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# 2-LOCAL DERIVATIONS ON ALGEBRAS OF LOCALLY MEASURABLE OPERATORS

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ABSTRACT. The paper is devoted to 2-local derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type $\mathrm{I}_\infty$ von Neumann algebra $M$. We prove that every 2-local derivations on any $*$-subalgebra $\mathcal{A}$ in $LS(M)$, such that $M \subseteq \mathcal{A}$, is a derivation.

## 1. INTRODUCTION

Given an algebra $\mathcal{A}$, a linear operator $D : \mathcal{A} \to \mathcal{A}$ is called a *derivation*, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$ (the Leibniz rule). Each element $a \in \mathcal{A}$ implements a derivation $D_a$ on $\mathcal{A}$ defined as $D_a(x) = [a, x] = ax - xa$, $x \in \mathcal{A}$. Such derivations $D_a$ are said to be *inner derivations*. If the element $a$, implementing the derivation $D_a$, belongs to a larger algebra $\mathcal{B}$ containing $\mathcal{A}$, then $D_a$ is called *a spatial derivation* on $\mathcal{A}$.

There exist various types of linear operators which are close to derivations [8, 9, 14]. In particular R. Kadison [8] has introduced and investigated so-called local derivations on von Neumann algebras and some polynomial algebras.

A linear operator $\Delta$ on an algebra $\mathcal{A}$ is called a *local derivation* if given any $x \in \mathcal{A}$ there exists a derivation $D$ (depending on $x$) such that $\Delta(x) = D(x)$. The main problems concerning this notion are to find conditions under which local derivations become derivations and to present examples of algebras with local derivations that are not derivations [8]. In particular Kadison [8] has proved that each continuous local derivation from a von Neumann algebra $M$ into a dual $M$-bimodule is a derivation.

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In 1997, P. Semrl [14] introduced the concept of 2-local derivations and automorphisms. A map $\Delta : \mathcal{A} \to \mathcal{A}$ (not linear in general) is called a 2-*local derivation* if for every $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \to \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$. Local and 2-local derivations have been studied on different operator algebras by many authors [2, 3, 4, 5, 7, 8, 9, 10, 11, 14].

In [14], P. Semrl described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space $H$. A similar description for the finite-dimensional case appeared later in [9]. In the paper [11] 2-local derivations have been described on matrix algebras over finite-dimensional division rings. J. H. Zhang and H. X. Li [17] described 2-local derivations on symmetric digraph algebras and constructed a 2-local derivation on the algebra of all upper triangular complex $2 \times 2$-matrices which is not a derivation. In [3] first two authors considered 2-local derivations on the algebra $B(H)$ of all linear bounded operators on an arbitrary (no separability is assumed) Hilbert space $H$ and proved that every 2-local derivation on $B(H)$ is a derivation.

The present paper is devoted to study 2-local derivations on $*$-subalgebras of the algebra $LS(M)$ of all locally measurable operators with respect to type $\mathrm{I}_\infty$ von Neumann algebra $M$. We prove that every 2-local derivations on any $*$-subalgebra $\mathcal{A}$ in $LS(M)$, such that $M \subseteq \mathcal{A}$, is a derivation.

## 2. Algebra of locally measurable operators

Let $B(H)$ be the $*$-algebra of all bounded linear operators on a Hilbert space $H$, and let $\mathbf{1}$ be the identity operator on $H$. Consider a von Neumann algebra $M \subset B(H)$. Denote by $P(M) = \{p \in M : p = p^2 = p^*\}$ the lattice of all projections in $M$ and by $P_{fin}(M)$ the set of all finite projections in $P(M)$.

A linear subspace $\mathcal{D}$ in $H$ is said to be *affiliated* with $M$ (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary $u$ from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra $M$.

A linear operator $x : \mathcal{D}(x) \to H$, where the domain $\mathcal{D}(x)$ of $x$ is a linear subspace of $H$, is said to be *affiliated* with $M$ (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$ and for every unitary $u \in M'$.

A linear subspace $\mathcal{D}$ in $H$ is said to be *strongly dense* in $H$ with respect to the von Neumann algebra $M$, if

- $\mathcal{D}\eta M$;
- there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^{\perp} = \mathbf{1} - p_n$ is finite in $M$ for all $n \in \mathbf{N}$, where $\mathbf{N}$ is the set of all natural numbers.

A closed linear operator $x$ acting in the Hilbert space $H$ is said to be *measurable* with respect to the von Neumann algebra $M$, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in $H$.

Denote by $S(M)$ the set of all linear operators on $H$, measurable with respect to the von Neumann algebra $M$. If $x \in S(M)$, $\lambda \in \mathbf{C}$, where $\mathbf{C}$ is the field of complex numbers, then $\lambda x \in S(M)$ and the operator $x^*$, adjoint to $x$, is also measurable with respect to $M$ (see [13]). Moreover, if $x, y \in S(M)$, then the operators $x + y$ and $xy$ are defined on dense subspaces and admit closures that are called, correspondingly, the strong sum and the strong product of the operators $x$ and $y$, and are denoted by $x \dot{+} y$ and $x * y$. It was shown in [13] that $x \dot{+} y$ and $x * y$ belong to $S(M)$ and these algebraic operations make $S(M)$ a $*$-algebra with the identity $\mathbf{1}$ over the field $\mathbf{C}$. It is clear that, $M$ is a $*$-subalgebra of $S(M)$. In what follows, the strong sum and the strong product of operators $x$ and $y$ will be denoted in the same way as the usual operations, by $x + y$ and $xy$.

A closed linear operator $x$ in $H$ is said to be *locally measurable* with respect to the von Neumann algebra $M$, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in $M$ such that $z_n \uparrow \mathbf{1}$ and $z_n x \in S(M)$ for all $n \in \mathbf{N}$ (see [16]).

Denote by $LS(M)$ the set of all linear operators that are locally measurable with respect to $M$. It was proved in [16] that $LS(M)$ is a $*$-algebra over the field $\mathbf{C}$ with the identity $\mathbf{1}$ under the strong addition and the strong multiplication. In such a case, $S(M)$ is a $*$-subalgebra in $LS(M)$. In the case where $M$ is a finite von Neumann algebra or a factor, the algebras $S(M)$ and $LS(M)$ coincide. This is not true in the general case. In [12] the class of von Neumann algebras $M$ has been described for which the algebras $LS(M)$ and $S(M)$ coincide.

We say that a measure $\mu$ on a measure space $(\Omega, \Sigma, \mu)$ has the direct sum property if there is a family $\{\Omega_i\}_{i \in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$, $\mu(A) < \infty$, there exist a countable subset $J_0 \subset J$ and a set $B$ with zero measure such that $A = \bigcup\limits_{i \in J_0} (A \cap \Omega_i) \cup B$.

It is well-known (see e.g. [13]) that for each commutative von Neumann algebra $M$ there exists a measure space $(\Omega, \Sigma, \mu)$ with $\mu$ having the direct sum property such that $M$ is $*$-isomorphic to the algebra $L^{\infty}(\Omega, \Sigma, \mu)$ of all (equivalence classes of) complex essentially bounded measurable functions on $(\Omega, \Sigma, \mu)$ and in this case $LS(M) = S(M) \cong L^0(\Omega, \Sigma, \mu)$, where $L^0(\Omega, \Sigma, \mu)$ is the algebra of all (equivalence classes of) complex measurable functions on $(\Omega, \Sigma, \mu)$.

Further we consider the algebra $S(Z(M))$ of operators which are measurable with respect to the center $Z(M)$ of the von Neumann algebra $M$. Since $Z(M)$ is an abelian von Neumann algebra, it is $*$-isomorphic to $L^{\infty}(\Omega, \Sigma, \mu)$ for an appropriate measure space $(\Omega, \Sigma, \mu)$. Therefore the algebra $S(Z(M))$ coincides with $Z(LS(M))$ and can be identified with the algebra $L^0(\Omega, \Sigma, \mu)$.

Let $M$ be a von Neumann algebra. Given an element $x \in LS(M)$, the smallest projection $p$ in $M$ with $xp = x$ is called the *right support* of $x$ and denoted by $r(x)$. The *left support* $l(x)$ is the smallest projection $p$ in $M$ with $px = x$. For a $*$-subalgebra $\mathcal{A} \subset LS(M)$ denote

$$\mathcal{F}(\mathcal{A}) = \{x \in \mathcal{A} : l(x) \in P_{fin}(M)\}.$$

From the definition of the algebra $\mathcal{F}(\mathcal{A})$ we have that the following properties are equivalent:

(1) $x \in \mathcal{F}(\mathcal{A})$;
(2) $\exists\, p \in P_{fin}(M)$ such that $px = x$;
(3) $\exists\, p \in P_{fin}(M)$ such that $xp = x$;

(4) $\exists\, p \in P_{fin}(M)$ such that $pxp = x$.

Note that $\mathcal{F}(A)$ is an $*$-ideal in $\mathcal{A}$. Moreover the algebra $\mathcal{F}(\mathcal{A})$ is semi-prime, i.e. if $a \in \mathcal{F}(\mathcal{A})$ and $a\mathcal{F}(\mathcal{A})a = \{0\}$ then $a = 0$. Indeed, let $a \in \mathcal{F}(\mathcal{A})$ and $a\mathcal{F}(\mathcal{A})a = \{0\}$, i.e. $axa = 0$ for all $x \in \mathcal{F}(\mathcal{A})$. In particular for $x = a^*$ we have $aa^*a = 0$ and hence $a^*aa^*a = 0$, i.e. $|a|^4 = 0$. Therefore $a = 0$.

Recall the definition of the faithful normal semifinite extended center valued trace on the algebra $M$ (see [15]).

Let $M$ be an arbitrary von Neumann algebra with the center $Z(M) \equiv L^\infty(\Omega, \Sigma, \mu)$ and let $M_+$ be the set of all positive elements of $M$. By $L_+$ we denote the set of all measurable functions $f : (\Omega, \Sigma, \mu) \to [0, \infty]$ (modulo functions equal to zero $\mu$-almost everywhere). Then there exists a map $\Phi : M_+ \to L_+$ with the following properties:

(1) $\Phi(x + y) = \Phi(x) + \Phi(y)$ for $x, y \in M_+$;
(2) $\Phi(ax) = a\Phi(x)$ for $a \in Z(M)_+, x \in M_+$;
(3) $\Phi(xx^*) = \Phi(x^*x)$;
(4) $\Phi(x^*x) = 0 \Rightarrow x = 0$;
(5) $\Phi\left(\sup\limits_{i\in J} x_i\right) = \sup\limits_{i\in J} \Phi(x_i)$ for any bounded increasing net $\{x_i\}$ in $M_+$.

This map $\Phi : M_+ \to L_+$, is called the *extended center valued trace* on $M$.

The set $\{x \in M : \Phi(x^*x) \in Z(M)\}$ is an ideal of $M$. If this ideal is $\sigma$-weakly dense in $M$, then $\Phi$ is said to be *semifinite*.

It is well-known (see e.g. [15]) that a von Neumann algebra $M$ is semifinite if and only if $M$ admits a faithful, semifinite, normal extended center valued trace.

Let us remark that a projection $p \in M$ is finite if and only if $\Phi(p) \in S(Z(M))$. Hence for any $x \in \mathcal{F}(LS(M)) \cap M_+$ we have that $\Phi(x) \in S(Z(M))$.

Note that the algebra $LS(M)$ has the following remarkable property: given any family $\{z_i\}_{i\in I}$ of mutually orthogonal central projections in $M$ with $\bigvee\limits_{i\in I} z_i = \mathbf{1}$ and a family of elements $\{x_i\}_{i\in I}$ in $LS(M)$ there exists a unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$. This element is denoted by $x = \sum\limits_{i\in I} z_i x_i$ (see [12]). Conversely if $M$ is a type I von Neumann algebra then for an arbitrary element $x \in LS(M)$ there exists a sequence $\{z_n\}$ of mutually orthogonal central projections with $\bigvee\limits_{n\in\mathbb{N}} z_n = \mathbf{1}$ such that $z_n x \in M$ for all $n \in \mathbb{N}$ (see [1]). For $0 \le x \in \mathcal{F}(LS(M))$ set

$$\Phi(x) = \sum_{n\in\mathbb{N}} z_n \Phi(z_n x). \tag{2.1}$$

Since the trace $\Phi$ is $Z(M)$-homogeneous, the equality (2.1) gives a well-defined map from $\mathcal{F}(LS(M))_+$ into $S(Z(M))$.

Since each element of $\mathcal{F}(LS(M))$ is a finite linear combination of positive elements from $\mathcal{F}(LS(M))$ we can naturally extend $\Phi$ to a $S(Z(M))$-valued trace on $\mathcal{F}(LS(M))$.

Now let $\mu$ be an arbitrary faithful normal semifinite trace on $Z(M)$. Put $\tau = \mu \circ \Phi$. Then by [15, Lemma 2.16] we have that

$$\tau(xy) = \tau(yx)$$

for all $x \in M$, $y \in \mathcal{F}(LS(M)) \cap M$. Therefore

$$\Phi(xy) = \Phi(yx)$$

for all $x \in LS(M)$, $y \in \mathcal{F}(LS(M))$. Since the trace $\Phi$ maps the set $\mathcal{F}(LS(M))$ into $S(Z(M))$ and $\mathcal{F}(LS(M))$ is an ideal in $LS(M)$ we have

$$\Phi(axy) = \Phi((ax)y) = \Phi((ya)x) = \Phi(xya),$$

i.e.

$$\Phi(axy) = \Phi(xya) \tag{2.2}$$

for all $a, x \in LS(M)$, $y \in \mathcal{F}(LS(M))$.

## 3. Main results

Let $D$ be a derivation on $LS(M)$. Then $D$ maps the ideal $\mathcal{F}(LS(M))$ into itself. Indeed, for any $x \in \mathcal{F}(LS(M))$ there exists a finite projection $p \in M$ such that $x = xp$. Then

$$D(x) = D(xp) = D(x)p + xD(p),$$

and therefore $D(x) \in \mathcal{F}(LS(M))$. Hence any 2-local derivation on $LS(M)$ also maps $\mathcal{F}(LS(M))$ into itself.

**Lemma 3.1.** *Let $M$ be an arbitrary semifinite von Neumann algebra and let $b \in LS(M)$ be an arbitrary element. If $\Phi(xb) = 0$ for all $x \in \mathcal{F}(LS(M))$ then $b = 0$.*

*Proof.* Let $b \in LS(M)$. For any finite projection $e \in LS(M)$ we have $eb^* \in \mathcal{F}(LS(M))$ and therefore by the assumption of the lemma it follows that $\Phi(eb^*b) = 0$. Thus

$$0 = \Phi(eb^*b) = \Phi(e^2b^*b) = \Phi(eb^*be) = \Phi((be)^*(be)),$$

i.e.

$$\Phi((be)^*(be)) = 0.$$

Since the trace $\Phi$ is faithful, we obtain $(be)^*(be) = 0$, i.e. $be = 0$.

Now take a family of finite projections $\{e_\alpha\}_{\alpha \in J}$ in $M$ such that $e_\alpha \uparrow \mathbf{1}$. Then

$$0 = be_\alpha b^* \uparrow bb^*,$$

i.e. $bb^* = 0$. Thus $b = 0$. The proof is complete. □

**Lemma 3.2.** *Let $M$ be an arbitrary von Neumann algebra of type $I_\infty$ and let $\Delta : LS(M) \to LS(M)$ be a 2-local derivation. Then*

(1) *$\Delta$ is $S(Z(M))$-homogenous, i.e. $\Delta(cx) = c\Delta(x)$ for all $c \in S(Z(M))$, $x \in LS(M)$;*

(2) *$\Delta(x^2) = \Delta(x)x + x\Delta(x)$ for all $x \in LS(M)$.*

*Proof.* (1) For each $x \in LS(M)$, and for $c \in S(Z(M))$ there exists a derivation $D_{x,cx}$ such that $\Delta(x) = D_{x,cx}(x)$ and $\Delta(cx) = D_{x,cx}(cx)$. Since $M$ is a type $\mathrm{I}_\infty$ then by [1, Theorem 2.7] every derivation on $LS(M)$ is inner, in particular, $S(Z(M))$-linear. Therefore

$$\Delta(cx) = D_{x,cx}(cx) = cD_{x,cx}(x) = c\Delta(x).$$

Hence, $\Delta$ is $S(Z(M))$-homogenous.

(2) For each $x \in LS(M)$, there exists a derivation $D_{x,x^2}$ such that $\Delta(x) = D_{x,x^2}(x)$ and $\Delta(x^2) = D_{x,x^2}(x^2)$. Then

$$\Delta(x^2) = D_{x,x^2}(x^2) = D_{x,x^2}(x)x + xD_{x,x^2}(x) = \Delta(x)x + x\Delta(x)$$

for all $x \in LS(M)$. The proof is complete. □

**Lemma 3.3.** *Let $M$ be an arbitrary von Neumann algebra of type $I_\infty$. If $\Delta : LS(M) \to LS(M)$ is a 2-local derivation such that $\Delta|_{\mathcal{F}(LS(M))} \equiv 0$, then $\Delta \equiv 0$.*

*Proof.* Let $\Delta : LS(M) \to LS(M)$ be a 2-local derivation such that $\Delta|_{\mathcal{F}(LS(M))} \equiv 0$. For arbitrary $x \in LS(M)$ and $y \in \mathcal{F}(LS(M))$ there exists a derivation $D_{x,y}$ on $LS(M)$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$. By [1, Theorem 2.7] there exists element $a \in LS(M)$ such that

$$[a, xy] = D_{x,y}(xy) = D_{x,y}(x)y + xD_{x,y}(y) = \Delta(x)y + x\Delta(y),$$

i.e.

$$[a, xy] = \Delta(x)y + x\Delta(y).$$

Since $y \in \mathcal{F}(LS(M))$ we have $\Delta(y) = 0$, and therefore $[a, xy] = \Delta(x)y$. By the equality (2.2) we obtain that

$$0 = \Phi(axy - xya) = \Phi([a, xy]) = \Phi(\Delta(x)y),$$

i.e. $\Phi(\Delta(x)y) = 0$ for all $y \in \mathcal{F}(LS(M))$. By Lemma 3.1 we have that $\Delta(x) = 0$. The proof is complete. □

**Lemma 3.4.** *Let $M$ be an arbitrary von Neumann algebra of type $I_\infty$ and let $\Delta : LS(M) \to LS(M)$ be a 2-local derivation. Then the restriction $\Delta|_{\mathcal{F}(LS(M))}$ of the operator $\Delta$ on $\mathcal{F}(LS(M))$ is additive.*

*Proof.* Let $\Delta : LS(M) \to LS(M)$ be a 2-local derivation. For each $x, y \in \mathcal{F}(LS(M))$ there exists a derivation $D_{x,y}$ on $LS(M)$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$. By [1, Theorem 2.7] there exists an element $a \in LS(M)$ such that

$$[a, xy] = D_{x,y}(xy) = D_{x,y}(x)y + xD_{x,y}(y) = \Delta(x)y + x\Delta(y),$$

i.e.

$$[a, xy] = \Delta(x)y + x\Delta(y).$$

Similarly as in Lemma 3.3 we have

$$0 = \Phi(axy - xya) = \Phi([a, xy]) = \Phi(\Delta(x)y + x\Delta(y)),$$

i.e. $\Phi(\Delta(x)y) = -\Phi(x\Delta(y))$. For arbitrary $u, v, w \in \mathcal{F}(LS(M))$, set $x = u + v$, $y = w$. Then from above we obtain

$$\Phi(\Delta(u + v)w) = -\Phi((u + v)\Delta(w)) =$$

$$= -\Phi(u\Delta(w)) - \Phi(v\Delta(w)) = \Phi(\Delta(u)w) + \Phi(\Delta(v)w) = \Phi((\Delta(u) + \Delta(v))w),$$

and so

$$\Phi((\Delta(u + v) - \Delta(u) - \Delta(v))w) = 0$$

for all $u, v, w \in \mathcal{F}(LS(M))$. Denote $b = \Delta(u+v) - \Delta(u) - \Delta(v)$ and put $w = b^*$. Then $\Phi(bb^*) = 0$. Since the trace $\Phi$ is faithful it follows that $bb^* = 0$, i.e. $b = 0$. Therefore

$$\Delta(u+v) = \Delta(u) + \Delta(v),$$

i.e. $\Delta$ is an additive map on $\mathcal{F}(LS(M))$. The proof is complete. □

The following theorem is the main result of this paper.

**Theorem 3.5.** *Let $M$ be an arbitrary von Neumann algebra of type $I_\infty$ and let $\mathcal{A}$ be a $*$-subalgebra of $LS(M)$ such that $M \subseteq \mathcal{A}$. Then every 2-local derivation $\Delta : \mathcal{A} \to \mathcal{A}$ is a derivation.*

*Proof.* First we consider the case $\mathcal{A} = LS(M)$. By Lemma 3.4 the restriction $\Delta|_{\mathcal{F}(LS(M))}$ of the operator $\Delta$ on $\mathcal{F}(LS(M))$ is additive. Further by Lemma 3.2 $\Delta$ is a homogeneous, and therefore $\Delta|_{\mathcal{F}(LS(M))}$ is a linear. Again by Lemma 3.4 we have $\Delta(x^2) = \Delta(x)x + x\Delta(x)$ for all $x \in LS(M)$. So the map $\Delta|_{\mathcal{F}(LS(M))}$ is a linear Jordan derivation on $\mathcal{F}(LS(M))$ in the sense of [6]. In [6, Theorem 1] it is proved that any Jordan derivation on a semi-prime algebra is a derivation. Since $\mathcal{F}(LS(M))$ is semiprime, therefore the linear operator $\Delta|_{\mathcal{F}(LS(M))}$ is a derivation on $\mathcal{F}(LS(M))$.

Since by Lemma 3.2 $\Delta$ is $S(Z(M))$-homogeneous then by [4, Corollary 3] the derivation $\Delta|_{\mathcal{F}(LS(M))} : \mathcal{F}(LS(M)) \to \mathcal{F}(LS(M))$ is spatial, i.e.

$$\Delta(x) = ax - xa, \; x \in \mathcal{F}(LS(M)) \tag{3.1}$$

for an appropriate $a \in LS(M)$.

Let us show that $\Delta(x) = ax - xa$ for all $x \in LS(M)$. Consider the 2-local derivation $\Delta_0 = \Delta - D_a$. Then from the equality (3.1) we obtain that $\Delta_0|_{\mathcal{F}(LS(M))} \equiv 0$. Now by Lemma 3.3 it follows that $\Delta_0 \equiv 0$. This means that $\Delta = D_a$.

Now let $\mathcal{A}$ be an arbitrary $*$-subalgebra of $LS(M)$ such that $M \subseteq \mathcal{A}$. Since $M$ is a type I von Neumann algebra for any element $x \in LS(M)$ there exists a sequence $\{z_n\}$ of mutually orthogonal central projections with $\bigvee\limits_{n\in\mathbb{N}} z_n = \mathbf{1}$ such that $z_n x \in M$ for all $n \in \mathbb{N}$. Set

$$\tilde{\Delta}(x) = \sum_{n\in\mathbb{N}} z_n \Delta(z_n x). \tag{3.2}$$

Since the map $\Delta$ is $Z(M)$-homogeneous, the equality (3.2) gives a well-defined 2-local derivation on $LS(M)$. From above we have that $\tilde{\Delta}$ is a derivation. Therefore $\Delta$ is a derivation. The proof is complete. □

**Corollary 3.6.** *Let $M$ be an arbitrary von Neumann algebra of type $I_\infty$. Then every 2-local derivation $\Delta : LS(M) \to LS(M)$ is a derivation.*

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