A SETTING FOR GENERALIZED ORTHOGONALIZATION

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This paper is dedicated to Professor Tsuyoshi Ando, with affection and admiration

Communicated by L. Székelyhidi

ABSTRACT. Let $\mathcal{H}$ be a complex Hilbert space. We study the geometry of the space of pairs (A, E), for A a (semidefinite bounded linear) positive operator on $\mathcal{H}$ and E a (bounded linear) projection on $\mathcal{H}$ such that $AE = E^*A$.

1. INTRODUCTION

An orthogonalization method in a Hilbert space $\mathcal{H}$ is a tool which converts a sequence of linearly independent vectors in a sequence of orthonormal vectors, with some additional properties. But one can also think that an orthogonalization method provides to every closed subspace $\mathcal{S}$ of $\mathcal{H}$ a projection with image $\mathcal{S}$ which is orthogonal with respect to a certain inner product. In this sense, a setting which unifies different orthogonalization methods is given by the set $\mathcal{Z}$ consisting of all pairs $(\langle\,,\,\rangle', E)$, where $\langle\,,\,\rangle'$ is a inner product or, more generally, a semi-inner product on $\mathcal{H}$ and E is a bounded linear projection acting on $\mathcal{H}$. In order to maintain the situation under control, we shall only admit bounded semi-inner products (with respect to the original inner product $\langle\,,\,\rangle$ of $\mathcal{H}$). Therefore, $\langle\,,\,\rangle'$ is indeed determined by a positive (semidefinite bounded linear) operator A acting on $\mathcal{H}$, by the rule $\langle\xi, \eta\rangle' = \langle A\xi, \eta\rangle$, where $\xi, \eta \in \mathcal{H}$. Thus, the object of our present study is the set

$$\mathcal{Z} = \{(A, E) \in L(\mathcal{H})^+ \times \mathcal{Q} : AE = E^*A\}$$

where $L(\mathcal{H})^+$ is the cone of positive operators and $\mathcal{Q}$ is the subset of all (bounded linear) projections acting on $\mathcal{H}$. The identity $AE = E^*A$ says, exactly, that the

projection E is Hermitian with respect to the semi-inner product $\langle \, , \, \rangle'$ determined by A.

For a fixed Hilbert space $\mathcal{H}$, we say that a closed subspace $\mathcal{S}$ is *compatible* with a positive (semidefinite bounded linear) operator A if there exists a (bounded linear) projection E on $\mathcal{H}$ with image $\mathcal{S}$ such that $AE = E^*A$. Thus, such E behaves like an orthogonal projection onto $\mathcal{S}$ with respect to the semi-inner product $\langle \, , \, \rangle_A$. Compatibility is automatic if $\mathcal{H}$ has finite dimension. However, in the infinite dimensional case, there exist pairs $(A, \mathcal{S})$ such that no such E exists. If

$$\mathcal{P}(A, \mathcal{S}) = \{E \in L(\mathcal{H}): \ E^2 = E, \ \mathcal{R}(E) = \mathcal{S}, \ AE = E^*A\}$$

then $(A, \mathcal{S})$ is compatible if $\mathcal{P}(A, \mathcal{S})$ is not empty. If A is invertible then $(A, \mathcal{S})$ is compatible for every $\mathcal{S}$ and $\mathcal{P}(A, \mathcal{S})$ contains a single projection, denoted by $P_{A, \mathcal{S}}$. The notion of compatibility has been implicitly used by A. Sard [21] in 1950' and later by Hassi and Nordström [12], for A Hermitian. In [1], Ando defined the notion of complementable matrices by a subspace, as a generalization of Schur complements. It turns out that compatibility and complementability for selfadjoint operators on a Hilbert space are equivalent, see [6]. For a complete discussion on compatibility matters amd applications, the reader is referred to the papers [5], [6], [7], [8], [16], [17]; see also [18].

Some relevant results are obtain if $A \in GL(\mathcal{H})^+$ of all positive invertible operators. In particular, many facts are proven not for $\mathcal{Z}$ but for the set

$$\mathcal{Z}^\circ = \{(A, E) \in \mathcal{Z}: \ A \in GL(\mathcal{H})^+\}.$$

In some sense, the main interest of the sets $\mathcal{Z}$ and $\mathcal{Z}^\circ$ is that every oblique projection $E \in \mathcal{Q}$ is paired with all the semi-inner products $\langle \, , \, \rangle_A$ under which E is orthogonal. This offers and interesting setting for problems where perturbations of the scalar product appear naturally. The reader is referred to the paper by Pasternak–Winiarski [19] with relevant bibliography. Also, in problems where the conjugate gradient method is used (see, e.g., [13]) a setting like the mentioned above may be useful.

The contents of the paper are the following. In Section 2 we collect some notation and known facts about Dixmier angles between closed subspaces of a Hilbert space. We also survey results about compatibility, referring the reader to the papers [5], [6], [7], [8], [16], [17] for the corresponding proofs. However, we include with a proof, a new result on compatibility, namely, that for a fixed $A \in L(\mathcal{H})^+$ every closed subspace $\mathcal{S}$ of $\mathcal{H}$ is compatible with A if and only if $\mathcal{R}(A)$ is closed and $\dim \mathcal{R}(A) < \infty$ or $\dim \mathcal{N}(A) < \infty$, where $\mathcal{R}(A)$ denotes the image of A and $\mathcal{N}(A)$ denotes its nullspace. In Section 3 we describe in several ways the set $\mathcal{Z}$ and its natural projections onto $L(\mathcal{H})^+$ and $\mathcal{Q}$. The fibers $\mathcal{Z}_A$ of the first projection are described in Section 4. Section 5 contains a similar study for the fibers $\mathcal{Z}^E$ of the second projection. However, it should be noticed that the similarity is only formal. In fact, each $\mathcal{Z}^E$ is convex and, so topologically quite simple. On the other side, each $\mathcal{Z}_A$ is as complex as the set $\mathcal{Q}$. Both sections 4 and 5 contain result describing some intersections of the type $\mathcal{Z}_{A_1} \cap \mathcal{Z}_{A_2}$, $\mathcal{Z}^{E_1} \cap \mathcal{Z}^{E_2}$, respectively. Finally, at Section 6 we study some topological facts about $\mathcal{Z}$ and $\mathcal{Z}^\circ$. In particular, we determine the connected components of $\mathcal{Z}$ and $\mathcal{Z}^\circ$.

2. Compatible pairs

Let $Gr = Gr(\mathcal{H})$ denote the Grassmann manifold of $\mathcal{H}$, i.e., the set of all closed subspaces $\mathcal{S}$ of $\mathcal{H}$. Let $L(\mathcal{H})$ denote the algebra of linear bounded operators acting on $\mathcal{H}$ and for $T \in L(\mathcal{H})$, denote by $\mathcal{R}(T)$ the range of T and by $\mathcal{N}(T)$ its nullspace. Given $A \in L(\mathcal{H})^+$ and $\mathcal{S} \in Gr$ we say that they are *compatible* (or that the pair $(A, \mathcal{S})$ is compatible) if there exists $E \in \mathcal{Q}$ such that $AE = E^*A$ and $\mathcal{R}(E) = \mathcal{S}$. As we said in the introduction, this means that E is Hermitian with respect to the semi inner product $\langle \, , \, \rangle_A$ defined as $\langle x, y \rangle_A := \langle Ax, y \rangle$, $\forall x, y \in \mathcal{H}$. We refer the reader to the papers [5], [6], [8] for the proofs of the results mentioned below. Theorem 2.1 is new and we present a proof. The pair $(A, \mathcal{S})$ is compatible if and only if $\mathcal{S} + (A\mathcal{S})^\perp = \mathcal{H}$. Now, if we defined the *Dixmier angle* between two closed subspaces $\mathcal{M}$, $\mathcal{W}$ as the unique $\alpha \in [0, \pi/2]$ such that

$$\cos \alpha = c_0(\mathcal{M}, \mathcal{W}) = \sup\{|\langle m, w \rangle| : m \in \mathcal{M}, w \in \mathcal{W}, \|m\| = \|w\| = 1\},$$

then it follows that $(A, \mathcal{S})$ is compatible if and only if $c_0(\mathcal{S}^\perp, \overline{A\mathcal{S}}) < 1$. If $A \in GL(\mathcal{H})^+$ then $(A, \mathcal{S})$ is compatible for every $\mathcal{S} \in Gr$. In fact in this case $\langle \, , \, \rangle_A$ is equivalent to the original $\langle \, , \, \rangle$ and then, $\mathcal{H}_A = (\mathcal{H}, \langle \, , \, \rangle_A)$ is also a Hilbert space, so that for every closed subspace $\mathcal{S}$ (of $\mathcal{H}$ or, indistinctly, $\mathcal{H}_A$) there exists a unique A-orthogonal projection $P_{A,\mathcal{S}}$ onto $\mathcal{S}$. It easily follows that $AP_{A,\mathcal{S}} = P_{A,\mathcal{S}}^*A$.

Concerning the existence of pairs $(A, \mathcal{S})$ which are not compatible, one can show some examples (see, for instance, [5]). However, the following result gives a complete answer.

Theorem 2.1. *Let $A \in L(\mathcal{H})^+$. Then $(A, \mathcal{S})$ is compatible for every $\mathcal{S} \in Gr$ if and only if $\mathcal{R}(A)$ is closed and $\dim \mathcal{R}(A) < \infty$ or $\dim \mathcal{N}(A) < \infty$.*

Proof. Suppose that $\mathcal{R}(A)$ is closed and consider $\mathcal{S} \in Gr$. If $\dim \mathcal{R}(A) < \infty$ then every subset of $\mathcal{R}(A)$ has finite dimension and therefore is closed. Then $c_0(\overline{A\mathcal{S}}, \mathcal{S}^\perp) = c_0(A\mathcal{S}, \mathcal{S}^\perp) < 1$, because $A\mathcal{S} \cap \mathcal{S}^\perp = \{0\}$ and $A\mathcal{S}$ has finite dimension. If $\dim \mathcal{N}(A) < \infty$ then $\mathcal{S} + \mathcal{N}(A)$ is automatically closed; this condition, if A has closed range, is equivalent to the compatibility of A and $\mathcal{S}$, see [5].

Conversely, suppose that $\mathcal{R}(A)$ is not closed; consider $y \in \overline{\mathcal{R}(A)}$ such that $y \notin \mathcal{R}(A)$, $y \neq 0$ and consider $\mathcal{S}^\perp = [y]$ the subspace generated by y. Then $(A\mathcal{S})^\perp = A^{-1}(\mathcal{S}^\perp) = A^{-1}([y] \cap \mathcal{R}(A)) = A^{-1}(\{0\}) = \mathcal{N}(A)$. Therefore, by taking orthogonal complement, $\overline{A\mathcal{S}} = \overline{\mathcal{R}(A)}$; in this case $\overline{A\mathcal{S}} \cap \mathcal{S}^\perp = \overline{\mathcal{R}(A)} \cap \mathcal{S}^\perp = \mathcal{S}^\perp = [y] \neq \{0\}$. Hence $c_0(\overline{A\mathcal{S}}, \mathcal{S}^\perp) = 1$ so that $(A, \mathcal{S})$ is not compatible. See [2].

If $\mathcal{R}(A)$ is closed and $\dim \mathcal{R}(A) = \dim \mathcal{N}(A) = \infty$, then it is possible to construct a closed subspace $\mathcal{S}$ such that $c(\mathcal{S}, \mathcal{N}(A)) = 1$, or equivalently, such that $\mathcal{S} + \mathcal{N}(A)$ is not closed, see [11]. Therefore $(A, \mathcal{S})$ is not compatible. This ends the proof. □

If $(A, \mathcal{S})$ is compatible, then the set

$$\mathcal{P}(A, \mathcal{S}) = \{E \in \mathcal{Q} : \mathcal{R}(E) = \mathcal{S}, \ AE = E^*A\}$$

is not empty. It is a closed affine manifold of $L(\mathcal{H})$, with a distinguished element $P_{A,\mathcal{S}}$ which corresponds to the direct decomposition $\mathcal{H} = \mathcal{S} \dotplus ((A\mathcal{S})^\perp \ominus \mathcal{S})$, where the sum is direct and $(A\mathcal{S})^\perp \ominus \mathcal{S} := (A\mathcal{S})^\perp \cap (\mathcal{S} \cap (A\mathcal{S})^\perp)^\perp$. It can be shown

that $\mathcal{P}(A, \mathcal{S})$ contains a single element if and only if $(A, \mathcal{S})$ is compatible and $\mathcal{N}(A) \cap \mathcal{S} = \{0\}$. In fact $\mathcal{P}(A, \mathcal{S})$ can be parametrized as

$$\mathcal{P}(A, \mathcal{S}) = P_{A,\mathcal{S}} + L(\mathcal{S}^\perp, \mathcal{S} \cap \mathcal{N}(A)).$$

3. $\mathcal{Z}$ AS A SET

This section is devoted to the study of the set

$$\mathcal{Z} = \{(A, E) : A \in L(\mathcal{H})^+, \ E \in \mathcal{Q}, \ AE = E^*A\},$$

with the natural projections

$$p_1 : \mathcal{Z} \longrightarrow L(\mathcal{H})^+, \quad p_1(A, E) = A,$$
$$p_2 : \mathcal{Z} \longrightarrow \mathcal{Q}, \quad p_2(A, E) = E,$$

and the corresponding fibers

$$p_1^{-1}(\{A\}), \ \text{for } A \in L(\mathcal{H})^+; \ p_2^{-1}(\{E\}), \text{for } E \in \mathcal{Q}.$$

Proposition 3.1. *For every $T \in L(\mathcal{H})$ and $E \in \mathcal{Q}$, the pair (T^*T, E) belongs to $\mathcal{Z}$ if and only if $T\mathcal{R}(E)$ is orthogonal to $T\mathcal{N}(E)$.*

To prove this proposition we will use the following lemma.

Lemma 3.2. *The following conditions are equivalent.*
   (1) *The pair (A, E) belongs to $\mathcal{Z}$.*
   (2) $\mathcal{N}(E) \subseteq A\mathcal{R}(E)^\perp$.
   (3) $A\mathcal{N}(E) \subseteq \mathcal{N}(E^*)$.
   (4) $A\mathcal{R}(E) \subseteq \mathcal{R}(E^*)$.

Proof. $1 \leftrightarrow 2$: Suppose that the pair $(A, E) \in \mathcal{Z}$ then $AE = E^*A$. If $x \in \mathcal{N}(E)$ then $0 = AEx = E^*Ax$. Hence $Ax \in \mathcal{N}(E^*) = \mathcal{R}(E)^\perp$, or equivalently, $x \in A^{-1}(\mathcal{R}(E)^\perp) = A(\mathcal{R}(E))^\perp$. Then $\mathcal{N}(E) \subseteq A(\mathcal{R}(E))^\perp$.

Conversely, if $\mathcal{N}(E) \subseteq A\mathcal{R}(E)^\perp$, then by taking orthogonal complement we get $\overline{A\mathcal{R}(E)} \subseteq \mathcal{R}(E^*)$. Therefore $\mathcal{R}(AE) \subseteq \mathcal{R}(E^*)$, which implies $AE = E^*AE$, and this shows that AE is Hermitian, i.e., $AE = E^*A$.

From this identity $A\mathcal{R}(E)^\perp = A^{-1}(\mathcal{R}(E)^\perp)$ the equivalences $2 \leftrightarrow 3$ and $3 \leftrightarrow 4$ are straightforward. $\qquad\square$

Proof. (of the proposition) Let $T = UA^{1/2}$ be the polar decomposition of T with $A = T^*T$ and U a partial isometry from $\overline{\mathcal{R}(A^{1/2})}$ onto $\overline{\mathcal{R}(T)}$. If $(A, E) \in \mathcal{Z}$ then, by the previous lemma, $\mathcal{H} = \mathcal{R}(E) \dotplus \mathcal{N}(E) \subseteq \mathcal{R}(E) + A^{-1}(\mathcal{R}(E)^\perp)$. Applying $A^{1/2}$ to both sides of the equality, $\mathcal{R}(A^{1/2}) = A^{1/2}\mathcal{R}(E) + A^{1/2}\mathcal{N}(E)$ and $A^{1/2}\mathcal{N}(E) \subseteq A^{1/2}A^{-1}(\mathcal{R}(E)^\perp) \subseteq A^{-1/2}(\mathcal{R}(E)^\perp) = A^{1/2}\mathcal{R}(E)^\perp$. Therefore the sum is orthogonal, i.e., $\mathcal{R}(A^{1/2}) = A^{1/2}\mathcal{R}(E) \oplus A^{1/2}\mathcal{N}(E)$. Finally, applying U to this equality we get that $\mathcal{R}(T) = T\mathcal{R}(E) \oplus T\mathcal{N}(E)$ because U preserves orthogonality when applied to subsets of $\mathcal{R}(A^{1/2})$.

Conversely, if $T\mathcal{R}(E)$ is orthogonal to $T\mathcal{N}(E)$ then $\mathcal{R}(T) = T\mathcal{R}(E) \oplus T\mathcal{N}(E)$. Therefore, $T\mathcal{N}(E) = T\mathcal{R}(E)^\perp \cap \mathcal{R}(T)$ and then $\mathcal{N}(E) = T^{-1}(T\mathcal{R}(E)^\perp \cap \mathcal{R}(T)) = T^{-1}(T\mathcal{R}(E)^\perp) = T^{-1}(T^{*-1}(\mathcal{R}(E)^\perp)) \subseteq (T^*T)^{-1}(\mathcal{R}(E)^\perp)$. $\qquad\square$

Remark 3.3. As we have shown in the proof of the proposition above, $(T^*T, E) \in \mathcal{Z}$ if and only if $T\mathcal{R}(E) \oplus T\mathcal{N}(E) = \mathcal{R}(T)$; in particular, $(A, E) \in \mathcal{Z}$ if and only if $A^{1/2}\mathcal{R}(E) \oplus A^{1/2}\mathcal{N}(E) = \mathcal{R}(A^{1/2})$. Observe that the subspaces $T\mathcal{R}(E)$ and $T\mathcal{N}(E)$ (resp. $A^{1/2}\mathcal{R}(E)$, $A^{1/2}\mathcal{N}(E)$) need not to be closed. However, if T is invertible, then, obviously, $T\mathcal{R}(E)$ and $T\mathcal{N}(E)$ are closed and it is easy to see that $(T^*T, E) \in \mathcal{Z}$ if and only if $T\mathcal{R}(E) \oplus T\mathcal{N}(E) = \mathcal{H}$.

We collect some facts about the projections p_1 and p_2.

Proposition 3.4. *The following assertions hold:*
  (1) *For every $E \in \mathcal{Q}$, $\theta(E) = E^*E + (I - E^*)(I - E) \in GL(\mathcal{H})^+$ and the mapping $s(E) = (\theta(E), E)$ verifies $s(E) \in \mathcal{Z}$ for all $E \in \mathcal{Q}$ and $p_2 \circ s = I_{\mathcal{Q}}$ (i.e. s is a global section of p_2).*
  (2) *$p_1 : \mathcal{Z} \longrightarrow L(\mathcal{H})^+$ is surjective.*
  (3) *$p_2 : \mathcal{Z} \longrightarrow \mathcal{Q}$ is surjective.*
  (4) *For every $E \in \mathcal{Q}$, $p_2^{-1}(\{E\})$ is convex.*

Proof. 1. It is easy to see that $\mathcal{N}(\theta(E)) = \mathcal{N}(E) \cap \mathcal{R}(E) = \{0\}$ so that $\theta(E)$ is injective. To see that $\theta(E)$ is surjective observe that $\mathcal{R}(E^*E) \subseteq \mathcal{R}(\theta(E))$. Since $\mathcal{R}(E)$ is closed, $\mathcal{R}(E^*E) = E^*(\mathcal{R}(E)) = E^*(\mathcal{N}(E^*)^\perp) = \mathcal{R}(E^*)$. Then $\mathcal{R}(E^*) \subseteq \mathcal{R}(\theta(E))$. In a similar way, $\mathcal{N}(E^*) = \mathcal{R}(I - E^*) \subseteq \mathcal{R}(\theta(E))$. Then $\mathcal{R}(\theta(E)) = \mathcal{H}$.

The rest of the assertion follows easily.

2. Given $A \in L(\mathcal{H})^+$, the pair $(A, I) \in \mathcal{Z}$ and $p_1(A, I) = A$.

3. Given $E \in \mathcal{Q}$, by 1, the pair $(\theta(E), E) \in \mathcal{Z}$ and $p_2(\theta(E), E) = E$. $\qquad\square$

From now on, we identify the fiber $p^{-1}(\{A\}) = \{(A, E) : (A, E) \in \mathcal{Z}\}$ with

$$\mathcal{Z}_A := \{E \in \mathcal{Q} : AE = E^*A\},$$

and $p_2^{-1}(\{E\}) = \{(A, E) : (A, E) \in \mathcal{Z}\}$ with

$$\mathcal{Z}^E := \{A \in L(\mathcal{H})^+ : AE = E^*A\}.$$

The next two sections are devoted to the characterization of fibers $\mathcal{Z}_A$ and $\mathcal{Z}^E$ for any $A \in L(\mathcal{H})^+$, $E \in \mathcal{Q}$. Thus, for $A \in L(\mathcal{H})^+$ the set $\mathcal{Z}_A$ contains all oblique projections which are "orthogonalized" by A, and, for $E \in \mathcal{Q}$ the set $\mathcal{Z}^E$ contains all positive operators which "orthogonalize" E.

4. The set $\mathcal{Z}_A$

Observe that

$$\mathcal{Z}_A = \bigcup_{\mathcal{S}} \mathcal{P}(A, \mathcal{S})$$

where $\mathcal{S}$ runs over Gr. Of course, a given $\mathcal{S}$ adds some projections only if A and $\mathcal{S}$ are compatible. Notice also that it is a disjoint union.

The following subset of $\mathcal{P}$ is helpful to characterize the set $\mathcal{Z}_A$. Define

$$\mathcal{P}_A = \{P \in \mathcal{P} : \mathcal{R}(PA^{1/2}) \subseteq \mathcal{R}(A^{1/2})\}.$$

Observe that $P \in \mathcal{P}_A$ if and only if the equation $PA^{1/2} = A^{1/2}X$ admits a solution.

For $A \in L(\mathcal{H})^+$ denote by P_A the orthogonal projection onto $\overline{\mathcal{R}(A)}$.

If $P \in \mathcal{P}_A$ then $P(\mathcal{R}(A^{1/2})) \subseteq \overline{\mathcal{R}(A)}$ so that $P(\overline{\mathcal{R}(A)}) \subseteq \overline{\mathcal{R}(A)}$ and then $PP_A = P_A PP_A$. Therefore, PP_A is positive; in particular PP_A and $P(I - P_A)$ are both (orthogonal) projections and $P = PP_A + P(I - P_A)$. Hence, the projections of $\mathcal{P}_A$ can be written as the sum of two orthogonal projections, one with range included in $\overline{\mathcal{R}(A)}$ and the other with range included in $\mathcal{N}(A)$.

Theorem 4.1. *Given $A \in L(\mathcal{H})^+$,*

$$\mathcal{Z}_A = \{ \begin{bmatrix} E_1 & 0 \\ W & E_2 \end{bmatrix} : E_1 = A^{1/2\dagger} P A^{1/2}|_{\overline{\mathcal{R}(A)}}, P \in \mathcal{P}_A, \ E_2^2 = E_2, \ E_2 W = W(I - E_1) \},$$

where the matrix representation is that induced by the decomposition $\mathcal{H} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$.

The proof of this theorem relies on the following lemmas.

Lemma 4.2. *Let $A \in L(\mathcal{H})^+$ and E an A-selfadjoint projection. Then $P_A E$ and $E(I - P_A)$ are A-selfadjoint projections.*

Proof. If $AE = E^*A$ then, $AP_A E = E^* P_A A = (P_A E)^* A$ and $P_A E$ is A-selfadjoint. Since $P_A E = A^{\dagger} E^* A$, it follows that $P_A E = P_A E P_A$. Then $(P_A E)^2 = P_A E P_A E = P_A E^2 = P_A E$.

Also $(I - P_A)E(I - P_A) = (E - P_A E)(I - P_A) = (E - P_A E P_A)(I - P_A) = E(I - P_A)$. Then, $E(I - P_A)$ is a projection and $AE(I - P_A) = A(I - P_A)E(I - P_A) = 0$, so that it is A-selfadjoint. $\square$

Lemma 4.3. *Let $A \in L(\mathcal{H})^+$. Then, E is an A-selfadjoint projection if and only if*

$$E = \begin{bmatrix} E_1 & 0 \\ W & E_2 \end{bmatrix},$$

where $E_1 \in L(\overline{\mathcal{R}(A)})$ is an A-selfadjoint projection, $E_2 \in L(\mathcal{N}(A))$ is an oblique projection and $W \in L(\mathcal{R}(A), \mathcal{N}(A))$ satisfies that $E_2 W = W(I - E_1)$.

Proof. Suppose that E is a projection such that $AE = E^*A$, then, by Lemma 4.2 $P_A E = P_A E P_A$, $E(I - P_A) = (I - P_A)E(I - P_A)$ so that $E = P_A E + E(I - P_A) + (I - P_A)EP_A = \begin{bmatrix} E_1 & 0 \\ W & E_2 \end{bmatrix}$, where $E_1 = P_A E|_{\overline{\mathcal{R}(A)}}$, $E_2 = E(I - P_A)|_{\mathcal{N}(A)}$ and $W = (I - P_A)EP_A|_{\overline{\mathcal{R}(A)}}$. By Lemma 4.2, E_1 and E_2 are A-selfadjoint projections. It is easy to see that $E_2 W = W(I - E_1) = (EP_A - EP_A E)|_{\overline{\mathcal{R}(A)}}$.

Conversely, if $E = \begin{bmatrix} E_1 & 0 \\ W & E_2 \end{bmatrix}$, with E_1 and E_2 projections and $E_2 W = W(I - E_1)$, then $E^2 = \begin{bmatrix} E_1 & 0 \\ WP + E_2 W & E_2 \end{bmatrix} = E$ and E is a projection. Also, $AE = \begin{bmatrix} AE_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_1^* A & 0 \\ 0 & 0 \end{bmatrix} = E^*A$, because E_1 is A-selfadjoint. $\square$

Lemma 4.4. *Let $A \in L(\mathcal{H})^+$ and $E \in \mathcal{Q}$ such that $R(E) \subseteq \overline{\mathcal{R}(A)}$. Then, E is A-selfadjoint if and only if $E = A^{1/2^\dagger} P A^{1/2}$, where $P \in \mathcal{P}_A$.*

Proof. Using Douglas' theorem, it is easy to verify that if $P \in \mathcal{P}_A$ then $E = A^{1/2^\dagger} P A^{1/2}$ is well defined and is a bounded A-selfadjoint projection. Conversely, applying Proposition 3.4 of [7] the converse follows immediately.   □

In particular, if A has closed range, the A-selfadjoint projections such $R(E) \subseteq \mathcal{R}(A)$, are given by $E = A^{1/2^\dagger} P A^{1/2}$, where $P \in \mathcal{P}$ verifies that $R(P) \subseteq \mathcal{R}(A)$.

Lemma 4.5. *Let $A \in L(\mathcal{H})^+$. Then $E \in \mathcal{Z}_A \cap \mathcal{P}$ if and only if $E = E_1 + E_2$, where $E_1, E_2 \in \mathcal{P}$, $\mathcal{R}(E_1) \subseteq \overline{\mathcal{R}(A)}$, $\mathcal{R}(E_2) \subseteq \mathcal{N}(A)$ and $\mathcal{R}(AE_1) \subseteq \mathcal{R}(E_1)$.*

Proof. Suppose that $E = E_1 + E_2$, with $E_1, E_2 \in \mathcal{P}$, $\mathcal{R}(E_1) \subseteq \overline{\mathcal{R}(A)}$, $\mathcal{R}(E_2) \subseteq \mathcal{N}(A)$ and $\mathcal{R}(AE_1) \subseteq \mathcal{R}(E_1)$. Then $E_1 E_2 = 0 = E_2 E_1$ so that $E^2 = E = E^*$. To see that $E \in \mathcal{Z}_A$, observe that $AE = AE_1 = E_1 AE_1$, because $\mathcal{R}(E_2) \subseteq \mathcal{N}(A)$ and $\mathcal{R}(AE_1) \subseteq \mathcal{R}(E_1)$. Then AE is positive so that $AE = E^*A$.

Conversely, if $E \in \mathcal{Z}_A$, by Lemma 4.3,

$$E = \begin{bmatrix} E_1 & 0 \\ W & E_2 \end{bmatrix},$$

where $E_1 \in L(\overline{\mathcal{R}(A)})$ is an A-selfadjoint projection, $E_2 \in L(\mathcal{N}(A))$ is an oblique projection and $W \in L(\mathcal{R}(A), \mathcal{N}(A))$ satisfies that $E_2 W = W(I - E_1)$. If E belongs also to $\mathcal{P}$ then $E_1^* = E_1$, $E_2^* = E_2$ and $W = 0$. Then $E = E_1' + E_2'$, where E_1' and E_2' are the orthogonal projections acting on $\mathcal{H}$ defined by $E_1' = E_1 P_A$ and $E_2' = E_2(I - P_A)$, with P_A the orthogonal projection onto $\overline{\mathcal{R}(A)}$. Finally E is A-selfadjoint if and only if $AE = EA$ or equivalently $AE_1' = E_1'A$ because $AE_2' = 0 = E_2'A$. Hence $\mathcal{R}(AE_1') \subseteq \mathcal{R}(E_1')$.   □

Proposition 4.6. *Let $A \in L(\mathcal{H})^+$, and $B \in GL(H)^+$ such that $\mathcal{Z}_A \subseteq \mathcal{Z}_B$, then there exists $\lambda > 0$ such that $B = \lambda A$. In particular A is invertible and $\mathcal{Z}_A = \mathcal{Z}_B$.*

Proof. First consider $B = I$. In this case $\mathcal{Z}_B = \mathcal{P}$. Suppose that $\mathcal{Z}_A \subseteq \mathcal{P}$. If $A = 0$ then $\mathcal{Z}_A = \mathcal{Q}$ which is not included in $\mathcal{P}$; therefore $A \neq 0$. Moreover A must be injective. Suppose on the contrary that $\mathcal{N}(B) \neq \{0\}$ and consider the operator E defined by its matrix decomposition in terms of $\overline{\mathcal{R}(A)}$ and $\mathcal{N}(A)$ as $E = \begin{bmatrix} 0 & 0 \\ W & I \end{bmatrix}$, where $W \neq 0$ (this is possible because $\mathcal{R}(A) \neq 0$ and $\mathcal{N}(A) \neq 0$). It is easy to see that E is a projection and $AE = 0 = E^*A$ so that E is A-selfadjoint, but $E \notin \mathcal{P}$. Let us see that A si also surjective: if $x \in \mathcal{H}$, $x \neq 0$, then the pair $(A, [x])$ is compatible because $[x]$ is finite dimensional. Since A is injective there exists only one projection onto $[x]$ which is A-selfadjoint and it is given by $P_{A,[x]} = P_{[x]//A[x]^\perp}$. Using that $\mathcal{Z}_A \subseteq \mathcal{P}$, it must hold that $A[x]^\perp = [x]^\perp$, or equivalently, $A[x] = [x]$. Therefore, there exists λ such that $Ax = \lambda x$ so that $x \in \mathcal{R}(A)$. Then $\mathcal{R}(A) = \mathcal{H}$ and A is invertible. In this case, by Theorem 4.1, the set $\mathcal{Z}_A = \{A^{-1/2} P A^{1/2} : P \in \mathcal{P}\} = A^{-1/2} \mathcal{P} A^{1/2}$. If $\mathcal{Z}_A \subseteq \mathcal{P}$, for every $P \in \mathcal{P}$,

$A^{-1/2}PA^{1/2} \in \mathcal{P}$. Hence, $A^{-1/2}PA^{1/2} = A^{1/2}PA^{-1/2}$ so that $PA = AP$ for every $P \in \mathcal{P}$. By [11], there exists $\lambda > 0$ such that $A = \lambda I$.

For any $B \in GL(\mathcal{H})^+$ the condition $\mathcal{Z}_A \subseteq \mathcal{Z}_B = B^{-1/2}\mathcal{P}B^{1/2}$ is equivalent to $B^{1/2}\mathcal{Z}_A B^{-1/2} \subseteq \mathcal{P}$. It holds that $B^{1/2}\mathcal{Z}_A B^{-1/2} = \mathcal{Z}_{B^{-1/2}AB^{-1/2}}$; hence $B^{-1/2}AB^{-1/2} = \lambda I$ or, equivalently $A = \lambda B$. □

Denote by $\mathcal{Q}_{\mathcal{S}}$ the subset of $\mathcal{Q}$ of oblique projections with range $\mathcal{S}$.

Proposition 4.7. *Let $A \in L(\mathcal{H})^+$.*

(1) $\mathcal{Z}_A \cap \mathcal{Q}_{\mathcal{S}} = \mathcal{P}(A, \mathcal{S})$.
(2) $\mathcal{Z}_A \cap \mathcal{Q}_{\mathcal{S}}$ *is not empty if and only if $(A, \mathcal{S})$ is compatible.*
(3) *If $A \in GL(\mathcal{H})^+$ then $\mathcal{Z}_A \cap \mathcal{Q}_{\mathcal{S}} = \{P_{A,\mathcal{S}}\}$.*
(4) $\mathcal{Z}_A \cap \mathcal{Q}_{\mathcal{S}}$ *contains a single element if and only if $(A, \mathcal{S})$ is compatible and $\mathcal{N}(A) \cap \mathcal{S} = \{0\}$.*

5. The set $\mathcal{Z}^E$

Given a fixed projection E we look for the positive operators A such that E is A-selfadjoint, i.e., the set $\mathcal{Z}^E = \{A \in L(\mathcal{H})^+ : AE = E^*A\}$. Observe that. by Proposition 3.4, $\mathcal{Z}^E$ is a convex set.

Theorem 5.1. *Given $E \in \mathcal{Q}$,*

$$\mathcal{Z}^E = \{A = A_1 + A_2 : A_1, A_2 \in L(\mathcal{H})^+, \mathcal{R}(A_1) \subseteq \mathcal{R}(E^*), \mathcal{R}(A_2) \subseteq \mathcal{N}(E^*)\}.$$

Proof. Suppose that $A = A_1 + A_2$, with A_1 and A_2 positive and such that $\mathcal{R}(A_1) \subseteq \mathcal{R}(E^*)$, $\mathcal{R}(A_2) \subseteq \mathcal{N}(E^*)$. Then A is positive. To see that E is A-selfadjoint, observe that $AE = (A_1 + A_2)E = A_1E = E^*A_1E$, because $\mathcal{R}(E) \subseteq \mathcal{N}(A_2)$ and $\mathcal{R}(A_1) \subseteq \mathcal{R}(E^*)$. Noticing that E^*AE is positive, $AE = E^*AE = E^*A$.

Conversely, if $A \in \mathcal{Z}^E$ then A is positive and $AE = E^*A = E^*AE$ so that $(I - E^*)AE = 0 = E^*A(I - E)$. Therefore, $A = E^*AE + (I - E^*)A(I - E)$. If $A_1 = E^*AE$ and $A_2 = (I - E^*)A(I - E)$, then $A = A_1 + A_2$, A_1 and A_2 are positive, $\mathcal{R}(A_1) \subseteq \mathcal{R}(E^*)$, $\mathcal{R}(A_2) \subseteq \mathcal{N}(E^*)$. □

Remark 5.2. By the above proposition, it is easy to see that

$$\mathcal{Z}^E = \{A = A_1 P_{\mathcal{R}(E^*)} + A_2 P_{\mathcal{N}(E^*)} : A_1 \in L(\mathcal{R}(E^*))^+, A_2 \in L(\mathcal{N}(E^*))^+\}.$$

Then the set $\mathcal{Z}^E$ can be identified with the product

$$L(\mathcal{R}(E^*))^+ \times L(\mathcal{N}(E^*))^+.$$

through the map

$$\mathcal{Z}^E \longrightarrow L(\mathcal{R}(E^*))^+ \times L(\mathcal{N}(E^*))^+,$$
$$A \longrightarrow (AE|_{\mathcal{R}(E^*)}, A(I - E)|_{\mathcal{N}(E^*)}).$$

For every $E \in \mathcal{Q}$ the set $\mathcal{Z}^E \cap \mathcal{P}$ is not empty. In fact, $P = P_{\mathcal{N}(E)^\perp}$ has the same nullspace as E and then $PE = P = E^*P$, which proves that $P \in \mathcal{Z}^E \cap \mathcal{P}$. The next proposition characterizes the set $\mathcal{Z}^E \cap \mathcal{P}$.

Proposition 5.3. *Let $E \in \mathcal{Q}$. Then*

$$\mathcal{Z}^E \cap \mathcal{P} = \{P = P_1 + P_2 : P_1, P_2 \in \mathcal{P}, P_1 P_2 = 0, \mathcal{R}(P_1) \subseteq \mathcal{R}(E^*), \mathcal{R}(P_2) \subseteq \mathcal{N}(E^*)\}.$$

Proof. If $P = P_1 + P_2$ with $P_i \in \mathcal{P}$, i=1,2, $P_1P_2 = 0$, $\mathcal{R}(P_1) \subseteq \mathcal{R}(E^*)$ and $\mathcal{R}(P_2) \subseteq \mathcal{N}(E^*)$ then P is an orthogonal projection. To see that $P \in \mathcal{Z}^E$ observe that $E^*P = E^*(P_1 + P_2) = P_1 = P_1^* = PE$.

Conversely, if $P \in \mathcal{Z}^E \cap \mathcal{P}$ then, by Proposition 5.1 $P = A_1 + A_2$, with A_1 and A_2 positive, $\mathcal{R}(A_1) \subseteq \mathcal{R}(E^*)$ and $\mathcal{R}(A_2) \subseteq \mathcal{N}(E^*)$. From $P^2 = P$ it follows that

$$A_1^2 + A_2^2 + A_1A_2 + A_2A_1 = A_1 + A_2. \tag{5.1}$$

Multiplying to the left by E^* and noticing that $E^*A_1 = A_1 = A_1E$ and $E^*A_2 = 0 = A_2E$ it follows that $A_1^2 + A_1A_2 = A_1$; multiplying by E to the right, $A_1^2 = A_1$. In a similar way, $A_2^2 = A_2$. Therefore A_1 and A_2 are orthogonal projections because they are positive. It follows from (5.1) that $A_1A_2 + A_2A_1 = 0$, then $A_1A_2 = 0 = A_2A_1$. $\square$

It is an old result by Penrose [20] and Greville [10] (see also an infinite dimensional treatment in [4]) that every $Q \in \mathcal{Q}$ can be decomposed as the Moore–Penrose inverse of a product of two orthogonal projections. More precisely, if E is the oblique projection with $\mathcal{R}(E) = \mathcal{W}$ and $\mathcal{N}(E) = \mathcal{M}^\perp$ then $E^\dagger = P_{\mathcal{M}^\perp}P_{\mathcal{W}}$ and therefore, $E = (P_{\mathcal{M}^\perp}P_{\mathcal{W}})^\dagger$. Thus

$$\mathcal{Z}^E \cap \mathcal{P} = \{P \in \mathcal{P} : \ PE = E^*P\} = \{P_{\mathcal{N}} \in \mathcal{P} : \ P_{\mathcal{N}}(P_{\mathcal{M}^\perp}P_{\mathcal{W}})^\dagger = (P_{\mathcal{W}}P_{\mathcal{M}^\perp})^\dagger P_{\mathcal{N}}\}.$$

The next two propositions exhibit a sort of disjoint behaviour of the fibers $\mathcal{Z}^E$.

Proposition 5.4. *Let $E, F \in \mathcal{Q}$. If $\mathcal{Z}^E \subseteq \mathcal{Z}^F$ then $EF = 0$ or $E(I - F) = 0$.*

Proof. Suppose first that E and F are selfadjoint projections. If $\mathcal{Z}^E \subseteq \mathcal{Z}^F$ then $E \in \mathcal{Z}^F$ because $E \in \mathcal{Z}^E$. Therefore $EF = FE$. In this case, $EF = P_{\mathcal{R}(E) \cap \mathcal{R}(F)}$, $E(I - F) = P_{\mathcal{R}(E) \cap \mathcal{N}(F)}$ and $E = EF + E(I - F)$.

Suppose that EF and $E(I - F)$ are both different from zero; let $x \in \mathcal{R}(E) \cap \mathcal{N}(F)$, $\|x\| = 1$, and $y \in \mathcal{R}(E) \cap \mathcal{R}(F)$, $\|y\| = 1/2$ and define $Bx = y$ and $B = 0$ in the orthogonal complement of x, $[y]^\perp$. The operator B is bounded and $\|B\| = 1/2$; also $\mathcal{R}(B) = [y] \subseteq \mathcal{R}(E)$; then $\mathcal{R}(E)^\perp \subseteq [y]^\perp = \mathcal{N}(B)$. Therefore, $\mathcal{R}(B^*) = \overline{\mathcal{R}(B^*)} = \mathcal{N}(B)^\perp \subseteq \mathcal{R}(E)$. Define $A = E + B + B^*$; then A is selfadjoint, $\mathcal{R}(A) \subseteq \mathcal{R}(E)$. In fact A is positive: first observe that $B + B^* = E(B + B^*)E$ because $\mathcal{R}(B + B^*) \subseteq R(E)$; then for $x \in \mathcal{H}$, $|\langle(B + B^*)x, x\rangle| = |\langle(B + B^*)Ex, Ex\rangle| \leq 2\|B\|\|Ex\|^2 = \|Ex\|^2$. Therefore $|\langle Ax, x\rangle| = |\langle(E + B + B^*)x, x\rangle| \geq \|x\|^2 - \|Ex\|^2 = 0$.

Observe that $L(\mathcal{R}(E))^+ \subseteq \mathcal{Z}^E$; in particular $A \in \mathcal{Z}^E$. But $A \notin \mathcal{Z}^F$: since $F = F^*$, a positive operator $A \in \mathcal{Z}^F$ if and only if $FA(I - F) = 0$ but, in this case, $FA(I - F) = F(E + B + B^*)(I - F) = FB(I - F) = B \neq 0$.

Therefore, $EF = 0$ or $E(I - F) = 0$, which ends the proof in the special case where E and F are orthogonal projections.

In the general case, observe that if F is an oblique projection and we consider an invertible $A \in \mathcal{Z}^F$, then, by the comment above, $B \in \mathcal{Z}^F$ if and only if $A^{-1/2}BA^{-1/2} \in \mathcal{Z}^{A^{1/2}FA^{-1/2}}$, where $A^{1/2}FA^{-1/2} \in \mathcal{P}$, i. e., $A^{-1/2}\mathcal{Z}^F A^{-1/2} = \mathcal{Z}^{A^{1/2}QA^{-1/2}}$. Let E and F be oblique projections such that $\mathcal{Z}^E \subseteq \mathcal{Z}^F$ and consider an invertible $A \in \mathcal{Z}^E$. Then $A \in Z^F$ so that $\mathcal{Z}^E \subseteq \mathcal{Z}^F$, i.e., $\mathcal{Z}^{A^{1/2}EA^{-1/2}} \subseteq \mathcal{Z}^{A^{1/2}FA^{-1/2}}$. But this last inclusion implies that $A^{1/2}EA^{-1/2} = A^{1/2}EFA^{-1/2}$

or $A^{1/2}EA^{-1/2} = A^{1/2}E(I - F)A^{-1/2}$, by the first part, and cancelling the A's, $E = EF$ or $E = E(I - F)$. □

Corollary 5.5. *Let E, $F \in \mathcal{Q}$ such that $E \neq 0$ and $F \neq I$. If $\mathcal{Z}^E \subseteq \mathcal{Z}^F$ then $F = E$ or $F = I - E$, so that $\mathcal{Z}^E = \mathcal{Z}^F$.*

Proof. As in the proof of the above proposition, we can suppose that $E, F \in \mathcal{P}$. If $\mathcal{Z}^E \subseteq \mathcal{Z}^F$ then by Proposition 5.4, $E = EF$ or $EF = 0$. Since $\mathcal{Z}^E = \mathcal{Z}^{(I-E)}$, we deduce that $I - E = (I - E)F$ or $(I - E)F = 0$. If $E = EF$ and $I - E = (I - E)F$ then $F = I$. If $E = EF$ and $(I - E)F = 0$ then $E = F$. If $EF = 0$ and $I - E = (I - E)F$ then $F = I - E$. Finally, if $EF = 0$ and $(I - E)F = 0$ then $F = 0$. □

Proposition 5.6. *Consider $A \in L(\mathcal{H})^+$ and $E \in \mathcal{Q}$, such that the representations of A and E induced by the decomposition $\mathcal{H} = \overline{\mathcal{R}(E)} \oplus \mathcal{R}(E)^\perp$ are $A = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$ and $E = \begin{bmatrix} 1 & r \\ 0 & 0 \end{bmatrix}$ respectively. Then the following conditions are equivalent.*

(1) $A \in \mathcal{Z}^E \cap \mathcal{Z}^{E^*}$.
(2) $AE = EA = AE^*$.
(3) $A = A_1 + A_2$ with A_1, $A_2 \in L(\mathcal{H})^+$ such that

$$\mathcal{R}(A_1) \subseteq \mathcal{R}(E) \cap \mathcal{N}(E)^\perp, \quad \mathcal{R}(A_2) \subseteq \mathcal{R}(E)^\perp \cap \mathcal{N}(E).$$

(4) $A = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$ with $ar = rc = 0$.

Proof. $1 \rightarrow 2$: $A \in \mathcal{Z}^E \cap \mathcal{Z}^{E^*}$ if and only if $AE = E^*A$ and $AE^* = EA$. Hence $A^2E = AE^*A = EA^2$. Since A is positive it follows that $AE = EA$ and the first equality holds. Taking adjoint to the last equality A and E^* also commute. Therefore $AE = E^*A = AE^*$ and the second equality follows.

$2 \rightarrow 3$: If $AE = EA = AE^*$ then $AE^* = E^*A$ and $A \in \mathcal{Z}^{E^*}$. By Proposition 5.1, $A \in \mathcal{Z}^{E^*}$ if and only if $A = A_1 + A_2$, with A_1, $A_2 \in L(\mathcal{H})^+$, $\mathcal{R}(A_1) \subseteq \mathcal{R}(E)$ and $\mathcal{R}(A_2) \subseteq \mathcal{N}(E)$. Also $AE = EA$ if and only if $A_1E + A_2E = EA_1$ because $EA_1 = A_1$ and $EA_2 = 0$, or equivalently $A_2E = A_1(I - E)$. Therefore $A_2E = 0$ and $A_1(I - E) = 0$ because $\mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\}$. Hence $\mathcal{R}(E) \subseteq \mathcal{N}(A_2)$ and $\mathcal{N}(E) \subseteq \mathcal{N}(A_1)$. Then $\mathcal{R}(A_2) \subseteq \mathcal{N}(E) \cap \mathcal{R}(E)^\perp$ and $\mathcal{R}(A_1) \subseteq \mathcal{R}(E) \cap \mathcal{N}(E)^\perp$.

$3 \rightarrow 4$: if $A = A_1 + A_2$ as in item 3 then $PA = PAP = A_1$, $PA(I - P) = 0$ and $(I - P)A(I - P) = A_2$, where P is the orthogonal projection onto $\mathcal{R}(E)$. Therefore $a = A_1|_{\mathcal{R}(E)}$, $b = 0$ and $c = A_2|_{\mathcal{N}(E)}$. Also $ar = A_1PE(I - P)|_{\mathcal{R}(E)^\perp} = A_1(E - P)|_{\mathcal{R}(E)^\perp} = 0$ because $A_1(E - P) = A_1E - A_1P = A_1 - A_1 = 0$; the last equality follows from the fact that $\mathcal{R}(E)^\perp$ and $\mathcal{N}(E)$ are subsets of $\mathcal{N}(A_1)$. The equality $rc = 0$ follows in a similar way.

$4 \rightarrow 1$: It is straightforward. □

Corollary 5.7. *If $E \neq E^*$ then $\mathcal{Z}^E \cap \mathcal{Z}^{E^*}$ contains no (positive) invertible operator. More generally, $\mathcal{Z}^E \cap \mathcal{Z}^{E^*}$ contains no injective $A \in L(\mathcal{H})^+$.*

Remark 5.8. If $P \in \mathcal{P}$ then $\mathcal{Z}^P = \{A \in L(\mathcal{H})^+ : AP = PA\} = (P')^+$, i.e., the cone of positive elements of the commutant algebra of P in $L(\mathcal{H})$. Consequently,

$$\bigcap_{P \in \mathcal{P}} \mathcal{Z}^P = \{\lambda I : \lambda \in \Re^+\}.$$

6. ON THE TOPOLOGY OF $\mathcal{Z}$

In order to study the topology of $\mathcal{Z}$, it is useful to consider the action of $GL(\mathcal{H})$ over $\mathcal{Z}$ given by

$$L : GL(\mathcal{H}) \times \mathcal{Z} \longrightarrow \mathcal{Z}$$
$$L(G, (A, E)) = (GAG^*, G^{*-1}EG^*).$$

By restriction, L defines actions over $\mathcal{Z}_A$ and $\mathcal{Z}^E$.

Lemma 6.1. *For every $A \in L(\mathcal{H})^+$, $E \in \mathcal{Q}$, $G \in GL(\mathcal{H})$ it holds*

$$\mathcal{Z}_{G^{*-1}AG^{-1}} = G\mathcal{Z}_A G^{-1},$$
$$\mathcal{Z}^{GEG^{-1}} = G^{*-1}\mathcal{Z}^E G^{-1}.$$

This shows that, by choosing convenient $A \in L(\mathcal{H})^+$, $E \in \mathcal{Q}$, one can get information on every $\mathcal{Z}_{A'}$, $\mathcal{Z}^{E'}$ for E' similar to E and A' congruent to A, provided that we have that type of information on $\mathcal{Z}_A$ or $\mathcal{Z}^E$.

Observe that $\mathcal{Z}$ is a closed subset of $L(\mathcal{H})^+ \times \mathcal{Q}$, if we provide $L(\mathcal{H})^+ \times \mathcal{Q}$ with the induced topology of $L(\mathcal{H}) \times L(\mathcal{H})$.

Our main concerns deal with

$$\mathcal{Z}^\circ = \{(A, E) \in \mathcal{Z} : A \in GL(\mathcal{H})^+\},$$

with the fibers

$$\mathcal{Z}_A^\circ = \{E \in \mathcal{Q} : (A, E) \in \mathcal{Z}^\circ\} = \mathcal{Z}_A$$

and

$$(\mathcal{Z}^\circ)^E = \{A \in GL(\mathcal{H})^+ : (A, E) \in \mathcal{Z}^\circ\},$$

for $A \in GL(\mathcal{H})^+$ and $E \in \mathcal{Q}$.

Proposition 6.2. *The mapping*

$$\phi : \mathcal{Z}^\circ \longrightarrow GL(\mathcal{H})^+ \times \mathcal{P}$$
$$(A, E) \longrightarrow (A, A^{1/2}EA^{-1/2})$$

is a homeomorphism with inverse

$$\psi : GL(\mathcal{H})^+ \times \mathcal{P} \longrightarrow \mathcal{Z}^\circ$$
$$(A, P) \longrightarrow (A, A^{-1/2}EA^{1/2}).$$

Proof. It suffices to compute $\phi \circ \psi$ and $\psi \circ \phi$.   □

Consider the map

$$\alpha : GL(\mathcal{H})^+ \times \mathcal{Q} \longrightarrow \mathcal{Q}$$
$$\alpha(A, E) = A^{1/2}EA^{-1/2}.$$

Proposition 6.3.
$$\alpha^{-1}(\mathcal{P}) = \mathcal{Z}^\circ.$$

Proof. Suppose that $(A, E) \in \alpha^{-1}(\mathcal{P})$. Then $\alpha(A, E) = P \in \mathcal{P}$; this is equivalent to $A^{1/2}EA^{-1/2} = A^{-1/2}E^*A^{1/2}$. Therefore $AE = E^*A$ so that $(A, E) \in \mathcal{Z}^\circ$. This shows that $\alpha^{-1}(\mathcal{P}) \subseteq \mathcal{Z}^\circ$. Conversely, if $(A, E) \in \mathcal{Z}^\circ$ it has been shown that the projection $A^{1/2}EA^{-1/2}$ is selfadjoint, or equivalently, $\alpha(A, E) \in \mathcal{Z}^\circ$. This proves the other inclusion. □

Denote again $\alpha = \alpha|_{\mathcal{Z}^\circ}$.

Corollary 6.4. *The map* $\alpha : \mathcal{Z}^\circ \longrightarrow \mathcal{P}$, $\alpha(A, E) = A^{1/2}EA^{-1/2} = P_{\mathcal{R}(A^{1/2}E)}$ *is continuous.*

Proposition 6.5. *For* $A \in GL(\mathcal{H})^+$, *consider the map* $\alpha_A = \alpha|_{\mathcal{Z}_A}$
$$\alpha_A : \mathcal{Z}_A \longrightarrow \mathcal{P}$$
$$\alpha_A(E) = A^{1/2}EA^{-1/2}$$
is a homeomorphism, with inverse $\alpha_A^{-1}(P) = A^{-1/2}PA^{1/2}$.

Corollary 6.6. *There is a natural bijection between the sets of connected components of* $\mathcal{Z}^\circ$ *and* $\mathcal{P}$.

Proof. In fact, $\mathcal{Z}^\circ$ is homeomorphic to $GL(\mathcal{H})^+ \times \mathcal{P}$ and $GL(\mathcal{H})^+$ is contractible, by Kuiper's theorem [15]. Thus, $\mathcal{Z}^\circ$ has the homotopy type of $\mathcal{P}$. In particular, ϕ induces a bijection between the set of connected components of $\mathcal{Z}^\circ$ and that of $\mathcal{P}$. □

We extend the map $\alpha : \mathcal{Z}^\circ \longrightarrow \mathcal{P}$ to $\mathcal{Z}$, but we are obliged to loose the continuity. The reason behind this lost is the fact that inversion can not be extended with continuity from $GL(\mathcal{H})$.

Proposition 6.7. *Let* $(A, E) \in \mathcal{Z}$. *Then* $P = A^{1/2^\dagger}E^*A^{1/2}$ *is well defined and* $P \in \mathcal{P}$. *Moreover,* $P = P_\mathcal{M}$, *where* $\mathcal{M} = \overline{\mathcal{R}(A^{1/2}E)}$.

Proof. If $(A, E) \in \mathcal{Z}$ then, by Lemma 4.2, $P_A E \in \mathcal{Z}_A$ and $\mathcal{R}(P_A E) \subseteq \overline{\mathcal{R}(A)}$. In this case, applying Proposition 3.4 of [7], $P_\mathcal{M}(\mathcal{R}(A^{1/2})) \subseteq \mathcal{R}(A^{1/2})$ and $P_A E = A^{1/2^\dagger}P_\mathcal{M}A^{1/2}$, where $\mathcal{M} = \overline{\mathcal{R}(A^{1/2}E)}$. Notice that the Moore–Penrose inverse of A is not necessarily bounded and its domain is the (dense) set $\mathcal{R}(A^{1/2}) \oplus \mathcal{N}(A)$. Therefore, $A^{1/2}E = A^{1/2}P_A E = P_\mathcal{M}A^{1/2}$, or $E^*A^{1/2} = A^{1/2}P_\mathcal{M}$. Hence, $A^{1/2^\dagger}E^*A^{1/2} = P_A P_\mathcal{M} = P_\mathcal{M}$. □

In view of Proposition 6.7, we can extend α to $\mathcal{Z}$, in the following way:
$$\tilde{\alpha} : \mathcal{Z} \longrightarrow \mathcal{P}, \ \tilde{\alpha}(A, E) = A^{1/2^\dagger}E^*A^{1/2} = P_{\overline{\mathcal{R}(A^{1/2}E)}},$$
where $(A, E) \in \mathcal{Z}$. However, $\tilde{\alpha}$ is no longer continuous.

Proposition 6.8. *For $A \in L(\mathcal{H})^+$ consider $\tilde{\alpha}_A : \mathcal{Z}_A \longrightarrow \mathcal{P}$, $\tilde{\alpha}_A(E) = \tilde{\alpha}(A, E) = P_{\overline{\mathcal{R}(A^{1/2}E)}}$. Then $\tilde{\alpha}_A(\mathcal{Z}_A) = \{P \in \mathcal{P}_A : \mathcal{R}(P) \subseteq \overline{\mathcal{R}(A)}\}$. In particular, if A has closed range then $\tilde{\alpha}_A(\mathcal{Z}_A) = \{P \in \mathcal{P} : \mathcal{R}(P) \subseteq \mathcal{R}(A)\}$.*

For a fixed $(I, P) \in \mathcal{Z}^\circ$ define the map

$$\pi : GL(\mathcal{H}) \longrightarrow \mathcal{Z}^\circ, \ \ \pi(G) = L(G, (I, P)) = (GG^*, G^{*-1}PG^*).$$

Given $(A, E) \in \mathcal{Z}^\circ$, the *orbit* of (A, E) given by the action L is the set

$$\mathcal{O}_{(A,E)} = \{(B, R) \in \mathcal{Z}^\circ : (B, R) = L_G((A, Q)) \text{ for } G \in GL(\mathcal{H})\}.$$

Observe that $\mathcal{O}_{(A,E)} = \mathcal{O}_{(I,E_0)}$, where $E_0 = A^{1/2}EA^{-1/2} \in \mathcal{P}$. If $\mathcal{U}(\mathcal{H})$ denotes the subgroup of $GL(\mathcal{H})$ of unitary operators, then:

Proposition 6.9. *Given $(A, E) \in \mathcal{Z}^\circ$ and $P \in \mathcal{P}$, then $(A, E) \in \mathcal{O}_{(I,P)}$ if and only if $E_0 := A^{1/2}EA^{-1/2} \in \mathcal{U}_P := \{UPU^* : U \in \mathcal{U}(\mathcal{H})\}$.*

Proof. Given $(A, E) \in \mathcal{Z}^\circ$ and $P \in \mathcal{P}$, suppose that $(A, E) \in \mathcal{O}_{(I,P)}$; then, there exists $G \in GL(\mathcal{H})$ such that $A = GG^*$ and $E = G^{*-1}PG^*$; or, equivalently, $A^{1/2} = |G^*|$, so that $G = A^{1/2}U$, with $U \in \mathcal{U}(\mathcal{H})$. Then, $E = A^{-1/2}UPU^*A^{1/2}$, or, $E_0 = A^{1/2}EA^{-1/2} = UPU^*$. Therefore, $E_0 \in \mathcal{U}_P$.

Conversely, if $E_0 \in \mathcal{U}_P$ then there exists $U \in \mathcal{U}(\mathcal{H})$ such that $E_0 = UPU^*$, or equivalently, $A^{1/2}EA^{-1/2} = UPU^*$, so that $E = A^{-1/2}UPU^*A^{1/2}$. Taking $G = A^{1/2}U$, then $GG^* = A$ and $E = G^{*-1}PG^*$. Hence, $(A, E) \in \mathcal{O}_{(I,P)}$. $\qquad\square$

Corollary 6.10. *Given $(A, E) \in \mathcal{Z}^\circ$ and $P \in \mathcal{P}$, then $(A, E) \in \mathcal{O}_{(I,P)}$ if and only if $\dim \mathcal{R}(E) = \dim \mathcal{R}(P)$ and $\dim \mathcal{N}(E) = \dim \mathcal{N}(P)$, or equivalently if $E \in \mathcal{O}_P := \{GPG^{-1} : G \in GL(\mathcal{H})\}$.*

Proposition 6.11. *The action L is locally transitive and the map π admits local cross sections.*

Proof. Let us see that the action L is locally transitive: given $(A_0, E_0) \in \mathcal{Z}^\circ$, we have to find an neighbourhood $V = V_{(A_0, E_0)}$ such that if $(A, E) \in V$ then there exists $G \in GL(\mathcal{H})$ such that $L(G, (A_0, E_0)) = (A, E)$. Suppose that such G exists. Then G verifies that

$$GA_0G^* = A, \ \text{ and } \ G^{*-1}E_0G^* = E.$$

Therefore, $|A_0^{1/2}G^*| = A^{1/2}$ so that $GA_0^{1/2} = A^{1/2}U$, with U unitary. Hence, $G = A^{1/2}UA_0^{-1/2}$.

Then, if $G^{*-1}E_0G^* = E$, it follows that $UA_0^{1/2}E_0A_0^{-1/2}U^* = A^{1/2}EA^{-1/2}$. Observe that the projections $P_0 = A_0^{1/2}E_0A_0^{-1/2}$ and $P = A^{1/2}EA^{-1/2}$ are selfadjoint because $(A_0, E_0), (A, E) \in \mathcal{Z}^\circ$.

If the pairs $(A_0, E_0), (A, E)$ are close enough, the projections P and P_0 verify that $\|P - P_0\| < 1$, because of the continuity of the functions involved and the fact that E_0 and E can be taken as close as necessary. Therefore it is possible to find an unitary operator U such that $P = UP_0U^*$: in fact $C = I - (P - P_0)^2$

provides a positive invertible operator such that the operator $U = PC^{-1/2}P_0 + (I - P)C^{-1/2}(I - P_0)$ is unitary and and $UP_0U^* = P$, see [14].

Using that L is locally transitive it is easy to see that π has local cross sections: in fact, if $s : U \longrightarrow GL(\mathcal{H})$, $s((A, E)) = A(EA^{-1/2}C^{-1/2}P + (I - E)A^{-1/2}C^{-1/2}(I - P))$, with $C = I - (A^{1/2}EA^{-1/2} - P)^2$ and U is a neighbourhood of P, given as before, then $\pi(s((A, E)) = (A, E)$, for all $(A, E) \in U$. To obtain a local cross section for another point of $\mathcal{Z}^\circ$ consider $(A_0, E_0) \in \mathcal{O}_{(I,P)}$. Then, there exists $G \in GL(\mathcal{H})$ such that $L_G(I, P) = (A_0, E_0)$. Therefore, $\tilde{s} = l_G \circ s \circ L_{G^{-1}}$ is a local section in the neighborhood GU of (A_0, E_0). $\qquad\square$

Proposition 6.12. *Let (A, E), $(A', E') \in \mathcal{Z}$. Then (A, E), (A', E') belong to the same arc component if and only if E and E' belong to the same arc component.*

Proof. Suppose that E and E' belong to the same arc component. Then there exists a continuous curve of projections $\gamma : [0, 1] \longrightarrow \mathcal{Q}$ such that $\gamma(0) = E$ and $\gamma(1) = E'$. Consider the curve $\beta(t) = \gamma(t)^*\gamma(t) + (I - \gamma(t)^*)(I - \gamma(t))$. Then $\beta(t) \subseteq GL(\mathcal{H})^+$ and $\beta(t)\gamma(t) = \gamma(t)^*\beta(t)$, for every $t \in [0, 1]$. Therefore the curve $(\beta(t), \gamma(t))$ connects the pairs $(\beta(0), E)$ and $(\beta(1), E')$ in $\mathcal{Z}$. But the positive operators $\beta(0)$ and A belong to $\mathcal{Z}^E$, which is a convex set. Then the segment $t\beta(0) + (1 - t)A$, for $0 \le t \le 1$, of positive operators joining them is included in $\mathcal{Z}^E$. Therefore the curve $(t\beta(0) + (1 - t)A, E)$ joins $(\beta(0), E)$ with (A, E) in $\mathcal{Z}$. In a similar way, the point $(\beta(1), E')$ are (A', E') are arc connected in $\mathcal{Z}$. This shows that (A, E), (A', E') are arc connected in $\mathcal{Z}$.

The converse is straightforward. $\qquad\square$

Remark 6.13. Observe that, for every $(A, E) \in \mathcal{Z}^\circ$ the fibres $\mathcal{Z}_A$, $\mathcal{Z}^E$ are homogeneous spaces: in the first case the unitary group acts on $\mathcal{P}$ by similarity and, in the second, the product group $GL(\mathcal{R}(E^*)) \times GL(\mathcal{N}(E^*))$ acts on $\mathcal{Z}^E$ by restriction of the map L defined above. See [3] for details on the geometric structure of positive operators on a Hilbert space. See also [9].

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