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KAPLANSKY'S AND MICHAEL'S PROBLEMS: A SURVEY

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ABSTRACT. I. Kaplansky showed in 1947 that every submultiplicative norm $\|.\|$ on the algebra $\mathcal{C}(K)$ of complex–valued functions on an infinite compact space K satisfies $\|f\| \geq \|f\|_K$ for every $f \in \mathcal{C}(K)$, where $\|f\|_K = max_{t\in K}|f(t)|$ denotes the standard norm on $\mathcal{C}(K)$. He asked whether all submultiplicative norms $\|.\|$ were in fact equivalent to the standard norm (which is obviously true for finite compact spaces), or equivalently, whether all homomorphisms from $\mathcal{C}(K)$ into a Banach algebra were continuous. This problem turned out to be undecidable in ZFC, and we will discuss here some recent progress due to Pham and open questions concerning the structure of the set of nonmaximal prime ideals of $\mathcal{C}(K)$ which are closed with respect to a discontinuous submultiplicative norm on $\mathcal{C}(K)$ when the continuum hypothesis is assumed. We will also discuss the existence of discontinuous characters on Fréchet algebras (Michael's problem), a long standing problem which remains unsolved. The Mittag–Leffler theorem on inverse limits of complete metric spaces plays an essential role in the literature concerning both problems.

1. Introduction and preliminaries

A linear seminorm $\|.\|$ on a real or complex algebra is said to be an algebra seminorm (or, equivalently, a submultiplicative seminorm) if $\|ab\| \leq \|a\|\|b\|$ for $a, b \in A$. Recall that a Fréchet algebra is a complex algebra A equipped with a family $(\|.\|_n)_{n\geq 1}$ of algebra seminorms satisfying the following conditions

(1) $\|a\|_n \leq \|a\|_{n+1}$ for $a \in A$, $n \geq 1$.
(2) $\cap_{n\geq 1} Ker\|.\|_n = \{0\}$.

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(3) every Cauchy sequence of elements of A is convergent

In the definition above a Cauchy sequence is a sequence $(a_n)_{n\geq 1}$ satisfying the usual Cauchy criterion with respect to the seminorm $\|.\|_n$ for every $n \geq 1$, which means that for every $\epsilon > 0$ and every $n \geq 1$ there exists a positive integer $N(\epsilon, n)$ such that $\|a_p - a_q\| < \epsilon$ for $p \geq N(\epsilon, n), q \geq N(\epsilon, n)$. Notice that if we set

$$d(a,b) := \sum_{n=1}^{+\infty} \frac{inf(1, \|a-b\|_n)}{2^n},$$

then d is a distance on A, and condition 3 means that (A, d) is a complete metric space.

A character on A is an algebra homomorphism $\chi : A \to \mathbb{C}$. The celebrated question of whether all characters on Fréchet characters are continuous is known as Michael's problem, and remains still open. We will discuss in section 3 some partial results due to Arens [4], and present the link, established by P.J. Dixon and the author in [18], between this question and a question concerning projective sequences $(\mathbb{C}^{p_n}, F_n)$, where $F_n : \mathbb{C}^{p_{n+1}} \to \mathbb{C}^{p_n}$ is entire for $n \geq 1$. The proof of these results are based on the Mittag–Leffler theorem on inverse limits of complete metric spaces, that we will present in section 2. The Mittag–Leffler theorem implies the Baire category theorem, but it is some sense stronger: we will give a very simple application of the Mittag–Leffler theorem which cannot be deduced from Baire's theorem. To illustrate the power of the Mittag–Leffler theorem we will use it in section 3 to prove the continuity of real characters on real Fréchet algebras, which follows from a paper of Shah [45] (the link between Michael's problem and projective systems of entire maps can be considered as the complex counterpart of this result).

Let K be a compact space, and let $\mathcal{C}(K)$ (resp. $\mathcal{C}_{\mathcal{R}}(K)$) be the algebra of complex valued (reps. real valued) continuous functions on K. I. Kaplansky showed in 1947 that every algebra norm $\|.\|$ on $\mathcal{C}(K)$ satisfies the condition

$$\|f\| \geq \|f\|_K \ \forall \in \mathcal{C}(K),$$

where $\|f\|_K = max_{t\in K}|f(t)|$ denotes the usual norm on $\mathcal{C}(K)$.

In the other direction he asked whether there always exists a constant $k > 0$, depending on the algebra norm, such that $\|f\|_K \geq k\|f\| \ \forall f \in \mathcal{C}(K)$. This question, known as Kaplansky's problem, is equivalent to the fact that every homomorphism from $\mathcal{C}(K)$ into a Banach algebra is continuous. This problem turns out to be undecidable in ZFC (Zermelo–Fraenkel set theory with the axiom of choice). Partial results were obtained by Badé and Curtis in [6], who showed in particular that for every homomorphism from $\mathcal{C}(K)$ into a Banach algebra there exists a finite set $S \subset K$ such that the restriction of ϕ to the algebra $\mathcal{A}_S$ is continuous, where $\mathcal{A}_S$ denotes the set of those functions which are constant on a neighborhood of s for every $s \in S$. This result is based on the "main boundedness theorem" of [6], which will be stated in section 4 in the form given by Badé and Curtis in [6].

The main boundedness theorem is one of the main tools in automatic continuity theory. Another important tool is the so–called stability lemma, which we will state in section 4 in the form given by Sinclair in [47], lemma 6 (see also [14], Chapter 5). Using this lemma, Sinclair showed in [48] that if ϕ is any homomorphism from a maximal ideal $\mathcal{M}$ of $\mathcal{C}(K)$ into a radical Banach algebra B then $\phi(f) \in [\phi(f^2)B]^-$ for every $f \in \mathcal{M}$.

Recall that an ideal I in a commutative ring A is said to be *semiprime* if I contains every $a \in A$ such that $a^n \in A$ for some positive integer n, which is equivalent to the fact that I equals the intersection of the prime ideals of A which contain it. We will say that a semiprime ideal I of a commutative unital algebra A is *pure* if for every maximal ideal $\mathcal{M}$ of A containing I there exists a prime ideal J of A such that $I \subset J \subsetneq \mathcal{M}$. It follows from Sinclair's result that the kernel of every homomorphism from $\mathcal{C}(K)$ into a Banach algebra is semiprime.

Our discussion of partial continuity properties of algebra semi norms on $\mathcal{C}(K)$ involve the following ideal.

Definition 1.1. Let q be an algebra seminorm on $\mathcal{C}(K)$. The **continuity ideal** $\mathcal{I}(q)$ of q is the set of all $f \in \mathcal{C}(K)$ such that there exists $k_f > 0$ satisfying $q(fg) \leq k_f \|g\|_K$ for every $g \in \mathcal{C}(K)$.

The following notion comes naturally from the main boundedness theorem of [6].

Definition 1.2. Let A be a commutative ring. A Badé–Curtis ideal of A is an ideal I of A such that, for every sequence $(f_n)_{n\geq 1}$ of elements of A such that $f_n f_m = 0$ for $n \neq m$, there exists $p \geq 1$ such that $f_n \in I$ for every $n \geq p$.

The following theorem follows then from the results of Badé–Curtis and Sinclair.

Theorem 1.3. *Let q be an algebra seminorm on $\mathcal{C}(K)$, and let $Prim(q)$ be the set of nonmaximal prime ideals of $\mathcal{C}(K)$ which are closed with respect to q.*

(i) There exists $k > 0$ such that $q(f) \leq k\|f\|_K$ for every $f \in \mathcal{I}(q)$, and the continuity ideal $\mathcal{I}(q)$ is the largest ideal I of $\mathcal{C}(K)$ such that the restriction of q to I is continuous with respect to the norm $\|.\|_K$.

(i) $\mathcal{I}(q) = \cap\{I : I \in Prim(q)\}$.

(iii) $\mathcal{I}(q)$ is a pure semiprime Badé–Curtis ideal of $\mathcal{C}(K)$.

It follows from this theorem that the existence of a discontinuous homomorphism from $\mathcal{C}(K)$ is equivalent to the existence of a nonmaximal prime ideal of $\mathcal{C}(K)$ such that the quotient algebra $\mathcal{C}(K)/I$ admits an algebra norm. Dales and the author showed independently that this is indeed the case if the continuum hypothesis $2^{\aleph_0} = \aleph_1$ (CH) is assumed, see [13, 21, 22, 23]. A summary of both constructions is given in [15]. In fact every complex algebra which is an integral domain of cardinality $\aleph_1$ possesses an algebra norm, see [23], a very general result if CH is assumed.

The author's construction was influenced by the seminal paper [1] where Allan showed that every commutative unital Banach algebra A such that $x^n \in [x^{n+1}A]^-$ for some nonnilpotent, quasinilpotent $x \in A$ contains a copy of $\mathbb{C}[[X]]$, the algebra of all formal power series with complex coefficients. The Mittag–Leffler theorem plays an essential role in Allan's embedding, and it plays also an essential role in the transcendental extensions involved in the author's construction [22].

In the other direction, Solovay and Woodin showed that there exists models of set theory including Martin's axiom in which all homomorphism from $\mathcal{C}(K)$ are continuous for every compact space K, see [16]. We will also discuss in section 5 models of set theory constructed independently by Franckiewicz–Sbierski [33] and Woodin [53] in which $2^{\aleph_0} = \aleph_2$ but in which there exists a discontinuous homomorphism from $\mathcal{C}(K)$ for every infinite compact space K.

We will also present in section 4 results concerning the set $Prim(q)$: it was shown by the author in [20] that every chain of prime ideals of $\mathcal{C}(K)$ which are closed with respect to some algebra seminorm on $\mathcal{C}(K)$ is well–ordered with respect to inclusion. This result is essentially best possible if the continuum hypothesis is assumed: given a well–ordered chain $\{I_\zeta\}_{\zeta<\omega}$ of closed nonmaximal prime ideals of $\mathcal{C}(K)$ such that $\text{card}(\mathcal{C}(K)/I_0) = \aleph_1$ there exists an algebra norm on $\mathcal{C}(K)$ such that $\{I_\zeta\}_{\zeta<\omega} \subset Prim(q)$, and it is even possible to arrange that $\{I_\zeta\}_{\zeta<\omega} = Prim(q)$ if $I_\zeta = \cup_{\eta<\zeta} I_\eta$ for every limit ordinal $\zeta < \omega$. A proof of this result, which has been known to the author for a long time, will be given in [29].

It was observed in [20] that if $f \in |f|$ for every $f \in \mathcal{C}(K)$ (in other terms, if K is a F–space) then the continuity ideal of every homomorphism from $\mathcal{C}(K)$ into a Banach algebra is the intersection if a finite family of prime ideals. This property holds in particular for $l^\infty \approx \mathcal{C}(\beta\mathbb{N})$. Answering a long standing question, Pham showed in [41] that this is not true in general. Denote by $\mathcal{I}(\mathcal{F})$ the intersection of a family $\mathcal{F}$ of prime ideals of $\mathcal{C}(K)$. A family $\mathcal{F}$ of prime ideals of $\mathcal{C}(K)$ is said to be *pseudofinite* if for every $I \in \mathcal{F}$ and every $f \in I$ the set $\{J \in \mathcal{F} \ : f \notin J\}$ is finite or empty. Pham observed in [41], assuming the continuum hypothesis, that if $\mathcal{F}$ is a pseudofinite family of pairwise noncomparable nonmaximal prime ideals then the quotient algebra $\mathcal{C}(K)/\mathcal{I}(\mathcal{F})$ possesses an algebra norm when $|\mathcal{C}(K)/\mathcal{I}(\mathcal{F})| = 2^{\aleph_0}$, and he constructed infinite families of this type (and even families with the power of the continuum, see [42]) for a large class of compact spaces which includes all perfect compact metric spaces. This shows that the continuity ideal of an homomorphism from $\mathcal{C}(K)$ into a Banach algebra can be very complicated, and that the family $Prim(q)$ is not necessarily a finite union of well-ordered chains. We conclude section 4 by a curious result which shows that the union of any family of elements of $Prim(q)$ is a finite union of prime ideals.

We present some open questions in section 5. It is known that the existence of discontinuous homomorphisms from $\mathcal{C}(K)$ is consistent with $2^{\aleph_0} = \aleph_2$, but does it imply that $2^{\aleph_0} \leq \aleph_2$?

A related question concerns the normability (i.e. the existence of an algebra norm) of "big" algebras of formal power series with exponents over the "minimal linearly ordered divisible η_1–group" $\mathbf{G}$. The formal power series over $\mathbf{G}$ with well–ordered countable support form a valued field $\mathbf{C}$ with values in $\mathbf{G}$ which is isomorphic to the quotient algebra $\mathbb{C}^{\mathbb{N}}/\mathcal{U}$ for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ if the continuum hypothesis is assumed. The fact that the algebra $\mathbf{C}^\sharp$ of bounded elements of $\mathbf{C}$ is normable is a theorem of ZFC. Now let $\widehat{\mathbf{C}}$ be the algebra of all formal power series with exponents over $\mathbf{G}$. The normability of the algebra $\widehat{\mathbf{C}}^\sharp$ of bounded elements of $\widehat{\mathbf{C}}$ is an open question, even if the continuum hypothesis is assumed, and so is the normability of the smaller algebra $\tilde{\mathbf{C}}^\sharp$, which is the complexification of the algebra $\tilde{\mathbf{R}}^\sharp$ discussed by Dales and Woodin in Chapter 3 of [17].

We will conclude section 5 and the paper with open questions concerning the continuity ideal $\mathcal{I}(q)$ and the set of nonmaximal prime q-closed ideals associated to a discontinuous algebra seminorm on $\mathcal{C}(K)$ when the continuum hypothesis is assumed.

Notice that both Kaplansky's and Michael's problem arise in ZFC. A deep theorem of Solovay [49] shows that the axiom "every subset of $\mathcal{R}$ is Lebesgue measurable" is consistent with ZF, and it is well kown [34] that this axiom implies the fact that every linear map from a complete metrizable topological vector space into a metrizable topological vector space is continuous. So characters on Fréchet algebras and homomorphisms from $\mathcal{C}(K)$ into a Banach algebra are obviously continuous, in "Solovayan" functional analysis, and the fact that derivations on commutative Banach algebras map into the radical [52] follow directly from the Singer–Wermer theorem since derivations are also obviously continuous in this context.

2. The Mittag–Leffler theorem

Let $(E_n)_{n\geq 1}$ be a family of sets. For $p \geq 1$ we will denote by $\pi_p : (x_n)_{n\geq 1} \to x_p$ the projection of $\Pi_{n=1}^\infty E_n$ onto E_p. If for each $n \geq 1$ a map $\theta_n : E_{n+1} \to E_n$ is given, the sequence $(E_n, \theta_n)_{n\geq 1}$ is called a projective sequence. The inverse limit of a projective sequence (E_n, θ_n) is given by the formula

$$\varprojlim (E_n, \theta_n) := \{(x_n)_{n\geq 1} \in \Pi_{n=1}^\infty E_n \mid x_n = \theta_n(x_{n+1}) \ \forall n \geq 1\}.$$

We now state the Mittag–Leffler theorem.

Theorem 2.1. *Let $(E_n, \theta_n)_{n\geq 1}$ be a projective sequence. Assume that E_n is a complete metric space with respect to a distance d_n, that $\theta_n : E_{n+1} \to E_n$ is continuous and that $\theta_n(E_{n+1})$ is dense in E_n for $n \geq 1$. Then, for every $p \geq 1$, $\pi_p(\varprojlim (E_n, \theta_n)$ is dense in E_p.*

We leave as an exercise the proof of this theorem, which can be found for example in [14] (theorem A.1.24 and corollary A.1.25). Baire's theorem can be easily obtained as a corollary of the Mittag–Leffler theorem.

Corollary 2.2. *Let (E, d) be a complete metric space, and let $(U_n)_{n\geq 1}$ be a sequence of dense open subsets of E. Then $\cap_{n\geq 1} U_n$ is dense in E.*

Proof: Set $V_n = U_1 \cap \cdots \cap U_n$. Then $(V_n)_{n\geq 1}$ is a non increasing sequence of dense open subsets of E. Denote by $i_n : x \to x$ the natural injection of V_{n+1} into V_n. We can assume that V_1 is strictly contained in E. Set, for $x, y \in V_n$,

$$d_n(x, y) = d(x, y) + \frac{1}{|d(x, \partial V_n) - d(y, \partial V_n)|},$$

where ∂V_n denotes the boundary of V_n.

Clearly, d_n is a distance on V_n which defines the same topology as the given distance d on V_n. Let $(x_m)_{m\geq 1}$ be a Cauchy sequence of elements of V_n with respect to d_n. Then $(x_m)_{m\geq 1}$ is also a Cauchy sequence with respect to d and its limit in $\overline{V}_n$ satisfies $d(x, \partial V_n) > 0$. Hence $x \in V_n$, $\lim_{m\to+\infty} d_n(x, x_m) = 0$, and (V_n, d_n) is a complete metric space. It follows then from the Mittag–Leffler theorem that $\cup_{n\geq 1} U_n = \cup_{n\geq 1} V_n = \pi_1(\underleftarrow{lim}(V_n, i_n)$ is dense in E. □

If A is a complex algebra, we set $A^{\#} = A$ if A is unital. If A is not unital we will denote by $A^{\#} = A \oplus \mathbb{C}e$ the algebra obtained by adding formally a unit e to A.

We now give an easy corollary of the Mittag–Leffler theorem, which cannot be deduced from Baire's theorem.

Corollary 2.3. *Let A be a Fréchet algebra, and let $(a_m)_{m\geq 1}$ be a sequence of elements of A such that $[a_n A]^- = A$ for $n \geq 1$. Then we have the following properties*

(i) $\cap_{m\geq 1} a_1 \cdots a_m A$ is dense in A.

(ii) There exists $b \in A$ such that $b - \sum_{m=1}^{n} a_1 \cdots a_m A \in a_1 \cdots a_{n+1} A^{\#}$ for $p \geq 1$.

Proof: (i) Set $E_n = A$ for $n \geq 1$, and set $\theta_n(u) = a_n u$ for $u \in A, n \geq 1$. It follows from the Mittag–Leffler theorem that $\cap_{m\geq 1} a_1 ... a_m A \subset \pi_1(\underleftarrow{\lim}(E_n, \theta_n)$ is dense in $E_1 = A$.

(ii) Set again $E_n = A$ for $n \geq 1$, and set $\theta_n(u) = a_n u + a_n$ for $u \in A$, $n \geq 1$. It follows from the Mittag–Leffler theorem that $\underleftarrow{\lim}(E_n, \theta_n) \neq \emptyset$. Let $(b_n)_{n\geq 1} \in \underleftarrow{\lim}(E_n, \theta_n)$, and set $b = b_1$. A routine induction shows that we have, for $n \geq 1$,

$$b = \sum_{m=1}^{n} a_1 \cdots a_m + a_1 \cdots a_n b_{n+1}$$

$$= \sum_{m=1}^{n} a_1 \cdots a_m + a_1 \cdots a_n (a_{n+1} + a_{n+1} b_{n+2}) \in a_1 \cdots a_{n+1} A^{\#}.$$

□

3. Continuity of characters on Fréchet algebras

Let $(A, (\|.\|)_{n\geq 1})$ be a Fréchet algebra. For $n \geq 1$ we denote by A_n the completion of the quotient algebra $A/Ker(\|.\|_n)$ with respect to the norm

$$\|.\|_n^* : f + Ker\|.\|_n \to \|f\|_n$$

on $A/Ker(\|.\|_n)$, and we denote by $\pi_n : a \to a + Ker\|.\|_n$ the natural homomorphism from A into $(A_n, \|.\|_n)$. The map $a + Ker\|.\|_{n+1} \to a + Ker\|.\|_n$ extends continuously to an homomorphism $\tilde{\pi}_n : A_{n+1} \to A_n$. It was observed by Michael [40] that the map $\pi : a \to ((\pi_n)(a))_{n\geq 1}$ defines a Fréchet algebra isomorphism between the given Fréchet algebra A and the algebra $\underleftarrow{lim}(A_n, \tilde{\pi}_n)$ considered as a closed subalgebra of the Fréchet algebra $\Pi_{n\geq 1} A_n$.[1]

[1]If $(B_n, p_n)_{n\geq 1}$ is a sequence of Banach algebras, then the product $\Pi_{m\geq 1} B_m$ is a Fréchet algebra with respect to the family $(q_n)_{n\geq 1}$ of seminorms defined by the formula $q_n((u_m)_{m\geq 1}) =$

A good presentation of known partial results concerning Michael's problem is given by Dales in section 4.10 of [14]. For example assume that A is a unital Féchet algebra, denote by $\sigma_A(u)$ the spectrum of $u \in A$, and we denote by $\widehat{A}$ the set of continuous characters on A. It was shown by Michael in [40] that we have, for $u \in A$,

$$\sigma_A(u) = \{\chi(u)\}_{\chi \in \widehat{A}}.$$

Hence if ϕ is a discontinuous character on A and if $u \in A$, there always exists a continuous character χ on A such that $\chi(u) = \phi(u)$.

Using the Mittag–Leffler theorem, Arens [4] showed that if $(u_1, \cdots, u_p) \in A^p$, and if $\pi_n(u_1)A_n + \cdots + \pi_n(u_p)A_n = A_n$ for every $n \geq 1$, then $u_1A + \cdots + u_nA = A$, see [14], theorem 4.10.8. Now consider the joint spectrum

$$\sigma_A(u_1, \cdots, u_p) := \{(\lambda_1, \cdots, \lambda_p) \in \mathbb{C}^p \mid (a_1 - \lambda_1)A + \cdots + (a_p - \lambda_p)A \subsetneq A\}.$$

It follows from theorem 4.10.8 of [14] that we have

$$\sigma_A(u_1, \cdots, u_p) = \{(\chi(u_1), \cdots, \chi(u_p))\}_{\chi \in \widehat{A}}.$$

In particular for every discontinuous character ϕ on A and every finite family $(u_1, \cdots, u_p)$ of elements of A, there exists a continuous character χ on A such that $\chi(u_j) = \phi(u_j)$ for $1 \leq j \leq p$. This result implies in particular that if a Fréchet algebra A is rationally generated by a finite set, then every character on A is continuous. This shows that every character on the algebra $\mathcal{H}(S)$ of holomorphic functions on a Stein manifold S is continuous (in fact for every character ϕ on $\mathcal{H}(S)$ there exists $s \in S$ such that $\phi(f) = f(s)$ for every $f \in \mathcal{H}(S)$).

Now let Ω be a σ–compact, non compact locally compact space. The fact that all characters on the Fréchet algebra $\mathcal{C}(\Omega)$ of continuous complex–valued functions on Ω is a consequence of a result of Michael, see theorem 4.10.13 of [14]. We present here a proof which is an interpretation via the Mittag–Leffler theorem of an argument of Shah [45].

Theorem 3.1. *Let A be a real Fréchet algebra. Then every homomorphism $\phi : A \to \mathbb{R}$ is continuous.*

Proof: We can assume without loss of generality that A is unital. Assume, if possible, that there exists a discontinuous homomorphism $\phi : A \to \mathbb{R}$. Then $\mathcal{M} := Ker(\phi)$ is dense in A. Equip $\mathcal{M}$ with the discrete topology, so that $\mathcal{M}$ is a complete metric space with respect to the trivial distance δ defined by the formulas

$$\begin{cases} \delta(u,v) = 1 \text{ if } u \neq v \\ \delta(u,v) = 0 \text{ if } u = v \end{cases}$$

Set $E_n = A \times \mathcal{M}^{n-1}$ for $n \geq 1$, and set, for $\mathbf{u} = (u, v_1, \cdots, v_{n-1}) \in E_n, \mathbf{u'} = (u', v'_1, \cdots, v'_{n-1}) \in E_n$,

$sup_{1 \leq j \leq n} p_j(u_j)$. The topology associated to this family of seminorms is the usual product topology.

$$d_n(\mathbf{u}, \mathbf{u}') = d(u, u') + \sum_{j=1}^{n-1} \delta(v_j, v_j'),$$

where d is a distance on A which defines the Fréchet topology of A with respect to which A is a complete metric space. Then (E_n, d_n) is a complete metric space. Now set, for $\mathbf{u} = (u, v_1, \cdots, v_n) \in E_{n+1}$, $n \geq 1$,

$$\theta_n(\mathbf{u}) = (u^2 + 1 - v_n, v_1, \cdots, v_{n-1}).$$

Then $\theta(n) : E_{n+1} \to E_n$ is continuous, and $\theta_n(E_{n+1})$ is dense in E_n since $\mathcal{M}$ is dense in A. It would then follow from the Mittag–Leffler theorem that $\varprojlim(E_n, \theta_n) \neq \emptyset$. Let $(\mathbf{u}_n)_{n\geq 1} \in \varprojlim(E_n, \theta_n)$. It follows from the definition of the maps θ_n thet there exists a sequence $(u_n)_{n\geq 1}$ of elements of A and a sequence $(v_n)_{n\geq 1}$ of elements of $\mathcal{M}$ such that we have, for $n \geq 1$,

$$\mathbf{u_n} = (u_n, v_1, \cdots, v_{n-1}),$$

and we have $u_n = u_{n+1}^2 + 1 - v_n$. Hence $\phi(u_n) = \phi(u_{n+1})^2 + 1$ for $n \geq 1$. But an induction on k shows then that $\phi(u_n) \geq 2^k$ for every $n \geq 1$ and every $k \geq 0$, which is impossible. □

Corollary 3.2. *Let Ω be a σ-compact, non compact locally compact space. Then all characters on the Fréchet algebra $\mathcal{C}(\Omega)$ are continuous.*

Proof: Let ϕ be a character on $\mathcal{C}(\Omega)$, denote by $\mathcal{C}_{\mathbb{R}}(\Omega)$ the algebra of continuous real–valued functions on Ω, and let $\tilde{\phi}$ be the restriction of ϕ to $\mathcal{C}_{\mathbb{R}}(\Omega)$. Since $\tilde{\phi}(f) = \phi(f) \in \sigma_{\mathcal{C}(\Omega)}(f) \subset \mathbb{R}$ for every $f \in \mathcal{C}_{\mathbb{R}}(\Omega)$, it follows from the theorem that $\tilde{\phi}$ is continuous on $\mathcal{C}_{\mathbb{R}}(\Omega)$. So there exists a compact subset K of Ω and $k > 0$ such that $|\phi(f)| \leq ksup_{t\in K}|f(t)|$ for $f \in \mathcal{C}_{\mathbb{R}}(\Omega)$. We obtain

$$|\phi(f)| = |\phi(Re(f)) + i\phi(Im(f))| \leq k\sqrt{2}sup_{t\in K}|f(t)| \;\; \forall f \in \mathcal{C}(K),$$

and so ϕ is continuous. □

With a little more work it is possible to show that for every character on $\mathcal{C}(\Omega)$ there exists $\tau \in K \subset \Omega$ such that $\phi(f) = f(\tau)$ for every $f \in \mathcal{C}(\Omega)$). The situation is a slightly more complicated for the algebra $\mathcal{C}(X)$ of continuous complex–valued functions on a completely regular space X. In general $\mathcal{C}(X)$ is not a Fréchet algebra, but it belongs to the class of locally multiplicatively convex complete algebras (complete LMC algebras) also studied by Michael in [40], which can be identified to the class of algebras isomorphic to a projective limit of a family of Banach algebras. It is possible to show that all characters on $\mathcal{C}(X)$ are bounded, but if X is not replete in the sense of definition 4.10.21 of [14] there exists characters on $\mathcal{C}(X)$ which do not have the form $\chi_\lambda : f \to f(\lambda)$ for some $\lambda \in X$. We refer to section 10.4 of [40] for more details.

Michael had asked in [40] whether all characters on complete LMC algebras were bounded. This question seems at first glance a little bit more general than the question of continuity of characters on Fréchet algebra (the class of complete LMC algebras contains the class of Fréchet algebras, and a character on a Fréchet algebra is continuous if and only if it is bounded), but Dixon and Fremlin showed in [19] that the two problems are in fact equivalent. Akkar gave in [3] a nice interpretation of this fact: the two problems are equivalent because every complete LMC algebra is, as a "bornological algebra", isomorphic to inductive limit of a family of Fréchet algebras. The equivalence of the two questions follows also from the fact that the existence of an unbounded character on a complete LMC algebra would imply the existence of a discontinuous character on the Fréchet algebras which are "test algebras" for Michael's problem, as Clayton's algebra [11] (Dixon and the author propose another algebra of this type in [15], but it seems to be similar to an unpublished example obtained by Mazur around 1937 [54]).

Denote by $\mathcal{H}(\mathbb{C}^p)$ the set of entire functions on $\mathbb{C}^p$, and let $f \in \mathcal{H}(\mathbb{C}^p)$. For $\alpha = (\alpha_1, \cdots, \alpha_p) \in [\mathcal{Z}^+]^p, \mathbf{z} = (z_1, \cdots, z_p) \in \mathbb{C}^p$, set $\mathbf{z}^\alpha = z_1^{\alpha_1} \cdots z_p^{\alpha_p}$, with the convention $0^0 = 1$, $\alpha! = \alpha_1! \cdots \alpha_p!$, $\partial^\alpha f = \frac{\partial^{\alpha 1 + \cdots + \alpha^p}(f)}{\partial z_1^{\alpha_1} \cdots \partial z_p^{\alpha_p}}, \mathbf{0} = 0_{\mathbb{C}^p}$. We have

$$f(\mathbf{z}) = \sum_{\alpha \in [\mathbb{Z}^+]^p} \frac{\partial^\alpha f(\mathbf{0})}{\alpha!} \mathbf{z}^\alpha.$$

Let A be a unital Fréchet algebra. For $\mathbf{a} = (a_1, \cdots, a_p) \in A^p$, set $\mathbf{a}^\alpha = a_1^{\alpha_1} \cdots a_p^{\alpha_p}$, with the convention $0^0 = 1_A$. We set, for $f \in \mathcal{H}(\mathbb{C}^p)$,

$$f(\mathbf{a}) = \sum_{\alpha \in [\mathbb{Z}^+]^p} \frac{\partial^\alpha f(\mathbf{0})}{\alpha!} \mathbf{a}^\alpha.$$

Now denote by $\mathcal{H}(\mathbb{C}^p, \mathbb{C}^q)$ the space of entire mappings from $\mathbb{C}^p$ into $\mathbb{C}^q$, i.e. the space of mappings $F = (f_1, \cdots, f_q) : \mathbb{C}^p \to \mathbb{C}^q$ such that $f_j \in \mathcal{H}(\mathbb{C}^p)$ for $1 \leq j \leq q$. We set, for $f \in \mathcal{H}(\mathbb{C}^p, \mathbb{C}^q)$, $\mathbf{a} \in A^p$,

$$F(\mathbf{a}) := (f_1(\mathbf{a}), \cdots, f_q(\mathbf{a})) \in A^q.$$

Also if χ is a character on A we set

$$\chi(a_1, \cdots, a_p) = (\chi(a_1), \cdots, \chi(a_p)).$$

It is easy to see that we have, for $F \in \mathcal{H}(\mathbb{C}^p, \mathbb{C}^q)$, $\mathbf{a} \in A^p$,

$$\chi(F(\mathbf{a})) = F(\chi(\mathbf{a})).$$

The complex counterpart of theorem 3.1, due to Dixon and the author, is given by the following result.

Theorem 3.3. *Assume that there exists a projective sequence $(\mathbb{C}^{p_n}, F_n)_{n \geq 1}$, where $p_n \geq 1$ and where $F_n : \mathbb{C}^{p_{n+1}} \to \mathbb{C}^{p_n}$ is an entire mapping for $n \geq 1$ such that $\varprojlim(\mathbb{C}^{p_n}, F_n) = \emptyset$. Then all characters on Fréchet algebras are continuous.*

It follows immediately from the big Picard theorem that $\varprojlim(\mathbb{C}, f_n) \neq \emptyset$ if $(f_n)_{n \geq 1}$ is any sequence of entire functions. On the other hand the "Poincaré–Fatou–Bieberbach phenomenon" shows that for every $p \geq 2$ there exists a one-to-one entire mapping $F : \mathbb{C}^p \to \mathbb{C}^p$ such that $F(\mathbb{C}^p)$ is not dense in $\mathbb{C}^p$, which suggests that there could exist a sequence $(F_n)_{n\geq 1}$ of entire mappings from $\mathbb{C}^p$ into itself such that $\cap_{n\geq 1}(F_1 \circ \cdots \circ F_n)(\mathbb{C}^p) = \emptyset$.

Some computer pictures of the intersection of the range of the original Bieberbach map from $\mathbb{C}^2$ into itself with two-dimensional real affine subspaces can be found in [18], p.149. These pictures suggest that the intersection of range of a one-to-one entire mapping $F \in \mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$ with two-dimensional real affine subspaces should be both either smooth or very irregular depending on the two-dimensional real affine subspace considered, but an important result of B. Stensones [50] shows that the range of such a map may have a $\mathcal{C}^\infty$-smooth boundary. A number of papers concerning Fatou–Bieberbach domains appeared since 1986, see [7, 8, 9, 10, 32, 31, 30, 35, 37, 39, 44, 51], but the existence of a sequence $(F_n)_{n\geq 1}$ of entire mappings from $\mathbb{C}^p$ into itself such that $\cap_{n\geq 1}(F_1 \circ ... \circ F_n)(\mathbb{C}^p) = \emptyset$. remains a big mystery for $p \geq 2$. Some more computer pictures of the range of concrete Fatou–Bieberbach maps would certainly be welcome, and the notion of Picard–Borel algebra, introduced in the author's "essay" on Michael's problem, Mittag–Leffler methods and Picard's theorem [27] might deserve further investigations.

4. Old and new results concerning homomorphisms from $\mathcal{C}(K)$ into a Banach algebra

In what follows K denotes an infinite compact space. If $s \in K$ we set $\mathcal{M}_s := \{f \in \mathcal{C}(K) \mid f(s) = 0\}$, $\mathcal{M}_K := \{\mathcal{M}_s\}_{s\in K}$, and we denote by $\mathcal{J}_s$ the set of continuous functions on K which vanish on some neighborhood of s. If B is a commutative Banach algebra, we will denote by $Rad(B)$ the Jacobson radical of B, i.e. the set of all quasinilpotent elements of B.

We introduce two notions which play a useful role in automatic continuity.

Definition 4.1. Let E and F be Banach spaces, ant let $T : E \to F$ be a linear operator. The separating space $\Delta(T)$ is the space of all $y \in F$ for which there exists a sequence $(x_n)_{n\geq 1}$ of elements of E satisfying

$$\lim_{n\to+\infty}\|x_n\| + \|y - T(x_n)\| = 0.$$

It follows from the closed graph theorem that T is continuous if and only if $\Delta(T) = \{0\}$.

Definition 4.2. Let $\phi : A \to B$ be a homomorphism from a Banach algebra A into a Banach algebra B. The continuity ideal $\mathcal{I}(\phi)$ of ϕ is the set of all $a \in A$ such that the map $\phi_a : x \to \phi(ax)$ is continuous.

Notice that it follows also from the closed graph theorem that if ϕ is a homomorphism from a Banach algebra A into a Banach algebra B, then $\phi(a)\Delta(\phi) = 0$ for every $a \in \mathcal{I}(\phi)$.

We now state the "main boundedness theorem" of Badé–Curtis [6].

Theorem 4.3. *("main boundedness theorem") Let A be a commutative Banach algebra and le $\phi : A \to B$ be a homomorphism from A into a Banach algebra*

B. Let $(g_n)_{n\geq 1}$ and $(h_n)_{n\geq 1}$ be two sequences of elements of $A \setminus \{0\}$ satisfying $g_nh_n = g_n$ for $n \geq 1$. Then we have

$$limsup_{n\to+\infty}\frac{\|\phi(g_n)\|}{\|g_n\|\|h_n\|} < +\infty.$$

Now let ϕ be a homomorphism from $\mathcal{C}(K)$ into a Banach algebra B. We can assume without loss of generality that $\phi(\mathcal{C}(K))$ is dense in B. It follows from the main boundedness theorem that if $(f_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{C}(K)$ such that $f_nf_m = 0$ for $n \neq m$, then there exists $N \geq 1$ such that $f_n^2 \in \mathcal{I}(\phi)$ for $n \geq N$. The main boundedness theorem also allowed Badé and Curtis to decompose ϕ as the sum of a "continuous" and a "singular" part.

Sinclair gave later in [47], theorem 10.3 a slightly more precise form of their result. Set $\Gamma := \{f \in \mathcal{C}(K) \mid \phi(f)\Delta(\phi) = 0\}$. Then the set $S := \{s \in K \mid \Gamma \subset \mathcal{M}_s\}$ is finite. Set $\mathcal{M}_S := \cap_{s\in S}\mathcal{M}_s$, denote by $\mathcal{J}_S$ the set of functions $f \in \mathcal{C}(K)$ vanishing on some open set containing S, and denote by $\mathcal{A}_S$ the set of functions $f \in \mathcal{C}(K)$ which are constant on some neighborhood of s for every $s \in S$. Then $\mathcal{J}_S$ is an ideal of $\mathcal{C}(K)$ contained in Λ which is dense in $\mathcal{M}_S$, $\mathcal{A}_S$ is a dense subalgebra of $\mathcal{C}(K)$, and we have the following properties.

(i) There exists a continuous homomorphism $\psi : \mathcal{C}(K) \to B$ such that $\psi(f) = \phi(f)$ for every $f \in \mathcal{A}_S$ and such that $\psi(gh) = \phi(gh)$ for every $g \in \Gamma$ and every $h \in \Gamma^-$;

(ii) $\psi(\mathcal{C}(K))$ is closed in B, $Rad(B) = \Delta(\phi)$, $\phi(\Gamma^-)Rad(B) = \{0\}$ and $B = \psi(\mathcal{C}(K)) \oplus Rad(B)$;

(iii) If we set $\theta = \phi-\psi$, then the restriction of θ to $\cap_{s\in S}\mathcal{M}_s$ is a homomorphism from $\cap_{s\in S}\mathcal{M}_s$ onto a dense subalgebra of $Rad(B)$;

(iv) There exists a family $(\theta_s)_{s\in S}$ of linear operators from $\mathcal{C}(K)$ into B such that

a) $\theta = \sum_{s\in S}\theta_s$,
b) $Rad(B) = \oplus_{s\in S}R_s$, where $R_s := [\theta_s(\mathcal{C}(K))]^-$,
c) $R_sR_t = \{0\}$ for $t \neq s$,
d) the restriction of θ_s to $\mathcal{M}_s$ is a homomorphism from $\mathcal{M}_s$ into $Rad(B)$, and $\theta_s(J_s) = \{0\}$.

The main boundedness, one of the key tools of automatic continuity theory, thus allowed to reduce the problem of existence of a discontinuous homomorphism of $\mathcal{C}(K)$ to the problem of existence of a nontrivial homomorphism from a maximal ideal of $\mathcal{C}(K)$ into a commutative radical Banach algebra. The next reduction was related to another key tool in automatic continuity theory, the so–called "stability lemma", which we state in the form given by Sinclair in lemma 1.6 of [47].

Lemma 4.4. *("stability lemma") Let E and F be Banach spaces, and let $(S_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ be sequences of bounded linear operators on E and F, respectively. If $T : E \to F$ is a linear operator satisfying $TS_n = R_nT$ for every $n \geq 1$. Then there exists and integer N satisfying, for every $n \geq N$,*

$$[R_1 \cdots R_n(\Delta(T))]^- = [R_1 \cdots R_N(\Delta(T))]^- .$$

Using the stability lemma applied to rational semigroups, Sinclair showed in [48] that if θ is a homomorphism from a maximal ideal $\mathcal{M}$ of $\mathcal{C}(K)$ into a commutative radical Banach algebra R, then $\theta(f) \in [\theta(f)^2 R]^-$ for every $f \in \mathcal{M}$, see theorem 11.7 in [47]. He also showed that if $f \in \mathcal{M} \setminus Ker(\theta)$ there exists a closed ideal L of B such that $\theta(f) \notin L$, such that $Ker(\pi \circ \theta)$ is prime and such that $\pi \circ \theta : \mathcal{M} \to B/J$ is a discontinuous homomorphism, where $\pi : u \to u + L$ demotes the canonical surjection from B onto B/L.

These results have far reaching-consequences.

(1) *Let ϕ be a homomorphism from $\mathcal{C}(K)$ into a Banach algebra B. Then $\mathcal{I}(\phi) = \{f \in \cap_{s \in S} \mathcal{M}_s \mid \phi(f)\Delta(\phi) = 0\} = \{f \in \cap_{s \in S} \mathcal{M}_s \mid \theta(f) = 0\}$, where θ denotes the "singular part" of ϕ in the sense of Badé and Curtis, and the restriction of ϕ to $\mathcal{I}(\phi)$ is continuous;*

(2) *if $f^n \in \mathcal{I}(\phi)$ for some $n \geq 2$, then $f \in \mathcal{I}(\phi)$ (in other terms, the continuity ideal $\mathcal{I}(\phi)$ is semiprime);*

(3) *if $(f_n)_{n \geq 1}$ is a sequence of elements of $\mathcal{C}(K)$ such that $f_n f_m = 0$ for $n \neq m$, then there exists $N \geq 1$ such that $f_n \in \mathcal{I}(\phi)$ for $n \geq N$;*

(4) *there exists a discontinuous homomorphism from $\mathcal{C}(K)$ into a Banach algebra if and only if there exists a prime ideal I of $\mathcal{C}(K)$ such that the quotient algebra $\mathcal{C}(K)/I$ possesses an algeba norm.*

Dales and the author showed independently that the quotient $l^\infty/\mathcal{U}$ is normable for every free ultrafilter $\mathcal{U}$ on $\mathcal{N}$ if the continuum hypothesis is assumed, which allowed them to construct discontinuous homomorphisms from $\mathcal{C}(K)$ for every infinite compact space K, see [15] for an outline of both constructions and detailed references. The author showed later that if the continuum hypothesis is assumed then every non–unital complex algebra of cardinality $2^{\aleph_0}$ which is an integral domain does possess an algebra norm, which shows that the quotient algebra $\mathcal{C}(K)/I$ possesses an algebra norm for every prime ideal I of $\mathcal{C}(K)$ such that $|\mathcal{C}(K)/I| = 2^{\aleph_0}$. We will present in the next section the author's approach, based on a "big" algebra of formal power series, and the alternative approach developed by Woodin to deduce the normability of non–unital integral domains of cardinality $2^{\aleph_0}$ from Dales' original construction of discontinuous homomorphisms from $\mathcal{C}(K)$.

In the other direction Solovay and Woodin constructed models of set theory including ZFC and Martin's axiom in which all homomorphisms from $\mathcal{C}(K)$ are continuous for every compact space K, see [16]. We will also briefly discuss the dependence of the answer to Kaplansky's problem on axioms of set theory in the next section.

Sinclair's results can be interpreted in terms on algebra seminorms, since $q_\phi : f \to \|\phi(f)\|$ is an algebra seminorm on $\mathcal{C}(K)$ for every homomorphism $\mathcal{C}(K)$ into a Banach algebra. [2] If we define the continuity ideal $\mathcal{I}(q)$ of an algebra seminorm q on $\mathcal{C}(K)$ to be the set of all f in $\mathcal{C}(K)$ such that $sup_{\|g\|_K \le 1} q(fg) < +\infty$, Sinclair's results show that the restriction of q to $\mathcal{I}(q)$ is continuous with respect to the usual norm $\|.\|_K$, that $\mathcal{I}(q)$ is semiprime and that for every $f \in \mathcal{C}(K) \setminus \mathcal{I}(q)$ there exists a prime ideal I of $\mathcal{C}(K)$ containing $\mathcal{I}(q)$ such that $f \notin I$. Also $\mathcal{I}(q)$ is a Badé–Curtis ideal in the sense of definition 1.2: for every sequence $(f_n)_{n\ge 1}$ of elements of $\mathcal{C}(K)$ such that $f_n f_m = 0$ for $m \ne n$ there exists an integer N such that $f_n \in \mathcal{I}(q)$ for every $n \ge N$.

The author developed in [20] an independent approach different of Sinclair's method, based on general properties of linear seminorms on $\mathbb{R}^{\mathbb{N}}$, and inspired by the work of Allan on elements of finite closed descent [2]. [3] He showed that in fact $[\phi(f)B]^- = [\phi(f)^2 B]^-$ for every $f \in \mathcal{C}(K)$ and every homomorphism ϕ from $\mathcal{C}(K)$ into a Banach algebra B, and considered general properties of the set of ideals of $\mathcal{C}(K)$ which are closed with respect to an algebra seminorm on $\mathcal{C}(K)$. We summarize these properties in the following theorem

Theorem 4.5. *Let q be an algebra seminorm on $\mathcal{C}(K)$, let $\mathcal{M}_K$ be the set of maximal ideals of $\mathcal{C}(K)$, and let $Prim(q)$ be the set of non maximal prime ideals of $\mathcal{C}(K)$ which are closed with respect to q.*

(i) For every ideal J of $\mathcal{C}(K)$, the closure $\overline{J}^q$ of J with respect to q satisfies

$$\overline{J}^q = \cap \{J \in Prim(q) \cup \mathcal{M}_K \ : \ J \subset I\};$$

(ii) the restriction of q to the continuity ideal $\mathcal{I}(q)$ is continuous with respect to the usual norm $\|.\|_K$, and the continuity ideal $\mathcal{I}(q)$ is the largest ideal I of $\mathcal{C}(K)$ such that the restriction of q to I is continuous with respect to the norm $\|.\|_K$.

(iii) $\mathcal{I}(q) = \cap\{I : I \in Prim(q)\}$;

(iv) $\mathcal{I}(q)$ is a pure semiprime Badé–Curtis ideal of $\mathcal{C}(K)$;

(v) every chain of elements of $Prim(q)$ is well–ordered with respect to inclusion.

Notice that property (v), proved by the author in [20], could also be obtained with the methods used by Sinclair in [48].

Recall that a compact space K is called a F–space if $f \in |f|\mathcal{C}(K)$ for every $f \in \mathcal{C}(K)$. For example, the Alexandroff and the Stone-Cěch compactifications of $\mathbb{N}$ are F–spaces. We obtain the following result [20].

Corollary 4.6. *Let K be a F–space, and let q be an algebra seminorm on $\mathcal{C}(K)$. Then the continuity ideal $\mathcal{I}(q)$ is the intersection of a finite family of non maximal prime ideals of $\mathcal{C}(K)$, and $Prim(q)$ is a finite union of well-ordered chains.*

The question whether corollary remained true for all infinite compact spaces remained open for many years. It was answered by the negative by Pham [41], assuming the continuum hypothesis. Recall that a family $(F_\lambda)_{\lambda\in\Lambda}$ of subsets of

[2] Conversely if q is an algebra seminorm on $\mathcal{C}(K)$ and if we denote by B the completion of the quotient algebra $\mathcal{C}(K)/Ker(q)$ with respect to the norm $\tilde{q} : f + Ker(q) \to q(f)$, then the map $\phi : f \to f + Ker(q)$ is a homomorphism from $\mathcal{C}(K)$ into B such that $q_\phi = q$.

[3] Both approaches involve implicitly or explicitly the Mittag–Leffler theorem

a set E is said to be pseudo–finite if for every $\lambda \in \Lambda$ and every $x \in F_\lambda$ the set $\{\mu \in \Lambda \mid x \notin F_\mu\}$ is finite or empty.

Also if Ω is a compact metric space, we denote by $\partial(\Omega)$ the set of all limit points of Ω, and we set

$$\partial_1(\Omega) = \partial(\Omega), \partial_n(\Omega) = \partial(\partial_{n-1}(\Omega) \text{ for } n \geq 2, \partial_\infty(\Omega) = \cap_{n\geq 1}\partial_n(\Omega).$$

Then either $\partial_n(\Omega) = \emptyset$ for some $n \geq 1$, or $\partial_\infty(\Omega) \neq \emptyset$. The following result was proved by Pham in [41].

Theorem 4.7. *(CH)*

(i) Let K be an infinite compact space, let $(I_\lambda)_{\lambda\in\Lambda}$ be an infinite, pseudo-finite family of nonmaximal prime ideals of $\mathcal{C}(K)$ such that $I_\lambda \nsubseteq I_\mu$ for $\lambda \neq \mu$, and set $\mathcal{I} = \cap_{\lambda\in\Lambda} I_\lambda$. If $|\mathcal{C}(K)/\mathcal{I}| = 2^{\aleph_0}$, then there exists an algebra seminorm q on $\mathcal{C}(K)$such that $Ker(q) = \mathcal{I}(q) = \mathcal{I}$.

(ii) Let Ω be an infinite compact metric space, and assume that $\partial_\infty(\Omega) \neq \emptyset$. Then for every $\omega \in \partial_\infty(\Omega)$, the maximal ideal $\mathcal{M}_\omega$ contains an infinite pseudo-finite family $(I_\lambda)_{\lambda\in\Lambda}$ of nonmaximal prime ideals of $\mathcal{C}(\Omega)$ such that $I_\lambda \nsubseteq I_\mu$ for $\lambda \neq \mu$.

Pham also showed in [42], assuming the continuum hypothesis, that there exists algebra seminorms on $\mathcal{C}([0, 1]$ such that $Ker(q) = \mathcal{I}(q)$ is not the intersection of any countable family of nonmaximal prime ideals.

Concerning the set $Prim(q)$, some new information will be given in [29], related to some results of [43]. We have the following observation.

Proposition 4.8. [29] *Let E be a set, let $\mathcal{F}$ be a family of subsets of E, and let $\mathcal{U}(\mathcal{F})$ be the family of all sets of the form $S_\mathcal{G} := \cup\{F \ : F \in \mathcal{G}\}$, where $\mathcal{G} \subset \mathcal{F}$. Then the following conditions imply each other:*

(i) Every chain of elements of $\mathcal{U}(\mathcal{F})$ is well–ordered with respect to inclusion.

(ii) Every sequence of elements of $\mathcal{F}$ possesses a pseudo–finite subsequence.

(iii) Every sequence of elements of $\mathcal{U}(\mathcal{F})$ possesses a pseudo–finite subsequence.

The following result, obtained independently by the author, follows from lemma 5.7 of [43], where the ideals $L_1, \cdots, L_m$ are called the "roofs" of the family $\mathcal{F}$.

Proposition 4.9. *Let $\mathcal{F}$ be a family of nonmaximal prime ideals of $\mathcal{C}(K)$ satisfying the equivalent conditions of proposition 4.8. Then for every $\mathcal{G} \subset \mathcal{F}$, there exists a finite family $L_1, \cdots, L_m$ of prime ideals of $\mathcal{C}(K)$ satisfying*

$$\cup\{I \ : I \in \mathcal{G}\} = L_1 \cup \cdots \cup L_m.$$

We will discuss at the end of the paper various characterizations of pure semiprime Badé–Curtis ideals of $\mathcal{C}(K)$. We give already here the following result from [29].

Theorem 4.10. *Let q be a discontinuous algebra seminorm on $\mathcal{C}(K)$. Then the set $Prim(q)$ satisfies the equivalent conditions of proposition 4.8.*

Corollary 4.11. *Let q be a discontinuous algebra seminorm on $\mathcal{C}(K)$, and let $\mathcal{F} \subset Prim(q)$. Then there exists a finite family $L_1, \cdots, L_m$ of prime ideals of $\mathcal{C}(K)$ satisfying*

$$\cup\{I \ : I \in \mathcal{F}\} = L_1 \cup \cdots \cup L_m.$$

We saw above that every chain of elements of $Prim(q)$ is well-ordered with respect to inclusion. The following result, to be proved in [29], shows that there is essentially no restriction on the ordinal type of these chains if the continuum hypothesis is assumed.

Theorem 4.12. *(CH) Let K be an infinite compact space, and let $\mathcal{F}$ be a well-ordered chain of nonmaximal prime ideals of $\mathcal{C}(K)$ such that $|\mathcal{C}(K)/\mathcal{I}| = 2^{\aleph_0}$, where $\mathcal{I} := \cap\{I \ : I \in \mathcal{F}\}$, and let $\mathcal{M}$ be the maximal ideal of $\mathcal{C}(K)$ containing all elements of $\mathcal{F}$.*

(i) There exists a discontinuous algebra seminorm q on $\mathcal{C}(K)$ such that $\mathcal{F} \subset Prim(q)$.

(ii) If, further, $\mathcal{F} \cup \{\mathcal{M}\}$ is stable under unions, there exists a discontinuous algebra seminorm q on $\mathcal{C}(K)$ such that $\mathcal{F} = Prim(q)$.

In fact the author obtained this result a long time ago, but was looking for a "natural" proof. Since such a natural proof still remains elusive, a correct but not very natural one will be given in [29].

5. Some open questions related to Kaplansky's problem

5.1. **Normability of big algebras of formal power series.** We first introduce some objects used by the author in his construction of discontinuous homomorphisms from $\mathcal{C}(K)$. A map ϕ from an ordered set E into an ordered set F will be said to be isotonic when it is order preserving. Let ω_1 be the smallest uncountable ordinal. We denote by $\mathbf{S} \subset \{0,1\}^{\omega_1}$ the set of all transfinite dyadic sequences $x = (x_\zeta)_{\zeta<\omega_1}$ for which there exists $\eta(x) < \omega_1$ such that $x_{\eta(x)} = 1$ and such that $x_\zeta = 0$ for every $\zeta > \eta(x)$.

Equipped with the lexicographic order, $\mathbf{S}$ is a linearly ordered set, and a classical result of Sierpiński [46], see also [28], shows that every linearly ordered set of cardinal $\leq \aleph_1$ is order-isomorphic to a subset of S.

To be more precise we need to recall the following notions (if A and B are nonempty subsets of a linearly ordered set E we will write $A < B$ when $x < y$ for every $x \in A$ and every $y \in B$).

Definition 5.1. Let $(E, <)$ be a linearly ordered set (resp. linearly ordered group, resp. linearly ordered field).

(i) E is said to be a β_1–set if there exist a chain $(E_\lambda)_{\lambda\in\Lambda}$ of subsets (resp. subgroups, resp. subfields) of E, with $E = \cup_{\lambda\in\Lambda} E_\lambda$, such that every subset of E_λ has a countable coinitial and cofinal subset for every $\lambda \in \Lambda$.

(ii) E is said to be a η_1–set if the two following conditions are satisfied

(a) E does not admit any countable coinitial and cofinal subset

(b) for every pair A, B of nonempty subsets of E, such that $|A| \leq \aleph_0$, $|B| \leq \aleph_0$, $A < B$ there exists $x \in E$ such that $A < x < B$.

The set $\mathbf{S}$ is a β_1–η_1 set, every β_1–set is order-isomorphic to a subset of $\mathbf{S}$, and every β_1–η_1 set is order isomorphic to $\mathbf{S}$.

Denote by $\mathbf{G} \subset \mathbf{S}^{\mathbb{R}}$ the set of all real–valued functions ϕ on $\mathbf{S}$ such that $Supp(\phi) := \{s \in \mathbf{S} \mid \phi(s) \neq 0\}$ is well–ordered and at most countable. For $\phi \in \mathbf{G} \setminus \{0\}$, denote by $v(\phi)$ the smallest element of $Supp(\phi)$. By definition, a nonzero element $\phi \in \mathbf{G}$ is said to be strictly positive if $\phi(v(\phi)) > 0$. Equipped with the linear structure inherited from the linear structure of $\mathbf{S}^{\mathbb{R}}$, $\mathbf{G}$ is a linearly ordered real vector space. In fact, $\mathbf{G}$ is a β_1–η_1 group which contains a copy of every β_1 group, and every linearly ordered divisible β_1–η_1 group is isomorphic as an ordered group to $\mathbf{G}$.

Now let G be a linearly ordered group, and let k be a field. We will denote by $\mathcal{F}(G,k)$ the set of all functions $f : G \to k$ such that $Supp(f) := \{\tau \in G \mid f(\tau \neq 0)\}$ is well–ordered, and we set

$$\mathcal{F}_{(1)}(G,k) := \{f \in \mathcal{F}(G,k) \mid |supp(g)| \leq \aleph_0\}.$$

Now let $f, g \in \mathcal{F}(G,k)$, and let $\tau \in G$. If $\tau \notin Supp(f) + Supp(g) := \{\alpha + \beta\}_{\alpha \in Supp(f), \beta \in Supp(g)}$, set $(fg)(\tau) = 0$. Otherwise set

$$(fg)(\tau) = \sum_{\substack{\alpha \in Supp(f), \beta \in Supp(g) \\ \alpha + \beta = \tau}} f(\alpha) g(\beta).$$

Then fg is well-defined, since the set $\{(\alpha,\beta) \in Supp(f) \times Supp(g) \mid \alpha + \beta = \tau\}$ is finite for every $\tau \in Supp(f) + Supp(g)$, and $fg \in \mathcal{F}(G,k)$. In fact Hahn observed in 1907 in [36] that $\mathcal{F}(G,k)$ is a field. Set $v(f) = inf(Supp(f))$ for $f \in \mathcal{F}(G,k) \setminus \{0\}$. Then v is a valuation on the field $\mathcal{F}(G,k)$, and the valued field $\mathcal{F}(G,k)$ is maximal: if U is a field containing $\mathcal{F}(G,k)$, and if w is a valuation on U with values in G such that $w(f) = v(f)$ for every $f \in \mathcal{F}(G,k) \setminus \{0\}$, then $U = \mathcal{F}(G,k)$.

Mac Lane showed in [38] that $\mathcal{F}(G,k)$ is algebraically closed if k is algebraically closed and if G is divisible, which means that the equation $nt = \tau$ has a solution in G for every $\tau \in G$ and every integer $n \geq 2$. Recall that a linearly ordered field k is said to be real–closed if $k(\sqrt{-1})$ is algebraically closed, or, equivalently, if every linearly ordered field strictly containing k contains an element which is transcendental over k. If k is real–closed, and if G is divisible, it follows from Mac Lane's result that $\mathcal{F}(G,k)$ is a linearly ordered real–closed field. In particular, $\mathcal{F}(G,\mathbb{C})$ and $\mathcal{F}_{(1)}(G,\mathbb{C})$ are algebraically closed fields and $\mathcal{F}(G,\mathbb{R})$ and $\mathcal{F}_{(1)}(G,\mathbb{R})$ are linearly ordered real–closed fields if G is a linearly ordered divisible group.

Now set $\mathbf{R} = \mathcal{F}_{(1)}(\mathbf{G},\mathbb{R})$, $\hat{\mathbf{R}} = \mathcal{F}(\mathbf{G},\mathbb{R})$, $\mathbf{C} = \mathcal{F}_{(1)}(\mathbf{G},\mathbb{R})$, $\hat{\mathbf{C}} = \mathcal{F}(\mathbf{G},\mathbb{C})$. Again, $\mathbf{R}$ is a real–closed β_1–η_1 field which contains a copy of every linearly ordered field of cardinal $\leq \aleph_1$, and every β_1–η_1 linearly ordered real–closed field is isomorphic to $\mathbf{R}$ (and also isomorphic to $\mathbf{R}$ as a real algebra). We denote by $\tilde{\mathbf{R}}$ the closure of $\mathbf{R}$ in $\hat{\mathbf{R}}$ with respect to the order topology, see [17], and we denote by $\tilde{\mathbf{C}} = \tilde{\mathbf{R}}(i) \subset \hat{\mathbf{C}}$ the algebraic closure of $\tilde{\mathbf{R}}$. We set, with the convention $v(0) = +\infty > \tau$ for every $\tau \in \mathbf{G}$,

$$\mathbf{R}^{\sharp} := \{f \in \mathbf{R} \mid v(f) \geq 0\}, \mathbf{C}^{\sharp} := \{f \in \mathbf{C} \mid v(f) \geq 0\},$$

and we define in a similar way $\hat{\mathbf{R}}^\sharp, \tilde{\mathbf{R}}^\sharp, \hat{\mathbf{C}}^\sharp$ and $\tilde{\mathbf{C}}^\sharp$.

The author's construction of a discontinuous homomorphism of $\mathcal{C}(K)$ was based on the following theorem

Theorem 5.2. *The algebra $\mathbf{C}^\sharp$ possesses an algebra norm.*

In fact, a commutative unital Banach algebra A contains a copy of $\mathbf{C}^\sharp$ if and only there exists a non nilpotent element $x \in Rad(A)$ such that $x^n \in [x^{n+1}Rad(A)]^-$ for some $n \geq 1$, which means that x has "finite closed descent" in the sense of Allan [2]. This shows in particular that if a commutative Banach algebra contains a copy of the algebra $\mathbb{C}[[X]]$ of all formal power series, it contains a copy of the much larger algebra $\mathbf{C}^\sharp$.

The maximal ideal of $\mathbf{C}^\sharp$ is "universal": if the continuum hypothesis is assumed it contains a copy of each complex nonunital algebra of cardinality $2^{\aleph_0}$ which is an integral domain. This shows that $\mathbf{C}^\sharp$ contains a copy of the quotient $\mathcal{C}(K)/I$ for every nonmaximal prime ideal I of $\mathcal{C}(K)$ such that $|\mathcal{C}(K)/I| = 2^{\aleph_0}$, which of course solves Kaplansky's problem assuming CH.

Our first open problem concerns "big algebras of formal power series".

Problem 5.3. Does the algebra $\hat{\mathbf{C}}^\sharp$ possess an algebra norm. If not, does the smaller algebra $\tilde{\mathbf{C}}^\sharp$ possess an algebra norm?

This problem is a major question left open by Dales and Woodin in [17]. The obstruction to the construction of an embedding from $\hat{\mathbf{C}}^\sharp$ into a Banach algebra is related to transcendental extensions: consider for example the convolution algebra $A = \mathbb{C}\delta_0 \oplus L^1(\mathbb{R}^+, e^{-t^2})$. There exists a "framework map" $\phi : \mathbf{G}^+ \to A$, i.e. a map satisfying $\phi(0) = \delta$, $\phi(\tau + \tau') = \phi(\tau)\phi(\tau')$ for $\tau \geq 0, \tau' \geq 0$, such that $\phi(\tau)$ generates a dense ideal of $L^1(\mathbb{R}^+, e^{-t^2})$ for every $\tau > 0$. Using this map, one can construct a one-to-one homomorphism $\theta : \mathbf{C}^\sharp$ such that $\theta(f) \in \phi(v(f))Inv(A)$ for every $f \in \mathbf{C}^\sharp \setminus \{0\}$. Now if D is a subalgebra of $\hat{\mathbf{C}}^\sharp$ containing the "monomials" X^τ for $\tau \in \mathbf{G}^+$, and if $\theta : D \to A$ is a (necessarily one-to-one) homomorphism satisfying $\theta(f) \in \phi(v(f))Inv(A)$ for every $f \in D \setminus \{0\}$ then it follows from the Arens–Calderon theorem [5] that there exists a homomorphism $\tilde{\theta} : K^\sharp \to A$ such that $\tilde{\theta}(f) \in \phi(v(f))Inv(A)$ for every $f \in K^\sharp \setminus \{0\}$ which extends θ, where K denotes the algebraic closure in $\hat{\mathbf{C}}$ of the field of fractions of D and where $K^\sharp := \{f \in K \mid v(f) > 0\}$. We refer to the last chapter of [14] for details. So the problem of extending a homomorphism from $\mathbf{C}^\sharp$ to the whole of $\hat{\mathbf{C}}^\sharp$ lies with transcendental extensions. In the case of $C^\sharp$, transcendental extensions are performed via a transfinite induction involving the fact that $\mathbf{C}^\sharp = \cup_{\zeta < \omega_1} \mathbf{C}^\sharp_\zeta$, where ω_1 is the smallest uncountable ordinal and where $\mathbf{C}_\zeta = \mathcal{F}(\mathbf{G}_\zeta, \mathbb{C})$, $\mathbf{G}_\zeta$ being a "β_1–group" for $\zeta < \omega_1$. The transcendental extensions within the algebras $\mathbf{C}^\sharp_\zeta$ can then be performed by using corollary 2.3 (ii), a consequence of the Mittag–Leffler theorem. We again refer the reader to the last chapter of [14] for details.

Of course we have here $|A| = 2^{\aleph_0}$, which can be strictly smaller than $|\hat{\mathbf{C}}^\sharp|$, but one could think that suitable Banach algebras extensions of A could allow to embed $\hat{\mathbf{C}}^\sharp$, or at least $\tilde{\mathbf{C}}^\sharp$, into some "big" Banach algebra, and problem 5.3

is probably the main open remaining question concerning normability of integral domains.

5.2. **Dependence on axioms of set theory.** We now briefly discuss the dependence of Kaplansky's problem to axioms of set theory. Recall that $\mathcal{U}$ is an filter on $\mathbb{N}$ if the following conditions are satisfied

(1) $\emptyset \notin \mathcal{U}$;
(2) If $A \in \mathcal{U}$, and if $A \subset B$, then $B \in \mathcal{U}$;
(3) If $A \in \mathcal{U}, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.

We will say that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ if $\mathbb{N} \setminus A \in \mathcal{U}$ for every subset A of $\mathbb{N}$ such that $A \notin \mathcal{U}$. An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is said to be a free ultrafilter if, or every $m \in \mathbb{N}, \mathcal{U} \neq \mathcal{U}_m := \{A \subset \mathbb{N} \mid m \in A\}$.

For $u = (u_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}}$, set $Z(u) := \{n \in \mathbb{N} \mid u_n = 0\}$ and if $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$ set $I_{\mathcal{U}} := \{u \in \mathbb{C}^{\mathbb{N}} \mid Z(u) \subset \mathcal{U}\}$. Then $I_{\mathcal{U}}$ is a maximal ideal of $\mathbb{C}^{\mathbb{N}}$, and $I_{\mathcal{U}} \cap \mathbb{R}^{\mathbb{N}}$ is a maximal ideal of $\mathbb{R}^{\mathbb{N}}$. We denote as usual by c_0 the algebra of complex sequences which converge to 0 as $n \to +\infty$ and by l^∞ the algebra of al bounded complex sequences, so that c_0 is a closed ideal of the Banach algebra $l^\infty \approx \mathcal{C}(\beta\mathbb{N})$. We set

$$\mathbb{C}^{\mathcal{N}}/\mathcal{U} := \mathbb{C}^{\mathcal{N}}/I_{\mathcal{U}}, \mathbb{R}^{\mathcal{N}}/\mathcal{U} := \mathbb{R}^{\mathcal{N}}/(I_{\mathcal{U}} \cap \mathbb{R}^{\mathcal{N}}),$$
$$l^\infty/\mathcal{U} := l^\infty/(I_{\mathcal{U}} \cap l^\infty), c_0/\mathcal{U} := c_0/(I_{\mathcal{U}} \cap c_0).$$

Before going into the dependence of Kaplansky's problem on axioms of set theory, we mention an interesting theorem of Woodin, which is true in ZFC, and is based upon the existence of "almost disjoint" subsets of $\mathbb{N}$, see [17].

Theorem 5.4. *For every non–unital complex algebra W which is an integral domain and satisfies $|B| = 2^{\aleph_0}$, there exists a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that the quotient algebra $c_0/\mathcal{U}$ contains a copy of W.*

If the continuum hypothesis is assumed, then all these quotient algebras $l^\infty/\mathcal{U}$ are isomorphic to the algebra $\mathbf{C}^\sharp$ introduced above, and one can use either the author's or Dales' construction of discontinuous homomorphisms of $\mathcal{C}(K)$ to deduce from Woodin's theorem that every nonunital complex algebra of cardinality $2^{\aleph_0}$ which is an integral domain possesses an algebra norm if the continuum hypothesis is assumed.

We will now outline the Solovay–Woodin proof of the consistency of the continuity of all homomorphisms from $\mathcal{C}(K)$, for all infinite compact spaces K, with ZFC. The following standard results can be found in [14].

(1) If there exists a discontinuous homomorphism from l^∞, then there exists a discontinuous homomorphism from $\mathcal{C}(K)$ for every compact space K.
(2) If there exists a discontinuous homomorphism from $\mathcal{C}(K)$ for some infinite compact space K, then there exists a discontinuous homomorphism from c_0.

It is well-known that if K is an infinite compact metric space, the existence of a discontinuous homomorphism from $\mathcal{C}(K)$ is equivalent to the existence of a discontinuous homomorphism from c_0, but the following problem seems to be still open.

Problem 5.5. Does the existence of a discontinuous homomorphism from c_0 imply the existence of a discontinuous homomorphism from l^∞?

Now assume that there exists a discontinuous homomorphism from $\mathcal{C}(K)$ for some infinite compact space K. Then there exists a discontinuous homomorphism from c_0, and so there exists a non modular prime ideal J of c_0 such that the quotient algebra c_0/J is normable. Denote by $c_0(\mathbb{R})$ the space of all real-valued sequences L which converge to 0, and set $L = J \cap c_0(\mathbb{R})$. Then the quotient algebra $c_0(\mathbb{R})/L$ is linearly ordered with respect to the quotient order induced by the natural partial order on $c_0(\mathbb{R})$. Let q be an algebra norm on c_0/L, and let $\mathcal{U}$ be the unique ultrafilter on $\mathbb{N}$ such that $I_\mathcal{U} \cap c_0(\mathbb{R}) \subset L$, set $c_0(\mathbb{R})\mathcal{U} := c_0(\mathbb{R})/(I_\mathcal{U} \cap c_0(\mathcal{R})$, and let $\pi : c_0(\mathcal{R})/\mathcal{U} \to c_0(\mathcal{R})/L$ be the canonical surjection. Set, for $\alpha \in c_0(\mathcal{R})/\mathcal{U}$,

$$\phi(\alpha) = (q(\pi(\alpha^n)))_{n \geq 1}.$$

Then $\phi(\alpha) \in c_0(\mathbb{R})$, since $\lim_{n\to+\infty} q(\pi(\alpha^n))^{\frac{1}{n}}$ $=\lim_{n\to+\infty} q(\pi(\alpha)^n)^{\frac{1}{n}} = 0$ for every $\alpha \in c_0/\mathcal{U}$, and if $\beta \in \alpha[c_0(\mathcal{R}/\mathcal{U}]$ then there exists $N \geq 1$ such that $q(\alpha) \prec q(\beta)$, where $\preccurlyeq$ is the partial order on $\mathbb{R}^\mathbb{N}$ defined by the formula

$$(x_n)_{n\geq 1} \preccurlyeq (y_n)_{n \geq 1} \text{ if and only if } x_n \leq y_n \text{ when } n \text{ is sufficiently large.}$$

Considering the set $\{\phi(\alpha)^{-1}\}_{\alpha \in \pi^{-1}([c_0(\mathbb{R})/L]\setminus\{0\})}$, we see that the existence of a discontinuous homomorphism from $\mathcal{C}(K)$ for some infinite compact space K would imply the existence of "very large" linearly ordered subsets of $(\mathbb{N}^\mathbb{N}, \preccurlyeq)$, and Solovay showed in 1976 that the impossibility of the existence of such very large linearly ordered subsets of $(\mathbb{N}^\mathbb{N}, \preccurlyeq)$ was consistent with ZFC, even if Martin's axiom, a popular alternative to the continuum hypothesis, is assumed. This construction uses Cohen's forcing theory, and we refer interested readers to the first monograph of Dales and Woodin [16].

The fact that the existence of discontinuous homomorphisms from $\mathcal{C}(K)$ does not imply the continuum hypothesis follows from the following theorem, proved independently by Frankiewicz–Zbierski and Woodin [33], [53].

Theorem 5.6. *The existence of a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that the quotient algebra $l^\infty/\mathcal{U}$ is isomorphic to the algebra $\mathbf{C}^\sharp$ is consistent with $2^{\aleph_0} = \aleph_2$.*

Since the algebra $\mathbf{C}^\sharp$ does possess an algebra norm, the Frankiewicz–Zbierski–Woodin theorem implies the existence of discontinuous homomorphisms from l^∞, which implies the existence of discontinuous homomorphisms from $\mathcal{C}(K)$ for every compact space K. This suggests the following problem (a negative answer to which would indeed necessitate the use of forcing theory).

Problem 5.7. Does the existence of a discontinuous homomorphism from $\mathcal{C}(K)$ for some compact space K imply that $2^{\aleph_0} \leq \aleph_2$?

5.3. **Characterization of the continuity ideal and of the set of closed nonmaximal prime ideals associated to a discontinuous algebra seminorm on $\mathcal{C}(K)$.** We will need the following notion

Definition 5.8. Let I be a prime ideal of $\mathcal{C}(K)$, set $A_I := \mathcal{C}(K)/I$, and let $\mathcal{J}(I)$ be the intersection of the set of all prime ideals of $\mathcal{C}(K)$ contained in I. A subalgebra B_I of $\mathcal{C}(K)$ is said to be a lifting of the quotient algebra A_I if the following two conditions are satisfied

(1) $B_I \cap I \subset \mathcal{J}(I)$
(2) $\mathcal{C}(K) = B_I + I$.

Notice that if B_I is a lifting of A_I then $\tilde{B}_I := B_I + \mathcal{J}(I)$ is also a lifting of A_I, which is obviously the largest lifting of A_I containing B_I. Also if I is a minimal prime ideal of $\mathcal{C}(K)$ then $\mathcal{C}(K)$ is obviously a lifting of A_I, and if $\mathcal{M}$ is a maximal ideal of $\mathcal{C}(K)$ then the algebra of constant functions is a lifting of the quotient algebra $\mathcal{A}_{\mathcal{M}} \approx \mathbb{C}$.

It is shown in [29] that the quotient algebra A_I does possess a lifting for every prime ideal I of $\mathcal{C}(K)$. More precisely let I be a prime ideal of $\mathcal{C}(K)$, and let J be a prime ideal of $\mathcal{C}(K)$ containing I. It proved in [29] that every lifting of A_I contains a lifting of A_J, and that, conversely, every lifting of A_J is contained in some lifting of A_I. This suggests the following problem.

Problem 5.9. Let I be a minimal prime ideal of $\mathcal{C}(K)$ and let $\mathcal{F}$ be the set of prime ideals of $\mathcal{C}(K)$ containing I. Does there exist a family $\{B_J\}_{J\in\mathcal{F}}$ of subalgebras of $\mathcal{C}(K)$ which possesses the following properties?

(i) B_J is a lifting of the quotient algebra A_J for every $J \in \mathcal{F}$.

(ii) If $J \in \mathcal{F}, L \in \mathcal{F}$, and if $J \subset L$, then $B_L \subset B_J$.

A positive answer to this problem would give a very simple proof of theorem 4.12 (i), but the author was not able so far to solve this question, despite periodic attempts during the last 35 years. The answer might depend on axioms of set theory (it follows from the structure of the algebra $\mathbf{C}^\sharp$ that the answer if yes if $K = \beta\mathbb{N}$ if the continuum hypothesis is assumed).

We now formulate another problem, for which we assume that the continuum hypothesis is true.

Problem 5.10. (CH) Let $\mathcal{G}$ be a family of nonmaximal prime ideals of $\mathcal{C}(K)$ such that every sequence of elements of $\mathcal{G}$ possesses a pseudo–finite subsequence and such that $|\mathcal{C}(K)/I_{\mathcal{G}} = 2^{\aleph_0}$, where $I_{\mathcal{G}}$ denotes the intersection of all elements of $\mathcal{G}$.

(i) Does there exist a discontinuous algebra seminorm on $\mathcal{C}(K)$ such that $\mathcal{G} \subset Prim(q)$?

(ii) Assume, further, that the union of every uncountable pseudo–finite family of elements of $\mathcal{G}$ and the union of every chain of elements of $\mathcal{G}$ without any countable cofinal subset which are not maximal ideals belong to $\mathcal{G}$. Does there exist a discontinuous algebra seminorm on $\mathcal{C}(K)$ such that $\mathcal{G} = Prim(q)$?

Notice that for this problem considering algebra norms instead of algebra seminorms make no difference, since it follows from Kaplansky' initial result that for every discontinuous algebra seminorm q on $\mathcal{C}(K)$ there exists a discontinuous algebra norm $\tilde{q}$ on $\mathcal{C}(K)$ such that $Prim(\tilde{q}) = Prim(q)$. Partial results concerning this problem can be found in [43]. The methods of [29] and [43] would certainly allow to give a positive answer to question (i) of the problem for families $\mathcal{G}$ of nonmaximal prime ideals of $\mathcal{C}(K)$ for which there exists a family $\{B_J\}_{J\in\mathcal{G}}$ of subalgebras of $\mathcal{C}(K)$ satisfying the following properties

(i) B_J is a lifting of the quotient algebra A_J for every $J \in \mathcal{G}$.

(ii) If $J \in \mathcal{F}, L \in \mathcal{G}$, and if $J \subset L$, then $B_L \subset B_J$,

which leads to questions more general than problem 5.9.

We saw that if ϕ is any discontinuous homomorphism from $\mathcal{C}(K)$ then the continuity ideal of ϕ is a pure semiprime Badé-Curtis ideal of $\mathcal{C}(K)$ (this notion coincides for ideals of $\mathcal{C}(K)$ with the notion of "abstract continuity ideal" introduced by Pham in [43], and various characterizations of semiprime pure Badé-Curtis ideals will be given in [29]). This suggests the following question, which can also be formulated in terms of continuity ideals of discontinuous algebra seminorms on $\mathcal{C}(K)$, or, equivalently, in terms of continuity ideals of discontinuous algebra norms on $\mathcal{C}(K)$.

Problem 5.11. (CH) Let I be a pure semiprime Badé–Curtis ideal of $\mathcal{C}(K)$ such that $|\mathcal{C}(K)/I| = 2^{\aleph_0}$. Does there exist a discontinuous homomorphism ϕ from $\mathcal{C}(K)$ into a Banach algebra such that the continuity ideal of ϕ is equal to I ?

Partial results on this question, which is less general than problem 5.10, are given by Pham in [43]. Problem 5.11 has a great heuristic importance: the fact that the continuity ideal of a homomorphism from $\mathcal{C}(K)$ is a pure semiprime Badé–Curtis ideal, which follows from Sinclair's work [48], is a consequence of the main boundedness theorem and the stability lemma, the two basic tools of automatic continuity theory. A positive answer to problem 5.11 would show that these two tools provide all the information about partial continuity properties of discontinuous homomorphisms from $\mathcal{C}(K)$, at least if the continuum hypothesis is assumed.

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