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ON LINEAR MAPS COMPRESSING OR DEPRESSING CERTAIN SUBSPACES

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ABSTRACT. Let X be a complex Banach space and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X. We characterize surjective linear maps $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ compressing or depressing any one of the range, the hyper-range, the analytic core and the kernel.

1. INTRODUCTION

There has been an interest in preserver problems that leave certain linear subspaces, invariant; see for instance [5, 6, 7, 12, 15]. In [15], the author characterized surjective additive maps $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserving the range or the kernel of operators. In [6], we obtained the descriptions of surjective additive maps that preserve the hyper-range, the analytic core, or the hyper-kernel of operators. Also, in [5], we determined the forms of all additive maps $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserving the local spectral subspace $X_T(\{\lambda\})$, i.e., $X_{\phi(T)}(\{\lambda\}) = X_T(\{\lambda\})$ for all $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$.

In this note, we treat surjective linear maps $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ that compress or depress certain subspaces of Banach space X. Namely, we determine the forms of maps ϕ which compress $\Delta(.)$ i.e., $\Delta(\phi(T)) \subset \Delta(T)$ for all $T \in \mathcal{L}(X)$ or depress $\Delta(.)$ i.e., $\Delta(T) \subset \Delta(\phi(T))$ for all $T \in \mathcal{L}(X)$ where $\Delta(.)$ denotes any one of $\mathrm{R}(.), \mathcal{R}^\infty(.), \mathrm{K}(.)$ and $\mathrm{N}(.)$.

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2. Notations and Preliminaries

Let X be a complex Banach space and let $\mathcal{L}(X)$ be the algebra of all bounded operators on X. For $T \in \mathcal{L}(X)$, we write $\mathrm{N}(T)$ for its kernel and $\mathrm{R}(T)$ for its range. The spectrum of T is denoted by $\sigma(T)$. The surjectivity spectrum $\sigma_s(T)$is defined by $\sigma_s(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective } \}$. We say that a map $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ is unital if $\phi(I) = I$, where I stands for the unit of $\mathcal{L}(X)$.

Let x be a nonzero vector in X and f be a nonzero functional in the topological dual X^* of X. We denote, as usual, by $x \otimes f$ the rank one operator given by $(x \otimes f)z = f(z)x$ for $z \in X$. Note that $x \otimes f$ is a projection if and only if $f(x) = 1$, and it is nilpotent if and only if $f(x) = 0$. The adjoint of such operator is given by $(x \otimes f)^* = f \otimes Jx$, where J is the natural embedding of X to X^{**}. We denote by span $\{x\}$ the subspace spanned by x. We write $\mathcal{F}_1(X)$ for the set of all rank one operators on X.

Recall that the hyper-range and the analytic core of an operator $T \in \mathcal{L}(X)$ are given, respectively, by $\mathcal{R}^\infty(T) := \bigcap\limits_{n \in \mathbb{N}} \mathrm{R}(T^n)$ and $\mathrm{K}(T) := \{x \in X : \text{ there exist } a > 0$ and a sequence $(x_n) \in X$ satisfying : $x_0 = x, Tx_{n+1} = x_n$ and $\| x_n \| \leq a^n \| x \|$, for all $n \geq 1\}$. Recall that $\mathcal{R}^\infty(T)$ and $\mathrm{K}(T)$ are the subspaces of X and $\mathrm{K}(T) \subset \mathcal{R}^\infty(T) \subset \mathrm{R}(T)$; see for example [1, 11, 14]. Note that

$$\mathrm{K}(T) = X \Leftrightarrow \mathcal{R}^\infty(T) = X \Leftrightarrow \mathrm{R}(T) = X$$

and

$$\mathrm{K}(x \otimes f) = \mathcal{R}^\infty(x \otimes f) = \mathrm{R}(x \otimes f) = \mathrm{span}\,\{x\}$$

where $x \in X$ and $f \in X^*$ such that $f(x) \neq 0$.

We start with the following lemma, see [4].

Lemma 2.1. *Let X and Y be complex Banach spaces. Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective linear map. Suppose that ϕ satisfy $\sigma_{su}(\phi(T)) \subset \sigma_{su}(T)$ for all $T \in \mathcal{L}(X)$ then either $\phi(F) = 0$ for all finite rank operator $F \in \mathcal{L}(X)$ or ϕ is injective. In the latter case, either*
(1) *there exists an invertible operator $A \in \mathcal{L}(X, Y)$ such that $\phi(T) = ATA^{-1}$ for all $T \in \mathcal{L}(X)$ or*
(2)*there exists an invertible operator $A \in \mathcal{L}(X^*, Y)$ such that $\phi(T) = AT^*A^{-1}$ for all $T \in \mathcal{L}(X)$. In the last case X and Y are reflexive.*

We need the following lemma about perturbations by rank one operators, so as to state the next lemma.

Lemma 2.2. *([16]) Let $T \in \mathcal{L}(X)$ be an invertible operator, let x be a nonzero vector in X, f be a nonzero functional in X^*. Then $T - x \otimes f$ is not invertible if and only if $f(T^{-1}x) = 1$.*

Lemma 2.3. *Let $A, B \in \mathcal{L}(X)$ be two invertible operators. If one of the two following assertions:*

(i) $\mathrm{R}(A + F) \subset \mathrm{R}(B + F)$ *for all* $F \in \mathcal{F}_1(X)$ *or*
(ii) $\mathrm{N}(A + F) \subset \mathrm{N}(B + F)$ *for all* $F \in \mathcal{F}_1(X)$

holds true then $A = B$.

Proof. Let $A, B \in \mathcal{L}(X)$ be two invertible operators. Let $x \in X$ and $f \in X^*$ such that $f(x) = 1$.

Suppose that (i) holds true. Let $F = -Bx \otimes f$. We have

$$\begin{aligned} \mathrm{R}(A - Bx \otimes f) &\subset \mathrm{R}(B - Bx \otimes f) \\ &= \mathrm{R}(I - Bx \otimes (B^{-1})^* f) \\ &= \mathrm{N}((B^{-1})^* f) \nsubseteq X. \end{aligned}$$

Then $A - Bx \otimes f$ is not surjective and so $A - Bx \otimes f$ is not invertible. By Lemma 2.2, we get that

$$f(A^{-1}Bx) = 1 = f(x).$$

This implies that $A^{-1}Bx = x$ and then $A = B$.

Now suppose that (ii) is yield and let $F = -Ax \otimes f$. We have

$$\begin{aligned} \operatorname{span}\{x\} &= \mathrm{N}(I - x \otimes f) \\ &= \mathrm{N}(A(I - x \otimes f)) \\ &= \mathrm{N}(A - Ax \otimes f) \\ &\subset \mathrm{N}(B - Ax \otimes f). \end{aligned}$$

Then $B - Ax \otimes f$ is not injective and so $B - Ax \otimes f$ is not invertible. Lemma 2.2, gives that

$$f(B^{-1}Ax) = 1 = f(x).$$

Consequently, $B^{-1}Ax = x$ and then $A = B$.

□

3. Main Results

Theorem 3.1. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective linear map such that $S := \phi(I)$ is invertible. Then the following assertions are equivalent:*

(i) $\mathrm{R}(\phi(T)) \subset \mathrm{R}(T)$ *for all* $T \in \mathcal{L}(X)$*;*
(ii) $\mathrm{R}(T) \subset \mathrm{R}(\phi(T))$ *for all* $T \in \mathcal{L}(X)$*;*
(iii) $\phi(T) = TS$ *for all* $T \in \mathcal{L}(X)$*.*

Proof. (i)$\Longrightarrow$(iii). Let $\psi(T) = \phi(T)S^{-1}$ for all $T \in \mathcal{L}(X)$, so we have

$$\mathrm{R}(\psi(T)) \subset \mathrm{R}(T) \quad \text{for all} \quad T \in \mathcal{L}(X).$$

Assume that there exists F a rank-one idempotent of $\mathcal{L}(X)$ such that $\psi(F) = 0$. We write $F = x \otimes f$ where $x \in X$, $f \in X^*$ such that $f(x) = 1$.
We have

$$X = \mathrm{R}(I) = \mathrm{R}(\psi(I)) = \mathrm{R}(\psi(I - F)) \subset \mathrm{R}(I - F) = \mathrm{N}(f)$$

a contradiction.
Then ψ does not annihilate all rank-one idempotents of $\mathcal{L}(X)$.

On the other hand, Let $F = x \otimes f$ where $x \in X$, $f \in X^*$. If $f(x) = 1$, we have

$$\{0\} \neq \mathrm{R}(\psi(F)) \subset \mathrm{R}(F) = \operatorname{span}\{x\}.$$

Then $\mathrm{R}(\psi(F)) = \operatorname{span}\{x\}$ and $\psi(F) = x \otimes g_f$ where g_f is a nonzero functional in X^*. We have

$$\mathrm{R}(I - x \otimes g_f) = \mathrm{R}(I - \psi(x \otimes f)) = \mathrm{R}(\psi(I - x \otimes f)) \subset \mathrm{R}(I - x \otimes f) = \mathrm{N}(f).$$

Then $z - g_f(z)x \in \mathrm{N}(f)$ for all $z \in X$ and so $g_f(z) = f(z)$ for all $z \in X$. It follows that $\psi(F) = F$. Thus, if $f(x) = \lambda \neq 0$, we have

$$\psi(x \otimes f) = \lambda\psi(\frac{1}{\lambda}x \otimes f) = \lambda\frac{1}{\lambda}x \otimes f = x \otimes f.$$

Now, let $0 \neq y \in X$ and $0 \neq g \in X^*$ such that $g(y) = 0$. Let $x \in X$ such that $g(x) = 1$. We have

$$\psi(y \otimes g) = \psi((x + y) \otimes g) - \psi(x \otimes g) = (x + y) \otimes g - x \otimes g = y \otimes g.$$

Therefore $\psi(F) = F$ for all $F \in \mathcal{F}_1(X)$.

Let $T \in \mathcal{L}(X)$ and $\lambda \notin \sigma(T) \cup \sigma(\psi(T))$. We have

$$\mathrm{R}(\psi(T) - \lambda + F) = \mathrm{R}(\psi(T - \lambda + F)) \subset \mathrm{R}(T - \lambda + F) \text{ for all } F \in \mathcal{F}_1(X).$$

Lemma 2.3 (i) gives that $\psi(T) = T$. As desired.

(ii)$\Longrightarrow$(iii). Consider $\psi(T) = \phi(T)S^{-1}$ for all $T \in \mathcal{L}(X)$, so we have

$$\mathrm{R}(T) \subset \mathrm{R}(\psi(T)) \quad \text{for all } T \in \mathcal{L}(X).$$

ψ is injective. Indeed, let $T \in \mathcal{L}(X)$ such that $\psi(T) = 0$, then $\mathrm{R}(T) \subset \mathrm{R}(\psi(T)) = \{0\}$ and so $T = 0$. Therefore ψ is bijective. Let ψ^{-1} the inverse of ψ then we have

$$\mathrm{R}(\psi^{-1}(T)) \subset \mathrm{R}(T) \text{ for all } T \in \mathcal{L}(X).$$

Since $\psi^{-1}(I) = I$ then, by Theorem 3.1 (i), it follows that $\psi^{-1}(T) = T$ for all $T \in \mathcal{L}(X)$. Consequently, $\phi(T) = TS$ for all $T \in \mathcal{L}(X)$.

(iii)$\Longrightarrow$(i) and (iii)$\Longrightarrow$(ii) are obvious.

□

Remark 3.2. (1) It turns out, from the hypothesis $\mathrm{R}(T) \subset \mathrm{R}(\phi(T))$ for all $T \in \mathcal{L}(X)$, that S is surjective.
(2) Note that (iii)$\Longrightarrow$(i) is valid without considering any condition on S.

Theorem 3.3. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective linear map such that $S := \phi(I)$ is invertible. Then the following assertions are equivalent:*

(i) *$\mathcal{R}^\infty(T) \subset \mathcal{R}^\infty(\phi(T))$ for all $T \in \mathcal{L}(X)$;*
(ii) *$\mathcal{R}^\infty(\phi(T)) \subset \mathcal{R}^\infty(T)$ for all $T \in \mathcal{L}(X)$;*
(iii) *there exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$.*

Proof. (i)$\Rightarrow$(iii). Consider $\psi(T) = \phi(T)S^{-1}$ for all $T \in \mathcal{L}(X)$. The surjective linear map ψ is unital and maps surjective operators to surjective operators then

$$\sigma_{su}(\psi(T)) \subset \sigma_{su}(T) \text{ for all } T \in \mathcal{L}(X).$$

We obtain by Lemma 2.1, that:
$\psi(F) = 0$ for all finite rank operator $F \in \mathcal{L}(X)$; or
ψ takes one of two following forms:

(1) there exists an invertible operator $A \in \mathcal{L}(X)$ such that $\psi(T) = ATA^{-1}$ for all $T \in \mathcal{L}(X)$; or
(2) there exists an invertible operator $A \in \mathcal{L}(X^*, X)$ such that $\psi(T) = AT^*A^{-1}$ for all $T \in \mathcal{L}(X)$. In this case X is reflexive.

Suppose that ψ annihilates all finite rank operators. Let $x \in X$ and $f \in X^*$ such that $f(x) = 1$, then we have

$$\begin{aligned} \text{span}\,\{x\} &= \mathcal{R}^\infty(x \otimes f) \subset \mathcal{R}^\infty(\phi(x \otimes f)) \\ &\subset \text{R}(\phi(x \otimes f)) = \text{R}(\psi(x \otimes f)) \\ &= \{0\}. \end{aligned}$$

A contradiction.

Suppose that ψ takes the form (2). Let $x \in X$ and $f \in X^*$ such that x and Af are linearly independent and $f(x) \neq 0$. We have

$$\begin{aligned} \text{span}\,\{x\} &= \mathcal{R}^\infty(x \otimes f) \subset \mathcal{R}^\infty(\phi(x \otimes f)) \\ &\subset \text{R}(\phi(x \otimes f)) = \text{R}(\psi(x \otimes f)) \\ &= \text{R}(Af \otimes (A^{-1})^* J_x) = \text{span}\,\{Af\}. \end{aligned}$$

Then $\text{span}\,\{x\} = \text{span}\,\{Af\}$. Consequently Af and x are linearly dependent, a contradiction.

Now, assume that ψ takes the form (1). Let $x \in X$ and $f \in X^*$ such that $f(x) \neq 0$. We have

$$\begin{aligned} \text{span}\,\{x\} &= \mathcal{R}^\infty(x \otimes f) \subset \mathcal{R}^\infty(\phi(x \otimes f)) \\ &\subset \text{R}(\phi(x \otimes f)) = \text{R}(\psi(x \otimes f)) \\ &= \text{R}(Ax \otimes (A^{-1})^* f) = \text{span}\,\{Ax\}. \end{aligned}$$

Therefore x and Ax are linearly dependent for all $x \in X$ and so $A = cI$ for some nonzero scalar $c \in \mathbb{C}$. Consequently $\psi(T) = T$ for all $T \in \mathcal{L}(X)$, thus $\phi(T) = TS$ for all $T \in \mathcal{L}(X)$.

Let $y \in X$ and $g \in X^*$ be such that $g(y) = 1$. We have
$\text{R}(I - y \otimes g) = \mathcal{R}^\infty(I - y \otimes g) \subset \mathcal{R}^\infty(\phi(I - y \otimes g)) \subset \text{R}(\phi(I - y \otimes g)) = \text{R}(\psi(I - y \otimes g)) = \text{R}(I - y \otimes g)$.
Hence, it follows that $\mathcal{R}^\infty(\phi(I - y \otimes g)) = \text{R}(\phi(I - y \otimes g))$. In particular we have

$$\text{R}((I - y \otimes g)S) = \text{R}(((I - y \otimes g)S)^2) = \text{R}((I - y \otimes g)S(I - y \otimes g)).$$

Let $u \in X$ be such that $(I - y \otimes g)Sy = (I - y \otimes g)S(I - y \otimes g)u$.
Applying S^{-1} we obtain

$$\begin{aligned} y - g(Sy)S^{-1}y &= (S^{-1} - S^{-1}y \otimes g)(Su - g(u)Sy) \\ &= u - g(u)y - g(Su - g(u)Sy)S^{-1}y \end{aligned}$$

Applying g we obtain:

$$g(y) - g(Sy)g(S^{-1}y) = g(u) - g(u)g(y) - g(Su - g(u)Sy)g(S^{-1}y).$$

Therefore

$$(g(Sy) - g(Su - g(u)Sy))g(S^{-1}y) = 1$$

which implies that $g(S^{-1}y) \neq 0$. Consequently, y and $S^{-1}y$ are linearly dependent. Hence $S = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$. Finally $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$.

(ii)⇒(iii). Consider also here $\psi(T) = \phi(T)S^{-1}$ for all $T \in \mathcal{L}(X)$. It is easy to see that if $\psi(T)$ is surjective then T is surjective. The surjective linear map ψ is unital and then satisfy

$$\sigma_{su}(T) \subset \sigma_{su}(\psi(T)) \text{ for all } T \in \mathcal{L}(X).$$

We derive from [8, Corollary 8] that:
ψ takes one of two following forms:
(1) there exists an invertible operator $A \in \mathcal{L}(X)$ such that $\psi(T) = ATA^{-1}$ for all $T \in \mathcal{L}(X)$; or
(2) there exists an invertible operator $A \in \mathcal{L}(X^*, X)$ such that $\psi(T) = AT^*A^{-1}$ for all $T \in \mathcal{L}(X)$. In this case X is reflexive.

As in (i)⇒(iii) of the proof of this Theorem, we show that the form (2) of ψ can not be occur and we check, in the case where ψ takes the form (1), that $A = c^{'}I$ for some nonzero scalar $c^{'} \in \mathbb{C}$. We proceed similarly to the last step of (i)⇒(iii), but here we consider the operator $(I - y \otimes g)S^{-1}$ instead of $(I - y \otimes g)$ and then we obtain that $S = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$.

(iii)⟹(i) and (iii)⟹(ii) are obvious.

□

Theorem 3.4. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective linear map such that $S := \phi(I)$ is invertible. Then the following assertions are equivalent:*

(i) $\mathrm{K}(T) \subset \mathrm{K}(\phi(T))$ *for all* $T \in \mathcal{L}(X)$;
(ii) $\mathrm{K}(\phi(T)) \subset \mathrm{K}(T)$ *for all* $T \in \mathcal{L}(X)$;
(iii) *there exists a nonzero scalar* $\mu \in \mathbb{C}$ *such that* $\phi(T) = \mu T$ *for all* $T \in \mathcal{L}(X)$.

Proof. We proceed as in the proof of Theorem 3.3. Using the following properties,

$$\mathrm{K}(T) \subset \mathcal{R}^{\infty}(T) \text{ for all } \; T \in \mathcal{L}(X)$$

and

$$\mathrm{K}(T) = \mathcal{R}^{\infty}(T) \text{ if } \; T \in \mathcal{L}(X) \text{ is a projection or of rank one.}$$

□

Theorem 3.5. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective linear map such that $S := \phi(I)$ is invertible. Then the following assertions are equivalent:*

(i) $\mathrm{N}(T) \subset \mathrm{N}(\phi(T))$ *for all* $T \in \mathcal{L}(X)$;
(ii) $\mathrm{N}(\phi(T)) \subset \mathrm{N}(T)$ *for all* $T \in \mathcal{L}(X)$;
(iii) $\phi(T) = ST$ *for all* $T \in \mathcal{L}(X)$.

Proof. (i)⟹(iii). Let $\psi(T) = S^{-1}\phi(T)$ for all $T \in \mathcal{L}(X)$, so we have

$$\mathrm{N}(T) \subset \mathrm{N}(\psi(T)) \quad \text{for all } \; T \in \mathcal{L}(X).$$

Let $x \in X$ and $f \in X^*$ such that $f(x) = 1$, then we have

$$\mathrm{N}(f) = \mathrm{N}(x \otimes f) \subset \mathrm{N}(\psi(x \otimes f))$$

and

$$\mathrm{span}\,\{x\} = \mathrm{N}(I - x \otimes f) \subset \mathrm{N}(I - \psi(x \otimes f)).$$

Since $X = \mathrm{span}\,\{x\} \oplus \mathrm{N}(f)$, let $z \in X$ such that $z = \alpha x + y$ for some scalar α in $\mathbb{C}$ and y in $\mathrm{N}(f)$, so $f(z) = \alpha f(x) + f(y) = \alpha$. We have

$$\begin{aligned} \psi(x \otimes f)z &= \alpha\psi(x \otimes f)x + \psi(x \otimes f)y \\ &= \alpha x + 0 \quad (\text{ see the two inclusions above}) \\ &= f(z)x \\ &= (x \otimes f)z. \end{aligned}$$

Then $\psi(x \otimes f) = x \otimes f$. It follows, easily, that $\psi(x \otimes f) = x \otimes f$ for all $x \in X$ and $f \in X^*$ such that $f(x) \neq 0$.

Now, in the case where $f(x) = 0$, there exist two non-nilpotent operators F_1 and F_2 such that $x \otimes f = F_1 + F_2$ and then

$$\begin{aligned} \psi(x \otimes f) &= \psi(F_1 + F_2) = \psi(F_1) + \psi(F_2) \\ &= F_1 + F_2 = x \otimes f. \end{aligned}$$

Thus $\psi(F) = F$ for all $F \in \mathcal{F}_1(X)$.

Let $T \in \mathcal{L}(X)$ and $\lambda \notin \sigma(T) \cup \sigma(\psi(T))$. We have

$$\mathrm{N}(T - \lambda + F) \subset \mathrm{N}(\psi(T - \lambda + F)) = \mathrm{N}(\psi(T) - \lambda + F) \text{ for all } F \in \mathcal{F}_1(X).$$

Lemma 2.3 (ii) gives that $\psi(T) = T$.

(ii)$\Longrightarrow$(iii). Consider again $\psi(T) = S^{-1}\phi(T)$ for all $T \in \mathcal{L}(X)$, so we have

$$\mathrm{N}(\psi(T)) \subset \mathrm{N}(T) \quad \text{for all } T \in \mathcal{L}(X).$$

ψ is injective. Indeed, let $T \in \mathcal{L}(X)$ such that $\psi(T) = 0$, then $X = \mathrm{N}(\psi(T)) \subset \mathrm{N}(T)$ and so $T = 0$. Therefore ψ is bijective. Let ψ^{-1} the inverse of ψ then we have

$$\mathrm{N}(T) \subset \mathrm{N}(\psi^{-1}(T)) \text{ for all } T \in \mathcal{L}(X).$$

Since $\psi^{-1}(I) = I$ then, by Theorem 3.5 (i), we get that $\psi^{-1}(T) = T$ for all $T \in \mathcal{L}(X)$. Consequently, $\phi(T) = ST$ for all $T \in \mathcal{L}(X)$.

(iii)$\Longrightarrow$(i) and (iii)$\Longrightarrow$(ii) are obvious. □

Some authors interested in some problems of maps that preserve certain functions of operator products; see for example, [2, 7, 9, 10, 13]. The following corollary concerns linear maps compressing or depressing $\Delta(.)$ of operator products.

Corollary 3.6. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective linear map such that $S := \phi(I)$ is invertible. Then the following assertions are equivalent:*

(i) $\mathrm{R}(AB) \subset \mathrm{R}(\phi(A)\phi(B))$ *for all* $A, B \in \mathcal{L}(X)$*;*
(ii) $\mathrm{R}(\phi(A)\phi(B)) \subset \mathrm{R}(AB)$ *for all* $A, B \in \mathcal{L}(X)$*;*
(iii) $\mathrm{N}(AB) \subset \mathrm{N}(\phi(A)\phi(B))$ *for all* $A, B \in \mathcal{L}(X)$*;*

(iv) $\mathrm{N}(\phi(A)\phi(B)) \subset \mathrm{N}(AB)$ *for all* $A, B \in \mathcal{L}(X)$*;*
(v) $\mathcal{R}^\infty(AB) \subset \mathcal{R}^\infty(\phi(A)\phi(B))$ *for all* $A, B \in \mathcal{L}(X)$*;*
(vi) $\mathcal{R}^\infty(\phi(A)\phi(B)) \subset \mathcal{R}^\infty(AB)$ *for all* $A, B \in \mathcal{L}(X)$
(vii) $\mathrm{K}(AB) \subset \mathrm{K}(\phi(A)\phi(B))$ *for all* $A, B \in \mathcal{L}(X)$*;*
(viii) $\mathrm{K}(\phi(A)\phi(B)) \subset \mathrm{R}(AB)$ *for all* $A, B \in \mathcal{L}(X)$*;*
(ix) *there exists a nonzero scalar* $\mu \in \mathbb{C}$ *such that* $\phi(T) = \mu T$ *for all* $T \in \mathcal{L}(X)$.

Proof. (i)$\Longrightarrow$(ix). Suppose that $\mathrm{R}(AB) \subset \mathrm{R}(\phi(A)\phi(B))$ for all $A, B \in \mathcal{L}(X)$. For $B = I$, we have

$$\mathrm{R}(A) \subset \mathrm{R}(\phi(A)S) = \mathrm{R}(\phi(A)) \text{ for all } \ A \in \mathcal{L}(X).$$

Then Theorem 3.1 (i) gives that $\phi(A) = AS$ for all $A \in \mathcal{L}(X)$. We have so $\mathrm{R}(AB) \subset \mathrm{R}(\phi(A)\phi(B)) = \mathrm{R}(ASBS) = \mathrm{R}(ASB)$ for all $A, B \in \mathcal{L}(X)$. Taking $A = I$ and $B = x \otimes f$ where $x \in X$ and $f \in X^*$ such that $f(x) = 1$, we get that

$$\mathrm{span}\,\{x\} = \mathrm{R}(x \otimes f) \subset \mathrm{R}(Sx \otimes f) = \mathrm{span}\,\{Sx\}.$$

This implies that x and Sx are linearly dependent and then $S = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$.

(ii)$\Longrightarrow$(ix) is similar to (i)$\Longrightarrow$(ix).

(iii)$\Longrightarrow$(ix). Suppose that $\mathrm{N}(AB) \subset \mathrm{N}(\phi(A)\phi(B))$ for all $A, B \in \mathcal{L}(X)$. For $A = I$, we have

$$\mathrm{N}(B) \subset \mathrm{N}(S\phi(B)) = \mathrm{N}(\phi(B)) \text{ for all } \ B \in \mathcal{L}(X).$$

Then Theorem 3.5 (i) gives that $\phi(B) = SB$ for all $B \in \mathcal{L}(X)$. We have so $\mathrm{N}(AB) \subset \mathrm{N}(\phi(A)\phi(B)) = \mathrm{N}(SASB) = \mathrm{N}(ASB)$ for all $A, B \in \mathcal{L}(X)$. Taking $B = I$ and $A = I - x \otimes f$ where $x \in X$ and $f \in X^*$ such that $f(x) = 1$, we get that
$\mathrm{span}\,\{x\} = \mathrm{N}(I - x \otimes f) \subset \mathrm{N}((I - x \otimes f)S) = \mathrm{N}(S(I - S^{-1}x \otimes S^*f)) = \mathrm{N}(I - S^{-1}x \otimes S^*f) = \mathrm{span}\,\{S^{-1}x\}$.
This implies that x and $S^{-1}x$ are linearly dependent and then $S = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$.

(iv)$\Longrightarrow$(ix) is similar to (iii)$\Longrightarrow$(ix).

(v)$\Longrightarrow$(ix). Suppose that $\mathcal{R}^\infty(AB) \subset \mathcal{R}^\infty(\phi(A)\phi(B))$ for all $A, B \in \mathcal{L}(X)$. For $B = I$, we have

$$\mathcal{R}^\infty(A) \subset \mathcal{R}^\infty(\phi(A)S) \text{ for all } \ A \in \mathcal{L}(X).$$

Let $\Phi(A) = \phi(A)S$ for all $A \in \mathcal{L}(X)$. We have so $\mathcal{R}^\infty(A) \subset \mathcal{R}^\infty(\Phi(A))$ for all $A \in \mathcal{L}(X)$ and $\Phi(I) = S^2$ is invertible, then by Theorem 3.3 (i), there exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\Phi(A) = \mu A$ for all $A \in \mathcal{L}(X)$. Therefore

$$\begin{aligned} \mathcal{R}^\infty(AB) \subset \mathcal{R}^\infty(\phi(A)\phi(B)) &= \mathcal{R}^\infty(\mu AS^{-1}\mu BS^{-1}) = \mathcal{R}^\infty(AS^{-1}BS^{-1}) \\ &\subset \mathrm{R}(\mathrm{AS^{-1}BS^{-1}}) = \mathrm{R}(\mathrm{AS^{-1}B}) \end{aligned}$$

for all $A, B \in \mathcal{L}(X)$. In particular for $A = I$ and $B = x \otimes f$ where $x \in X$ and $f \in X^*$ such that $f(x) \neq 0$, we have

$$\text{span}\,\{x\} = \mathcal{R}^\infty(x \otimes f) \subset \mathcal{R}^\infty(S^{-1}x \otimes f) = \text{span}\,\{S^{-1}x\}.$$

This completes the proof of (v)$\Longrightarrow$(ix).

(vi)$\Longrightarrow$(ix). We proceed as in (v)$\Longrightarrow$(ix) and we obtain that $\mathcal{R}^\infty(AS^{-1}BS^{-1}) \subset \mathcal{R}^\infty(AB)$ for all $A, B \in \mathcal{L}(X)$. Then $\mathcal{R}^\infty(AB) \subset \mathcal{R}^\infty(ASBS)$ for all $A, B \in \mathcal{L}(X)$ and $S = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$.

(vii)$\Longrightarrow$(ix) is similar to (v)$\Longrightarrow$(ix).

(viii)$\Longrightarrow$(ix) is similar to (vi)$\Longrightarrow$(ix).

□

Recall that the hyper-kernel of an operator $T \in \mathcal{L}(X)$ is given by

$$\mathcal{N}^\infty(T) := \bigcup_{n\in\mathbb{N}} \mathrm{N}(T^n).$$

Remark 3.7. Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective additive map. Suppose that ϕ satisfy one of the following assertions :

(i) $\mathrm{R}(T) = \mathrm{R}(\phi(T))$ for all $T \in \mathcal{L}(X)$
(ii) $\mathcal{R}^\infty(\phi(T)) = \mathcal{R}^\infty(T)$ for all $T \in \mathcal{L}(X)$
(iii) $\mathrm{K}(\phi(T)) = \mathrm{K}(T)$ for all $T \in \mathcal{L}(X)$
(iv) $\mathrm{N}(T) = \mathrm{N}(\phi(T))$ for all $T \in \mathcal{L}(X)$
(v) $\mathcal{N}^\infty(\phi(T)) = \mathcal{N}^\infty(T)$ for all $T \in \mathcal{L}(X)$.

then $\phi(I)$ is invertible. see [6, 15].

We finish this note with the following question:

Question 3.8. Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective linear map such that $S := \phi(I)$ is invertible. Does we have the equivalences between the following assertions :

(i) $\mathcal{N}^\infty(T) \subset \mathcal{N}^\infty(\phi(T))$ for all $T \in \mathcal{L}(X)$;
(ii) $\mathcal{N}^\infty(\phi(T)) \subset \mathcal{N}^\infty(T)$ for all $T \in \mathcal{L}(X)$;
(iii) there exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$.

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