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ON QUASI $*$-PARANORMAL OPERATORS

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ABSTRACT. An operator $T \in B(H)$ is called quasi $*$-paranormal if $||T^*Tx||^2 \leq ||T^3x||||Tx||$ for all $x \in H$. If μ is an isolated point of the spectrum of T, then the Riesz idempotent E of T with respect to μ is defined by

$$E := \frac{1}{2\pi i}\int_{\partial D}(\mu I - T)^{-1}d\mu,$$

where D is a closed disk centered at μ which contains no other points of the spectrum of T. Stampfli [Trans. Amer. Math. Soc., 117 (1965), 469–476], showed that if T satisfies the growth condition G_1, then E is self-adjoint and $E(H) = N(T-\mu)$. Recently, Uchiyama and Tanahashi [Integral Equations and Operator Theory, 55 (2006), 145–151] obtained Stampfli's result for paranormal operators. In general even though T is a paranormal operator, the Riesz idempotent E of T with respect to $\mu \in \text{iso}\sigma(T)$ is not necessary self-adjoint. In this paper 2×2 matrix representation of a quasi $*$-paranormal operator is given. Using this representation we show that if E is the Riesz idempotent for a nonzero isolated point λ_0 of the spectrum of a quasi $*$-paranormal operator T, then E is self-adjoint if and only if the null space of $T - \lambda_0$ satisfies $N(T - \lambda_0) \subseteq N(T^* - \overline{\lambda_0})$. Other related results are also given.

1. INTRODUCTION AND PRELIMINARIES

Let $B(H)$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space H. Let T be an operator in $B(H)$. The operator T is said to be positive (denoted $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. The operator T is said to be a p-hyponormal operator if and only if $(T^*T)^p \geq (TT^*)^p$ for a positive number p. In [16], the class of log-hyponormal operators is defined as follows: T is a log-hyponormal operator if it is invertible

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and satisfies the following relation $\log T^*T \geq \log TT^*$. Class of p-hyponormal operators and class of log-hyponormal operators were defined as extension class of hyponormal operators, i.e, $T^*T \geq TT^*$. It is well known that every p-hyponormal operator is a q-hyponormal operator for $p \geq q > 0$, by the Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$", and every invertible p-hyponormal operator is a log-hyponormal operator since log is an operator monotone function. An operator T is paranormal if $||Tx||^2 \leq ||T^2x||||x||$ for all $x \in H$. It is also well known that there exists a hyponormal operator T such that T^2 is not a hyponormal operator (see [8]). In [6] authors, Furuta, Ito and Yamazaki introduced the class A operators, respectively class $A(k)$ operators defined as follows: for each $k > 0$, an operator T is $A(k)$ class operator if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2, \tag{1.1}$$

(for $k = 1$ it defines the class A operators) which includes the class of log-hyponormal operators (see Theorem 2, in [6]) and is included in the class of paranormal operators, in case where $k = 1$ (see Theorem 1 in [6]). An operator $T \in B(H)$ is called (p,k)-quasihyponormal for a positive number $0 < p \leq 1$ and a positive integer k, if

$$T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0.$$

I.H. Kim [11] introduced (p,k)-quasihyponormal operators and proved many interesting properties of (p,k)-quasihyponormal operators. It is shown [3] that T is paranormal if and only if

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0, \text{ for all } \lambda > 0.$$

An operator $T \in B(H)$ is said to be $*$-paranormal if $||T^*x||^2 \leq ||T^2x||$ for all unit vector x in H.

Hyponormal operators are paranormal and $*$-paranormal. An operator $T \in B(H)$ is said to be normoloid if $||T|| = r(T)$ (the spectral radius of T) . Paranormal operators are normaloid and $*$-paranormal operators are normaloid ([1, 7, 9, 14]). The class of paranormal operators was defined by Istrătescu, Saitō and Yoshino [9] as class (N). Furuta [4] renamed this class from class (N) to paranormal. The class of $*$-paranormal operators was defined by S.M. Patel [14]. The class of k-$*$-paranormal operators was defined by M.Y. Lee, S.H. Lee and C.S. Ryoo [12]. In order to extend the class of paranormal and $*$-paranormal operators we introduce the class of quasi-$*$ paranormal operators defined as follows:

Definition 1.1. An operator T is called quasi $*$- paranormal if it satisfies the following inequality:

$$||T^*Tx||^2 \leq ||T^3x||||Tx||$$

for all $x \in H$

It is well known that for any operators A, B and C,

$$A^*A - 2\lambda B^*B + \lambda^2 C^*C \geq 0 \text{ for all } \lambda > 0 \Leftrightarrow ||Bx||^2 \leq ||Ax||||Cx|| \text{ for all } x \in H.$$

Thus we have. An operator $T \in B(H)$ is quasi $*$-paranormal if and only if

$$T^*(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T \geq 0, \quad \text{for all } \lambda > 0.$$

It is well known that T is $*$-paranormal if and only if

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0, \text{ for all } \lambda > 0.$$

Thus every $*$-paranormal operator is quasi $*$-paranormal and we have the following implications:

$$\begin{aligned} &\text{Hyponormal} \Rightarrow *\text{-paranormal} \\ &\Rightarrow \text{ quasi } * \text{-paranormal.} \end{aligned}$$

If $T \in B(H)$, write $\sigma(T)$, $\sigma_p(T)$ for the spectrum of T and for the approximate point spectrum of T,respectively. Let $T \in B(H)$. $N(T)$ denotes the null space of T and $R(T)$ denotes the range of T. T is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. Let $\mu \in \mathbb{C}$ be an isolated point of the spectrum of T. Then the Riesz idempotent E of T with respect to μ is defined by

$$E := \frac{1}{2\pi i} \int_{\partial D} (\mu I - T)^{-1} d\mu,$$

where D is a closed disk centered at μ which contains no other points of the spectrum of T. It is well known that the Riesz idempotent satisfies $E^2 = E$, $ET = TE$, $\sigma(T|_{E(H)}) = \{\mu\}$ and $N(T - \mu I) \subseteq E(H)$. In [17], Stampfli showed that if T satisfies the growth condition G_1, then E is self-adjoint and $E(H) = N(T - \mu)$. Recently, Jeon and Kim [10] and A. Uchiyama [18] obtained Stampfli's result for quasi-class A operators and paranormal operators. In [13] the author obtained Jeon, Kim and Uchiyama results for k-quasi-paranormal operators. In general even though T is a paranormal operator, the Riesz idempotent E of T with respect to $\mu \in \text{iso}\sigma(T)$ is not necessary self-adjoint.

In this paper 2×2 matrix representation of a quasi $*$-paranormal operator is given. Using this representation we show that if E is the Riesz idempotent for a nonzero isolated point λ_0 of the spectrum of a quasi $*$-paranormal operator T, then E is self-adjoint if and only if the null space of $T - \lambda_0$ satisfies $N(T - \lambda_0) \subseteq N(T^* - \overline{\lambda_0})$.

2. Main Results

Lemma 2.1. *Let $T \in B(H)$ be quasi $*$-paranormal. If $R(T)$ is not dense, then*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*) \text{ and } A = T|_{\overline{R(T)}} \text{ is } {}^*\text{-paranormal.}$$

Proof. Since T is quasi $*$-paranormal and T does not have dense range, we can represent T as the upper triangular matrix

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*).$$

We shall show that A is $*$-paranormal. Since T is quasi $*$-paranormal, we have

$$T^*(T^{2*}T^2 - 2\lambda TT^* + \lambda^2)T \geq 0 \text{ for all } \lambda > 0.$$

Therefore

$$\langle (T^{2*}T^2 - 2\lambda TT^* + \lambda^2)x, x\rangle = \langle (A^{2*}A^2 - 2\lambda AA^* - 2\lambda BB^* + \lambda^2)x, x\rangle \geq 0,$$

for all $\lambda > 0$ and for all $x \in \overline{R(T)}$. Hence $\langle (A^{2*}A^2 - 2\lambda AA^* + \lambda^2)x, x\rangle \geq 2\langle \lambda BB^*x, x\rangle \geq 0$ for all $\lambda > 0$. Hence A is $*$-paranormal.

□

It is easily seen that if T is quasi $*$-paranormal and $R(T)$ is dense, then T is $*$-paranormal. Thus we have the following proposition:

Proposition 2.2. *Let $T \in B(H)$ be quasi $*$-paranormal. If $R(T)$ is dense, then T is $*$-paranormal.*

Proposition 2.3. *Let M be a closed T-invariant subspace of H. Then the restriction $T_{|M}$ of a quasi $*$-paranormal operator T to M is a quasi $*$-paranormal operator.*

Proof. Let
$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \quad \text{on} \quad H = M \oplus M^{\perp}.$$
Since T is quasi $*$-paranormal, we have
$$T^{*3}T^3 - 2\lambda T^*TT^*T + \lambda^2 T^*T \geq 0 \quad \text{for all } \lambda > 0.$$
Hence
$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^* \Big\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{*2} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^2 - 2\lambda \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^* + \lambda^2 \Big\} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \geq 0$$
for all $\lambda > 0$.

Therefore
$$\begin{pmatrix} A^*(A^{*2}A^2 - 2\lambda(AA^* + CC^*) + \lambda^2)A & E \\ F & G \end{pmatrix},$$
for some operators E, F and G. Hence
$$A^*(A^{*2}A^2 - 2\lambda AA^* + \lambda^2)A \geq A^*(2\lambda CC^*)A \geq 0,$$
for all $\lambda > 0$. This implies that $A = T_{|M}$ is quasi $*$-paranormal. □

We will denote the ascent of T by $p(T)$ and the descent of T by $q(T)$. In the following theorem we will give a necessary and sufficient condition for the Riesz idempotent E of a quasi $*$-paranormal operator to be self-adjoint. For this we need the following lemma.

Theorem 2.4. *Let $T \in B(H)$ be quasi $*$-paranormal. If μ is a non-zero isolated point of $\sigma(T)$, then μ is a simple pole of the resolvent of T.*

Proof. Assume that $R(T)$ is dense. Then T is $*$-paranormal and μ is a simple pole of the resolvent of T [19]. So we may assume that T does not have dense range. Then by Lemma 2.1 the operator T can be decomposed as follows:
$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*),$$

where A is $*$-paranormal. Now if μ is a non-zero isolated point of $\sigma(T)$, then $\mu \in \mathrm{iso}\sigma(A)$ because $\sigma(T) = \sigma(A) \cup \{0\}$. Therefore μ is a simple pole of the resolvent of A and the $*$-paranormal operator A can be written as follows:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ on } R(T) = N(A-\mu) \oplus R(A-\mu),$$

where $\sigma(A_1) = \{\mu\}$. Therefore

$$T - \mu = \begin{pmatrix} 0 & 0 & B_1 \\ 0 & A_2 - \mu & B_2 \\ 0 & 0 & -\mu \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & F \end{pmatrix} \text{ on } H = N(A-\mu) \oplus R(A-\mu) \oplus N(T^*),$$

where

$$F = \begin{pmatrix} A_2 - \mu & B_2 \\ 0 & -\mu \end{pmatrix}.$$

We claim that F is an invertible operator on $R(A-\mu) \oplus N(T^*)$. Indeed,

(1) $A_2 - \mu I$ is invertible. If not, then μ will be an isolated point in $\sigma(A_2)$. Since A_2 is $*$-paranormal, μ is an eigenvalue of A_2 and so $A_2 x = \mu x$ for some non-zero vector x in $R(A - \mu I)$. On the other hand, $Ax = A_2 x$ implying x is in $N(A - \mu I)$. Hence x must be a zero vector. This contradicts leads to (1).

(2) F is invertible. Indeed, by (1) and [8, Problem 71], $(A_2 - \mu I)(-\mu I)$ is invertible. It is easy to show that $p(T-\mu) = q(T-\mu) = 1$. Hence μ is a simple pole of the resolvent of T. □

Theorem 2.5. *Let $T \in B(H)$ be quasi $*$-paranormal. Assume $0 \neq \mu \in \mathrm{iso}\sigma(T)$ and E is the Riesz idempotent of T with respect to μ. Then E is self-adjoint if and only if $N(T-\mu) \subseteq N(T^* - \overline{\mu})$.*

Proof. Since E is the Riesz idempotent of T with respect to μ and T is quasi $*$-paranormal, it results from Theorem 2.1 that

$$R(E) = N(T-\mu) \text{ and } N(E) = R(T-\mu).$$

Assume that E is self-adjoint. Then E is an orthogonal projection. Hence $R(E^{\perp}) = N(E)$. Therefore we get

$$N(T-\mu) \subseteq N(T^* - \overline{\mu})$$

by using the equality

$$R(T-\mu) = N(T^* - \overline{\mu})^{\perp}.$$

Conversely, assume that

$$N(T-\mu) \subseteq N(T^* - \overline{\mu}).$$

Then $N(T-\mu)$ and $R(T-\mu)$ are orthogonal. Hence $R(E)^{\perp} = N(E)$, and so E is self-adjoint. □

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