}{n}\sum_{i=0}^{n-1} T^i x)| = \frac{1}{n}|\sum_{i=0}^{n-1} L(T^i x)| = |L(x)|.$$

Hence $|L(x)| \leq \lim_{n\to\infty}\left(\sup_j \frac{1}{n}\left\|\sum_{i=0}^{n-1} x_{i+j}\right\|_V\right) = p(x)$. □

Classical Banach limit on bounded real sequences is a generalization of ordinary convergence, so item (iv) is always included in the definition of Banach limit. From our view point of Banach limit functional here, Banach limit actually is the Banach limit functional induced from linear functional $f(x) = x$, $\forall x \in \mathbb{R}$. Moreover, the following proposition shows that in some sense item (iv) can be implied from item (ii) and (iii). Hence, our definition of Banach limit functional is essentially a simplification of Banach limit.

Proposition 3.13. *If $L \in \mathfrak{L}(V)$ and $x = \{x_n\}_{n=1}^{\infty} \in c(V)$ with $\lim_{n\to\infty} x_n = v \in V$, then $L(x) = L(\widetilde{v})$.*

Proof. Since $\lim_{n\to\infty}(x_n - v) = 0$, it follows from Lemma 3.4 and Lemma 3.5 that $p(x-\widetilde{v}) = 0$. From Theorem 3.12, $|L(x-\widetilde{v})| \leq p(x-\widetilde{v}) = 0$, i.e., $L(x) = L(\widetilde{v})$. □

Remark 3.14. We remark that in the definition of classical Banach limit of bounded real sequences, item (i)(positivity) could be implied from item (iii) and (iv), so this item could be excluded. Moreover, due to Proposition 3.13, item (iv) could be replaced by (iv') $L(\widetilde{1}) = 1$. We leave the proofs as easy exercises to interested readers.

Now we are ready to define the strong almost convergence in terms of Banach limit functionals.

Definition 3.15. A sequence $x = \{x_n\}_{n=1}^{\infty} \in l^{\infty}(V)$ is called *strongly almost convergent* to $v \in V$ if for any Banach limit functional $L \in \mathfrak{L}(V)$, it holds that $L(x) = L(\widetilde{v})$. Let us denote it by $x_n \xrightarrow{s.a.} v$, and v is called *strong almost limit* of x.

Next we will give an equivalent characterization of strong almost convergence, and show that our strong almost convergence is equivalent to almost convergence given by Hajduković[7]. Moreover, as an immediate corollary, his quasi-almost convergence is equivalent too.

Lemma 3.16. *Suppose $x = \{x_n\}_{n=1}^{\infty} \in l^{\infty}(V)$. $p(x) = 0$ if and only if $L(x) = 0$, $\forall L \in \mathfrak{L}(V)$.*

Proof. If $p(x) = 0$, then $\forall L \in \mathfrak{L}(V)$, it follows from Theorem 3.12 that $|L(x)| \leq p(x) = 0$. Hence $L(x) = 0$.

Conversely. Since $\forall L \in \mathfrak{L}(V)$ $L(x) = 0$, it suffices to find a particular Banach limit functional L_0 such that $L_0(x) = p(x)$. The following is the construction of such L_0. Let $M = \{\lambda x : \lambda \in \mathbb{C}\}$ be a subspace of $l^{\infty}(V)$. On M define

$f_0(\lambda x) = \lambda p(x)$, then $|f_0(y)| = p(y)\ \forall y \in M$. From Hahn-Banach Theorem, we can get an extension L_0 of f_0 on whole $l^\infty(V)$ such that $|L_0(y)| \le p(y)\ \forall y \in l^\infty(V)$. From Theorem 3.12, we know that L_0 is a Banach limit functional. So we are done. □

Remark 3.17. In fact, so far all statements concerning p in this section still hold for q, so we can see that quasi-almost convergence given by Hajduković is actually equivalent to strong almost convergence.

An immediate corollary of Lemma 3.16 is the following important theorem:

Theorem 3.18. *A sequence $x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$ is strongly almost convergent to $v \in V$ if and only if $p(x - \widetilde{v}) = 0$.*

Proposition 3.19. *(i) If $x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$ is strongly almost convergent in V, then its strong almost limit is unique.*

(ii) Suppose $x = \{x_n\}_{n=1}^\infty, y = \{y_n\}_{n=1}^\infty \in l^\infty(V)$. If $x_n \xrightarrow{s.a.} u$ and $y_n \xrightarrow{s.a.} v$, then for any $\lambda, \mu \in \mathbb{C}$, $\lambda x_n + \mu y_n \xrightarrow{s.a.} \lambda u + \mu v$.

(iii) If $\{x_n\}_{n=1}^\infty$ is a sequence from V such that $\lim_{n\to\infty} x_n = v \in V$, then $x_n \xrightarrow{s.a.} v$.

Proof. (i) If $x_n \xrightarrow{s.a.} v_1$ and $x_n \xrightarrow{s.a.} v_2$ simultaneously, then it follows from Theorem 3.18 that $\|v_1 - v_2\|_V = p(v_1 - v_2) \le p(\widetilde{v_1} - x) + p(x - \widetilde{v_2}) = 0 + 0 = 0$. Hence $v_1 = v_2$.

(ii) $p(\lambda x + \mu y - \lambda\widetilde{u} - \mu\widetilde{v}) \le |\lambda| p(x - \widetilde{u}) + |\mu| p(y - \widetilde{v})$.

(iii) From Theorem 3.13. □

Remark 3.20. Please notice that if $x_n \xrightarrow{s.a.} v \in V$, it doesn't mean that each subsequence of $x = \{x_n\}_{n=1}^\infty$ is also strongly almost convergent, let alone strongly almost convergent to the same vector. For example, consider bounded real sequence $x = \{1, 0, 1, 0, \ldots\}$. Then $x_n \xrightarrow{s.a.} 1/2$. However, $\lim_{k\to\infty} x_{2k-1} = 1$, while $\lim_{k\to\infty} x_{2k} = 0$.

Lemma 3.21. *Suppose $x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$ and $p(x) = 0$, then*

$$\lim_{n\to\infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = 0$$

uniformly in j.

Proof. Since $p(x) = 0$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V < \varepsilon$$

when $n > N$. In other words, for any $j \in \mathbb{N}$, when $n > N$

$$\frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V < \varepsilon.$$

So

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{i=0}^{n-1}x_{i+j}\right\|_V=0$$

uniformly in j. □

Theorem 3.22. *Suppose $x=\{x_n\}_{n=1}^{\infty}\in l^{\infty}(V)$. $x_n\xrightarrow{s.a.}v\in V$ if and only if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}x_{i+j}=v$$

uniformly in j.

Proof. $x_n\xrightarrow{s.a.}v\iff p(x-\widetilde{v})=0$. From Lemma 3.21,

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{i=0}^{n-1}(x_{i+j}-v)\right\|_V=\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=0}^{n-1}x_{i+j}-v\right\|_V=0$$

uniformly in j, i.e.,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}x_{i+j}=v$$

uniformly in j. □

This theorem shows that strong almost convergence is equivalent to almost convergence in [7], and so is quasi-almost convergence.

Remark 3.23. In the definition of strong almost convergence, we require $x=\{x_n\}_{n=1}^{\infty}$ to be bounded. Actually this is not constrained, because from

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}x_{i+j}=v$$

uniformly in j, we can easily imply that $\{x_n\}_{n=1}^{\infty}$ is bounded.

Corollary 3.24. *Suppose $x=\{x_n\}_{n=1}^{\infty}\in l^{\infty}(V)$. If $x_n\xrightarrow{s.a.}v\in V$, then $v\in\overline{co}\{x_n:n\in\mathbb{N}\}$.*

Definition 3.25. V is a normed vector space and $A\subseteq V$. A is called *s.a.-sequentially closed* if $\forall\{x_n\}_{n=1}^{\infty}$ from A such that $x_n\xrightarrow{s.a.}v\in V$, then $v\in A$.

Theorem 3.26. *Suppose V is a normed vector space and $A\subseteq V$ is convex. A is (norm) closed if and if A is s.a.-sequentially closed. In particular, a subspace of V is (norm) closed if and if it is s.a.-sequentially closed.*

Proof. Suppose A is s.a.-sequentially closed. If $\{x_n\}_{n=1}^{\infty}\subseteq A$ and $\lim_{n\to\infty}x_n=v\in V$. From Theorem 3.19 (iii), $x_n\xrightarrow{s.a.}v\in A$. Hence A is (norm) closed.

Conversely, suppose A is (norm) closed. If $\{x_n\}_{n=1}^{\infty}\subseteq A$ and $x_n\xrightarrow{s.a.}v\in V$. From Corollary 3.24, $v\in\overline{co}\{x_n:n\in\mathbb{N}\}\subseteq\overline{A}=A$. Hence A is s.a.-sequentially closed. □

Definition 3.27. A bounded sequence $\{x_n\}_{n=1}^{\infty}$ of normed vector space V is called an *s.a.-Cauchy sequence* if for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any $j \in \mathbb{N}$ if $n, m > N$, then $\left\|\frac{1}{n}\sum_{i=0}^{n-1} x_{i+j} - \frac{1}{m}\sum_{i=0}^{m-1} x_{i+j}\right\|_V < \varepsilon$. V is called *s.a.-complete* if every s.a.-Cauchy sequence in V is strongly almost convergent to a vector in V.

Corollary 3.28. *A normed vector space V is (norm) complete if and only if it is s.a.-complete.*

Remark 3.29. This shows that though strong almost convergence is weaker than (norm) convergence, considering completion, it doesn't enlarge the space further.

In the end, we will explain why we use the terminology strong almost convergence.

Definition 3.30 (J. Kurtz[9])**.** Suppose $x = \{x_n\}_{n=1}^{\infty} \in l^{\infty}(V)$. We say that $x = \{x_n\}_{n=1}^{\infty}$ is *weakly almost convergent* to $v \in V$ if for any $f \in V^*$, $\widehat{f}(x) := \{f(x_n)\}_{n=1}^{\infty} \in l^{\infty}(\mathbb{C})$ is almost convergent to $f(v)$. Let us denote it by $x_n \xrightarrow{w.a.} v$.

Remark 3.31. From the definition, it is immediate that any weakly convergent sequence is weakly almost convergent to its weak limit.

Theorem 3.32. *Suppose $x = \{x_n\}_{n=1}^{\infty} \in l^{\infty}(V)$ and $v \in V$. If $x_n \xrightarrow{s.a.} v$, then $x_n \xrightarrow{w.a.} v$.*

Proof. From Theorem 3.18, we just need to show that for any $f \in V^*$, $p(\widehat{f}(x) - \widetilde{f(v)}) = 0$. Since $p(x - \widetilde{v}) = 0$, we have

$$\begin{aligned} p(\widehat{f}(x) - \widetilde{f(v)}) &= \lim_{n\to\infty}\left(\sup_j \frac{1}{n}\left|\sum_{i=1}^{n-2}(f(x_{i+j}) - f(v))\right|\right) \\ &= \lim_{n\to\infty}\left(\sup_j \frac{1}{n}\left|f\left(\sum_{i=1}^{n-2}(x_{i+j} - v)\right)\right|\right) \\ &\leq \lim_{n\to\infty}\left(\sup_j \frac{1}{n}\|f\|\left\|\sum_{i=1}^{n-2}(x_{i+j} - v)\right\|_V\right) \\ &= \|f\| p(x - \widetilde{v}) = 0. \end{aligned}$$

□

Theorem 3.33 (J. Kurtz[9])**.** *Suppose $x = \{x_n\}_{n=1}^{\infty} \in l^{\infty}(V)$ and $v \in V$. If $\{x_n : n \in \mathbb{N}\}$ is precompact and $x_n \xrightarrow{w.a.} v$, then $x_n \xrightarrow{s.a.} v$.*

Remark 3.34. When $V = \mathbb{C}$, strong almost convergence and weak almost convergence coincide, since each bounded sequence in $\mathbb{C}$ is precompact. So we just say almost convergence there.

References

1. B.Q. Feng and J. L. Li, *Some estimations of Banach limits*, J. Math. Anal. Appl. **323** (2006), no. 1, 481–496.

2. G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math., **80** (1948), 167–190.
3. M. Mursaleen, *On some new invariant matrix methods of summability*, Quart. Jour. Math. Oxford (2) , **34** (1983), no. 1, 77–86.
4. Z.U. Ahmad and M. Mursaleen, *An Application of Banach limits*, Proc. Amer. Math. Soc., **103** (1988), 244–246.
5. M. Mursaleen, *On A-invariant mean and A-almost convergence*, Analysis Mathematica, **37(3)** (2011), 173–180.
6. L. Sucheston, *Banach limits*, Amer. Math. Monthly, **74** (1967), 308-311.
7. D. Hajduković, *Quasi-almost convergence in a normed space*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **13** (2002), 36–41.
8. S. Shaw and S. Lin, *Strong and absolute almost-convergence of vector functions and related Tauberian theorems*, J. Math. Anal. Appl. **334** (2007), 1073–1087.
9. J.C. Kurtz, *Almost convergent vector sequences*, Tôhoku Math. Journ. **22** (1970), 493–498.
10. L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, New York, Wiley, 1974.

Natural Science Research Center, Academy of Fundamental and Interdisciplinary Sciences, Harbin Institute of Technology, Harbin 150001, Heilongjiang, P. R. China.

E-mail address: youchao@hit.edu.cn