MANHATTAN PRODUCTS OF DIGRAPHS: CHARACTERISTIC POLYNOMIALS AND EXAMPLES

NOBUAKI OBATA

Communicated by M. S. Moslehian

Abstract. We study spectral properties of the Manhattan products of the path graphs and show the concentration of zero-eigenvelues.

1. Introduction and Preliminaries

A *digraph (directed graph)* is a pair $G = (V, E)$, where V is a non-empty set and E is a subset of $V \times V$. An element $x \in V$ is called a *vertex* and $e = (x, y) \in E$ an *arc (arrow)* from the *initial vertex* x to the *final vertex* y. In that case we also write $x \to y$. By definition a digraph may have a *loop*, i.e., an arc from a vertex to itself. Throughout this paper a digraph means a finite digraph, i.e., with finite number of vertices.

The *adjacency matrix* of a digraph $G = (V, E)$ is a matrix A with index set $V \times V$ defined by

$$(A)_{xy} = \begin{cases} 1, & \text{if } x \to y, \\ 0, & \text{otherwise.} \end{cases}$$

Then A becomes a $\{0, 1\}$-matrix. Conversely, every $\{0, 1\}$-matrix with index set $V \times V$ defines a digraph with vertex set V. A digraph is called *symmetric* if its adjacency matrix is symmetric. A symmetric digraph with no loops is nothing else but a *graph* in the usual sense.

The eigenvalues of a digraph G is defined to be

$$\text{ev}\, G = \{\lambda_1, \lambda_2, \ldots, \lambda_s\},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_s$ are distinct eigenvalues of the adjacency matrix A of G. The characteristic polynomial of A, also referred to as the *characteristic polynomial of G*, is factorized as follows:

$$\varphi_G(x) = \det(x - A) = \prod_{i=1}^{s}(x - \lambda_i)^{m_i}, \qquad m_i \geq 1.$$

Then m_i is called the *algebraic multiplicity* of λ_i. While, the dimension l_i of the eigenspace associated with λ_i is called the *geometric multiplicity*. It is obvious that $1 \leq l_i \leq m_i$. Note that $l_i < m_i$ may happen for a general digraph and that $l_i = m_i$ for a symmetric digraph. Thus we need to distinguish the *algebraic spectrum* and *geometric spectrum* defined by

$$\mathrm{ASpec}\,(G) = \begin{pmatrix} \cdots & \lambda_i & \cdots \\ \cdots & m_i & \cdots \end{pmatrix}, \qquad \mathrm{GSpec}\,(G) = \begin{pmatrix} \cdots & \lambda_i & \cdots \\ \cdots & l_i & \cdots \end{pmatrix},$$

respectively.

There is a long history of spectral analysis of graphs and digraphs with many relevant topics, e.g., [2], [6], [7], see also [1] for a concise review for digraphs. In the recent years the profound relation has been investigated between the product structures of (undirected) graphs and various concepts of independence in quantum probability, see e.g., [8]. It is therefore an interesting direction to extend this relation to digraphs. In this line the Manhattan product of digraphs $G_1\#G_2$, introduced by Comellas, Dalfó and Fiol [5], is considered as the first non-trivial case to be studied in detail. The purpose of this note is to add a few results on spectral analysis of Manhattan products. So far an explicit and concrete result on spectrum is known only for the (2-dimensional) Manhattan street network, i.e., the Manhattan product of cycles $C_m\#C_n$ with even m, n, by Comellas *et al.* [3, 4]. In this paper we compute characteristic polynomials of the Manhattan products, in particular, of the path graphs $P_n\#P_2$ and $P_n\#P_3$, and show the concentration of zero-eigenvelues.

2. Bipartite Digraphs and Manhattan Products

A digraph $G = (V, E)$ is called *bipartite* if the vertex set admits a partition

$$V = V^{(0)} \cup V^{(1)} \quad V^{(0)} \neq \emptyset, \quad V^{(1)} \neq \emptyset, \quad V^{(0)} \cap V^{(1)} = \emptyset$$

such that every arc has its initial vertex in $V^{(0)}$ and final vertex in $V^{(1)}$, or initial vertex in $V^{(1)}$ and final vertex in $V^{(0)}$. By definition a bipartite digraph has no loops.

Example 2.1. For $n \geq 1$ let P_n denote the directed path with n vertices, i.e., $V = \{1, 2, \ldots, n\}$ and $E = \{(1, 2), (2, 3), \ldots, (n - 1, n)\}$. P_n is bipartite for all n.

Example 2.2. For $n \geq 2$ let C_n denote the directed cycle with n vertices, i.e., $V = \{1, 2, \ldots, n\}$ and $E = \{(1, 2), (2, 3), \ldots, (n - 1, n), (n, 1)\}$. C_n is bipartite if and only if n is even.

The adjacency matrix of a bipartite digraph may be expressed in the form:

$$A = \begin{bmatrix} O & C \\ D & O \end{bmatrix}, \tag{2.1}$$

where C is a $\{0,1\}$-matrix with index set $V^{(0)} \times V^{(1)}$ and D is a $\{0,1\}$-matrix with index set $V^{(1)} \times V^{(0)}$.

Proposition 2.3. *Let G be a bipartite digraph with adjacency matrix* (2.1). *Then the characteristic polynomial is given by*

$$\varphi_G(x) = \det(x - A) = x^{m-n} \det(x^2 - DC),$$

where $m = |V^{(0)}|$ and $n = |V^{(1)}|$ with $m \geq n$.

Proof. Straightforward by elementary knowledge of linear algebra. □

Let $G = (V, E)$ be a bipartite digraph. Given a partition $V = V^{(0)} \cup V^{(1)}$, which is not uniquely determined though, we define the parity function $\pi = \pi_G : V \to \{0,1\}$ by

$$\pi(x) = \pi_G(x) = \begin{cases} 0, & x \in V^{(0)}, \\ 1, & x \in V^{(1)}. \end{cases}$$

For an arc $(x, y) \in E$ we have $\pi(x) + \pi(y) = 1$. Moreover, the parity of the length of a path from x to y (whenever exists) is independent of the choice of such a path.

For $i = 1, 2$ let $G_i = (V_i, E_i)$ be a bipartite digraph with parity function $\pi = \pi_i$. Consider the direct product

$$V = V_1 \times V_2 = \{(x, y) \,;\, x \in V_1, \, y \in V_2\}$$

and let E consist of pairs of vertices $((x, y), (x', y'))$ satisfying one of the following two conditions:

    (i) $y = y'$, and $(x, x') \in E_1$ or $(x', x) \in E_1$ according as $\pi_2(y) = 0$ or $\pi_2(y) = 1$;
    (ii) $x = x'$, and $(y, y') \in E_2$ or $(y', y) \in E_2$ according as $\pi_1(x) = 0$ or $\pi_1(x) = 1$.

Following Comellas, Dalfó and Fiol [5], the digraph $G = (V, E)$ is called the *Manhattan product* and is denoted by

$$G = G_1 \# G_2.$$

Although not explicitly indicated, the Manhattan product depends on the choice of the partitions $V_i = V_i^{(0)} \cup V_i^{(1)}$, or equivalently on the choice of the parity functions π_i. The Manhattan product of two bipartite digraphs is again bipartite.

Proposition 2.4. *Let G_i be a bipartite digraph with the adjacency matrix A_i, $i = 1, 2$. Then the adjacency matrix A of the Manhattan product $G = G_1 \# G_2$ verifies*

$$(A)_{(x,y)(x',y')} = \delta_{xx'}(t^{\pi_1(x)}(A_2))_{yy'} + (t^{\pi_2(y)}(A_1))_{xx'}\delta_{yy'}, \quad x, x' \in V_1, \, y, y' \in V_2,$$

where $t(A) = A^T$ stands for the transposition and π_i is the parity function of G_i.

3. A Simple Example: $G \# C_2$

Let $G = (V, E)$ be a bipartite digraph and consider the Manhattan product $G \# C_2$. Let B be the adjacency matrix of G. Then the adjacency matrix A of $G \# C_2$ is given by

$$A = \begin{bmatrix} B & I \\ I & B^T \end{bmatrix}, \tag{3.1}$$

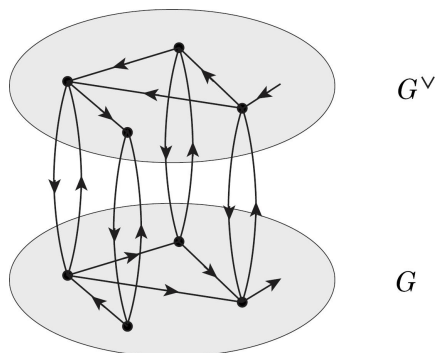where I is the identity matrix indexed by $V \times V$.

FIGURE 1. $G \# C_2$ ($G^\vee$: the opposite graph of G)

Lemma 3.1. *Let $G = (V, E)$ be a bipartite digraph with adjacency matrix B. Then the characteristic polynomial of the Manhattan product $G \# C_2$ is given by*

$$\varphi(x) = \det((x - B)(x - B^T) - I). \tag{3.2}$$

Moreover, if

$$B = \begin{bmatrix} O & C \\ D & O \end{bmatrix},$$

then we have

$$\varphi(x) = \det \begin{bmatrix} (x^2 - 1)I + CC^T & -x(C + D^T) \\ -x(C^T + D) & (x^2 - 1)I + DD^T \end{bmatrix}. \tag{3.3}$$

Proof. Let A be the adjacency matrix of the Manhattan product $G \# C_2$. Then the characteristic polynomial is given by

$$\varphi(x) = \det(x - A) = \det \begin{bmatrix} x - B & -I \\ -I & x - B^T \end{bmatrix}.$$

Applying the standard formula:

$$\det \begin{bmatrix} X & I \\ I & Y \end{bmatrix} = \det(XY - I) = \det(YX - I),$$

where X, Y are $n \times n$ matrices and I is the identity matrix, we obtain (3.2). Then (3.3) follows by direct computation. $\square$

Remark 3.2. In fact, $G\#C_2$ may be defined without assuming that G is bipartite, see Fig. 1. In that case too, $G\#C_2$ keeps the typical properties of the Manhattan street networks and the formula (3.2) remains valid. Another derivation and some relevant discussion are found in [9].

Theorem 3.3. *For* $n = 1, 2, \ldots$ *we have*

$$\mathrm{ev}\,(P_n\#\,C_2) = \left\{ 2\cos\frac{k\pi}{n+2}\,;\, k = 1, 2, \ldots, n+1 \right\} \cup \{0\},$$

where every non-zero eigenvalue has algebraic multiplicity one.

Proof. The adjacency matrix of P_n is given by

$$B = \begin{bmatrix} 0 & 1 & 0 & & & & \\ & 0 & 1 & & & & \\ & & 0 & \ddots & & & \\ & & & \ddots & 1 & 0 & \\ & & & & 0 & 1 & \\ & & & & & 0 & \end{bmatrix}. \tag{3.4}$$

For $n \geq 1$ the characteristic polynomial of $P_n\#\,C_2$ is denoted by φ_n. It then follows from Lemma 3.1 that

$$\varphi_n(x) = \det((x - B)(x - B^T) - I).$$

Applying cofactor expansion we obtain

$$\varphi_n(x) = x^2\varphi_{n-1}(x) - x^2\varphi_{n-2}(x).$$

Then, comparing with the recurrence relation of the Chebyshev polynomials of the second kind [8], we come to

$$\varphi_n(x) = x^{n-1}\tilde{U}_{n+1}(x),$$

where

$$\tilde{U}_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}\,.$$

Consequently,

$$\mathrm{ev}\,(P_n\#\,C_2) = \left\{ 2\cos\frac{k\pi}{n+2}\,;\, k = 1, 2, \ldots, n+1 \right\} \cup \{0\},$$

where every non-zero eigenvalue has algebraic multiplicity one.   □

The asymptotic spectral distribution of $P_n\#C_2$ as $n \to \infty$ is obtained explicitly, where we observe the concentration of zero-eigenvalues.

Theorem 3.4. *The asymptotic (algebraic) spectral distribution of* $P_n\#\,C_2$ *is given by*

$$\frac{1}{2}\,\delta_0 + \frac{1}{2}\,\rho(x)dx,$$

where

$$\rho(x) = \frac{1}{\pi\sqrt{4 - x^2}}\,\chi_{(-2,2)}(x).$$

Proof. It is sufficient to show that

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{2\cos\frac{k\pi}{n+1}}$$

tends to $\rho(x)dx$ as $n \to \infty$. Let $f(x)$ be a bounded continuous function. Then we have

$$\int_{-\infty}^{+\infty} f(x)\mu_n(dx) = \frac{1}{n} \sum_{k=1}^{n} f\left(2\cos\frac{k\pi}{n+1}\right) \to \int_0^1 f(2\cos\pi t)dt, \quad \text{as } n \to \infty,$$

which follows by the definition of Riemann integral. By change of variable, one gets

$$\int_0^1 f(2\cos\pi t)dt = \int_{-2}^{2} f(x)\frac{dx}{\pi\sqrt{4-x^2}}.$$

Consequently,

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f(x)\mu_n(dx) = \int_{-2}^{2} f(x)\frac{dx}{\pi\sqrt{4-x^2}} = \int_{-\infty}^{+\infty} f(x)\rho(x)dx,$$

which completes the proof. $\qquad\square$

Remark 3.5. The probability distribution $\rho(x)dx$ in Theorem 3.4 is called the *arcsine law* (with mean 0 and variance 2).

4. The Manhattan Product $P_n \# P_2$

Let B denote the adjacency matrix of P_n as in (3.4). We define $n \times n$ matrices by

$$P = \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{bmatrix}.$$

Note that $P + Q = I$. Then the adjacency matrix of $P_n \# P_2$ becomes

$$A = \begin{bmatrix} B & P \\ Q & B^T \end{bmatrix}.$$

Hence the characteristic polynomial of $P_n \# P_2$ is given by

$$\begin{aligned}
\varphi_n(x) &= \det(x - A) \\
&= \det\begin{bmatrix} x - B & -P \\ -Q & x - B^T \end{bmatrix} \\
&= \det(x - B)\det(x - B^T - (-Q)(x-B)^{-1}(-P)) \\
&= x^n \det(x - B^T - Q(x-B)^{-1}P).
\end{aligned}$$

Lemma 4.1. *It holds that*

$$\varphi_1(x) = x^2,$$
$$\varphi_2(x) = x^4,$$
$$\varphi_3(x) = x^6 - x^2,$$
$$\varphi_4(x) = x^8 - x^4,$$
$$\varphi_n(x) = x^4\varphi_{n-2}(x) - x^4\varphi_{n-4}(x), \quad n \geq 5.$$

Proof. Writing $\Delta_n(x) = \det(x - B^T - Q(x - B)^{-1}P)$ explicitly and applying the standard cofactor expansion, we obtain

$$\Delta_n(x) = x^2\Delta_{n-2}(x) - \Delta_{n-4}(x), \qquad n \geq 5.$$

While, $\Delta_n(x)$ for a smaller n is calculated directly. Then we obtain the recurrence relations for $\varphi_n(x)$. □

Lemma 4.2. *For $m \geq 1$ we have*

$$\varphi_{4m-3}(x) = (-1)^{m-1}mx^{4m-2} + (higher\ terms),$$
$$\varphi_{4m-2}(x) = (-1)^{m-1}mx^{4m} + (higher\ terms),$$
$$\varphi_{4m-1}(x) = (-1)^m x^{4m-2} + (higher\ terms),$$
$$\varphi_{4m}(x) = (-1)^m x^{4m} + (higher\ terms).$$

Proof. By induction on m using Lemma 4.1. □

Theorem 4.3. *Let α_n be the algebraic multiplicity of zero-eigenvalue of $P_n\#P_2$. Then for $n = 1, 2, \ldots$ it holds that*

$$\alpha_{2n-1} = 4\left[\frac{n+1}{2}\right] - 2, \quad \alpha_{2n} = 4\left[\frac{n+1}{2}\right].$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{2n}\alpha_n = \frac{1}{2}.$$
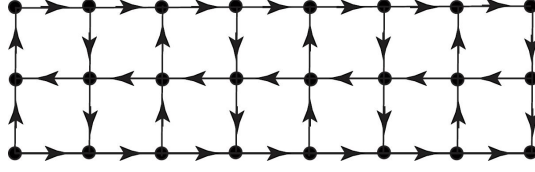
Proof. Straightforward from Lemma 4.2. □

5. THE MANHATTAN PRODUCT $P_n\#P_3$

Let B, P, Q be the $n \times n$ matrices defined in the previous sections. The adjacency matrix of $P_n\#P_3$ is given by

$$A = \begin{bmatrix} B & P & O \\ Q & B^T & P \\ O & Q & B \end{bmatrix}$$

and our task is to calculate

$$\varphi_n(x) = \det(x - A).$$

FIGURE 2. Manhattan product $P_8 \# P_3$

Lemma 5.1. *For $n = 1, 2, \ldots$ it holds that*

$$\varphi_n(x) = x^{2n} D_n(x),$$
$$D_n(x) = \det(x - B^T - Q(x - B)^{-1}P - P(x - B)^{-1}Q).$$

Proof. By definition we have

$$\varphi_n(x) = \det(x - A)$$
$$= \det \begin{bmatrix} x - B & -P & O \\ -Q & x - B^T & -P \\ O & -Q & x - B \end{bmatrix}$$
$$= \det(x - B) \det \left\{ \begin{bmatrix} x - B^T & -P \\ -Q & x - B \end{bmatrix} - \begin{bmatrix} -Q \\ O \end{bmatrix} (x - B)^{-1} \begin{bmatrix} -P & O \end{bmatrix} \right\}$$
$$= \det(x - B) \det \begin{bmatrix} x - B^T - Q(x - B)^{-1}P & -P \\ -Q & x - B \end{bmatrix}$$
$$= \det(x - B)^2 \det(x - B^T - Q(x - B)^{-1}P - P(x - B)^{-1}Q).$$

This proves $\varphi_n(x) = x^{2n} D_n(x)$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.2. *It holds that*

$$D_0(x) = 1,$$
$$D_1(x) = x,$$
$$D_2(x) = x^2 - x^{-2},$$
$$D_n(x) = x D_{n-1}(x) - x^{-1} D_{n-3}(x), \quad n \geq 3.$$

Proof. Write $D_n(x)$ explicitly and apply the standard cofactor expansion. $\quad\square$

Lemma 5.3. *For $m \geq 1$ we have*

$$D_{3m-1}(x) = (-1)^m x^{-(m+1)} + (higher\ terms),$$
$$D_{3m}(x) = (-1)^m (m + 1)x^{-m} + (higher\ terms),$$
$$D_{3m+1}(x) = \frac{(-1)^m}{2}(m + 1)(m + 2)x^{-(m-1)} + (higher\ terms).$$

Proof. By induction on m using Lemma 5.2. $\qquad\qquad\qquad\qquad\qquad\square$

Theorem 5.4. *Let α_n be the algebraic multiplicity of zero-eigenvalue of $P_n \# P_3$. Then for $n = 1, 2, \ldots$ it holds that*

$$\alpha_n = 3n - 4 \left[\frac{n+1}{3} \right].$$

Therefore,

$$\lim_{n \to \infty} \frac{1}{3n} \alpha_n = \frac{5}{9}\,.$$

Proof. Straightforward from Lemma 5.3. □

In [4] the spectrum of the Manhattan product $C_m \# C_n$ for even numbers m, n is obtained explicitly. We know that the algebraic multiplicity of zero-eigenvalue $mn/4$, i.e., the density is $1/4$ if $m, n \not\equiv 0$ (mod 4). We see from Theorem 4.3 that the density of zero-eigenvalue of $P_n \# P_2$ is $1/2$ asymptotically. Similarly, from Theorem 5.4 the density of zero-eigenvalue of $P_n \# P_3$ is $5/9$ asymptotically. Further systematic study on concentration of zero-eigenvalue is now in progress.

References

1. R.A. Brualdi, *Spectra of digraphs*, Linear Algebra Appl. **432** (2010), 2181–2213.
2. N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, 1994.
3. F. Comellas, C. Dalfó, M.A. Fiol and M. Mitjana, *A spectral study of the Manhattan networks*, Electronic Notes in Discrete Mathematics **29** (2007) 267–271.
4. F. Comellas, C. Dalfó, M. A. Fiol and M. Mitjana, *The spectra of Manhattan street networks*, Linear Algebra Appl. **429** (2008), 1823–1839.
5. F. Comellas, C. Dalfó and M.A. Fiol, *The Manhattan product of digraphs*, preprint.
6. D.M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, 1979.
7. D.M. Cvetković, P. Rowlinson and S. Simić, *Eigenspaces of Graphs*, Cambridge University Press, 2008.
8. A. Hora and N. Obata, *Quantum Probability and Spectral Analysis of Graphs*, Springer, 2007.
9. H.-O. Lee, *Product structures of networks and their spectra*, Master Thesis, Tohoku University, 2011.

Graduate School of Information Sciences, Tohoku University,
Sendai, 980-8579, Japan.

E-mail address: obata@math.is.tohoku.ac.jp