Ann. Funct. Anal. 1 (2010), no. 1, 123–132
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

ON SPLITTING OF EXTENSIONS OF RINGS AND TOPOLOGICAL RINGS

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Communicated by V. Valov

ABSTRACT. Several results on splitting of extensions of Banach algebras are generalized to the case of (not necessarily commutative, not necessarily unital) rings or topological rings. Detailed proofs of the results are provided.

1. INTRODUCTION AND DEFINITIONS

The *Wedderburn Principal Theorem* asserts that for any finite-dimensional algebra A over $\mathbb{C}$ there exists a subalgebra B of A such that A is the direct sum of B and the Jacobson radical rad A of A. In this case there exists an extension of A by rad A, i.e., there exists a short exact sequence

$$\theta_A \to \mathrm{rad}A \to A \to B \to \theta_B$$

of algebras and algebra homomorphisms. Several authors have studied more general problem, where they replaced the Jacobson radical by a two-sided ideal of A. The main source for this paper was [1, pp. 1–13], where the case of extensions of Banach algebras was studied. Some results on topological or algebraic extensions of algebras or groups could be found also in [4], [6] or [7]. While studying the proofs of the results in [1], it appeared that many of them work when we replace Banach algebras with topological rings or even rings without any topology. The aim of this paper is to give the results similar to the ones in [1] in more general setting, i.e., for rings or topological rings instead of Banach algebras. All rings and algebras in this paper are quite general, i.e., they are associative but the existence of a unital element or the commutativity of the multiplication is nowhere assumed nor needed.

Date: Received: 22 June 2010; Accepted: 24 October 2010.
2010 *Mathematics Subject Classification.* Primary 16S70; Secondary 16W80, 46H99, 54H13.
Key words and phrases. Ring, topological ring, extension of a ring, splitting.

Let R be a ring. In case R has the unit e, then we know what mean the sets $(e-p)R$ and $R(e-p)$, where $p \in R$. In case R does not have the unit, we define the sets $(e-p)R$ and $R(e-p)$ by $(e-p)R := \{r - pr : r \in R\}$ and $R(e-p) := \{r - rp : r \in R\}$, respectively. Notice that these definitions coincide with the definitions of $(e-p)R$ and $R(e-p)$ in unital case. In what follows, we use e just as a symbol. The actual existence of the unital element in R is not required.

Let S be a subset of a ring R, $n \in \mathbb{N}$ and $S_n := \{s_1 \cdot \ldots \cdot s_n : s_1, \ldots, s_n \in S\}$. Define a kind of "*linear span*" for S_n by setting

$$S^n := \bigcup_{l \in \mathbb{N}} \Big\{ \sum_{i=1}^{l} m_i r_i : m_1, \ldots, m_l \in \mathbb{N}, r_1, \ldots, r_l \in S_n \Big\}.$$

It is easy to see that if I is a one-sided or a two-sided ideal of R, then I^n is also an ideal of R of the same kind.

A ring is said to be *Artinian* (more precisely, *right Artinian*), if any non-empty set of its right ideals has a minimal element (see [3, pp. 18–19]). A ring with topology in which the addition is continuous and the multiplication is separately continuous will be called a *topological ring*. The *Jacobson radical* rad R of a (topological) ring R is the intersection of all (closed, in case of a topological ring) maximal regular left (or right) ideals of R. The ring R is *semisimple* if rad $R = \{\theta_R\}$, where θ_R stands for the zero element of R.

An *algebraic extension* of a ring (or algebra) R is a short exact sequence of rings (algebras) and homomorphisms

$$\Sigma(\mathcal{A}; I) : \theta_I \to I \xrightarrow{\iota} \mathcal{A} \xrightarrow{\pi} R \to \theta_R,$$

where ι is an inclusion and π is an epimorphism such that $\iota(I) = \ker\pi$. This condition implies that I is isomorphic to a two-sided ideal $\iota(I)$ of $\mathcal{A}$. In what follows, we take ι to be an identity map and hence, I to be a two-sided ideal of $\mathcal{A}$. The proofs do not depend on whether ι is an identity map or just an isomorpism between I and a two-sided ideal $\iota(I)$ of $\mathcal{A}$. An algebraic extension $\Sigma(\mathcal{A}; I)$ of R *splits algebraically* if there is a homomorphism $\Theta : R \to \mathcal{A}$ such that $\pi \circ \Theta = \mathrm{id}_R$, where id_R stands for the identity map on R. An algebraic extension $\Sigma(\mathcal{A}; I)$ of R is

a) *of dimension* m if the algebraic dimension of I is m, i.e., dim $I = m$;
b) *nilpotent*, if I is nilpotent;
c) *singular*, if $I^2 = \{\theta_I\}$.

The algebraic extension $\Sigma(\mathcal{A}; I)$ of R is a *topological extension* of a topological ring R if I and $\mathcal{A}$ are topological rings and π is continuous. A topological extension $\Sigma(\mathcal{A}; I)$ of a topological ring R *splits strongly* if it splits algebraically and the map Θ is continuous.

It can be shown that an algebraic extension $\Sigma(\mathcal{A}; I)$ of R splits algebraically if and only if there is a subring $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B} \cap I = \{\theta_{\mathcal{A}}\}$ and $\mathcal{A} = \mathcal{B} + I$. An extension $\Sigma(\mathcal{A}; I)$ of R splits strongly if and only if it splits algebraically so that $\mathcal{B}$ and I are closed subspaces of $\mathcal{A}$ (see, for example, [1, p. 9]).

An extension (algebraic or topological) $\Sigma(\mathcal{A}; I)$ of R is also called an (algebraic or topological) *extension of R by I*. For fixed I and R there could be several rings $\mathcal{A}$ for which $\Sigma(\mathcal{A}; I)$ is an extension of R by I.

2. Results

In what follows, we generalize the results of [1] for rings and topological rings.

Proposition 2.1. *Let $\Sigma(\mathcal{A}; I)$ be an algebraic extension of a ring R.*

(i) *Suppose that I contains a non-zero idempotent p such that $I = pI + Ip$. Then $\Sigma(\mathcal{A}; I)$ splits algebraically. Moreover, if I and R are topological rings and $\mathcal{A}$ is a topological ring which is also a Hausdorff space, then $\Sigma(\mathcal{A}; I)$ is a topological extension of R which splits strongly.*

(ii) *Suppose that I is semisimple and finite-dimensional complex algebra. Then $\Sigma(\mathcal{A}; I)$ splits algebraically. Moreover, if I and R are topological rings and $\mathcal{A}$ is a topological ring which is also a Hausdorff space, then $\Sigma(\mathcal{A}; I)$ is a topological extension of R which splits strongly.*

Proof. (i) Let $p \in I$ be a non-zero idempotent such that $I = pI + Ip$.

Take an arbitrary $a \in \mathcal{A}$. Then $pa \in I$ and $pa = p^2a = p(pa) \in pI$. Similarily, $ap \in I$ and $ap = ap^2 = (ap)p \in Ip$. Hence, $p\mathcal{A} \subseteq pI \subseteq I$ and $\mathcal{A}p \subseteq Ip \subseteq I$. Since $I \subseteq \mathcal{A}$, then also $Ip \subseteq \mathcal{A}p$ and $pI \subseteq p\mathcal{A}$. Therefore, $Ip = \mathcal{A}p, pI = p\mathcal{A}$ and $I = pI + Ip = p\mathcal{A} + \mathcal{A}p$.

We also have $p\mathcal{A}p = (p\mathcal{A})p \subseteq (pI)p \subseteq Ip \subseteq I$. Now,

$$I = p\mathcal{A} + \mathcal{A}p \subseteq p\mathcal{A} + \mathcal{A}p - p\mathcal{A}p \subseteq I + I + I \subseteq I.$$

Hence,

$$I = p\mathcal{A} + \mathcal{A}p = p\mathcal{A} + \mathcal{A}p - p\mathcal{A}p = p\mathcal{A}(e-p) + (e-p)\mathcal{A}p + p\mathcal{A}p.$$

Let $\mathcal{B} := (e-p)\mathcal{A}(e-p)$. Then it is easy to see that $\mathcal{B}$ is a subring of $\mathcal{A}$ (the product of any two elements $a - pa - ap + pap$ and $b - pb - bp + pbp$ of $\mathcal{B}$ is $(ab + apb) - p(ab + apb) - (ab + apb)p + p(ab + apb)p \in \mathcal{B}$) and $\mathcal{A} = \mathcal{B} + I$.

Take $b \in \mathcal{B} \cap I$. Then there exist $a \in \mathcal{A}$ and $c, d \in I$ such that $b = (e-p)a(e-p) = pc + dp$. Now,

$$\theta_{\mathcal{A}} = pc - pcp - p^2c + pcp + dp - dp^2 - pdp + pdp^2 = (e-p)pc(e-p) + (e-p)dp(e-p) =$$
$$= (e-p)(pc+dp)(e-p) = (e-p)b(e-p) = b - pb - bp + pbp = (a - pa - ap + pap) -$$
$$- p(a - pa - ap + pap) - (a - pa - ap + pap)p + p(a - pa - ap + pap)p = b.$$

Hence, $\mathcal{B} \cap I = \{\theta_I\}$, which means that $\Sigma(\mathcal{A}; I)$ splits algebraically.

Suppose, that $\mathcal{A}$ is a topological ring and take an element

$$b_0 \in \mathrm{cl}_{\mathcal{A}}\mathcal{B} = \mathrm{cl}_{\mathcal{A}}\big((e-p)\mathcal{A}(e-p)\big).$$

Since $(e-p)\mathcal{A}(e-p) = \{a - pa - ap + pap : a \in A\}$, then there exist nets $(b_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{B}$ and $(a_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{A}$ such that $b_\lambda = a_\lambda - pa_\lambda - a_\lambda p + pa_\lambda p$ for every $\lambda \in \Lambda$ and the net $(b_\lambda)_{\lambda\in\Lambda}$ converges to b_0. Exactly as we did before in case of $b \in \mathcal{B} \cap I$, we can now show that $b_\lambda - pb_\lambda - b_\lambda p + pb_\lambda p = b_\lambda$ for every $\lambda \in \Lambda$. The net $(b_\lambda - pb_\lambda - b_\lambda p + pb_\lambda p)_{\lambda\in\Lambda}$ converges to $b_0 - pb_0 - b_0p + pb_0p$. Since $\mathcal{B} \subseteq \mathcal{A}$, then $b_0 \in \mathrm{cl}_{\mathcal{A}}\mathcal{B} \subseteq \mathrm{cl}_{\mathcal{A}}\mathcal{A} = \mathcal{A}$. Hence, $b_0 - pb_0 - b_0p + pb_0p \in \mathcal{B}$. Because

$b_\lambda = b_\lambda - pb_\lambda - b_\lambda p + pb_\lambda p$ for every $\lambda \in \Lambda$, then the net b_λ converges also to $b_0 - pb_0 - b_0 p + pb_0 p$. Therefore, $b_0 = b_0 - pb_0 - b_0 p + pb_0 p \in \mathcal{B}$, which means that $\mathcal{B}$ is closed.

Suppose that $\mathcal{A}$ is also a Hausdorff space and I, R are topological rings. Then the set $\{\theta_{\mathcal{A}}\}$ is closed in $\mathcal{A}$. Define a map $T_p : \mathcal{A} \to \mathcal{A}$ by $T_p(a) := a - pa - ap + pap$ for every $a \in \mathcal{A}$. It is clear that T_p is a continuous linear map. Hence, the original of $\{\theta_{\mathcal{A}}\}$ by T_p is closed in $\mathcal{A}$. This means that $\ker T_p$ is closed in $\mathcal{A}$.

Since $I = p\mathcal{A} + \mathcal{A}p$ and

$$T_p(pa + bp) = pa + bp - p^2 a - pbp - pap - bp^2 + p^2 ap + pbp = \theta_{\mathcal{A}}$$

for every $a, b \in \mathcal{A}$, then $I \subseteq \ker T_p$. Take $a \in \ker T_p$. Then $a - pa - ap + pap = \theta_{\mathcal{A}}$. Hence, $a = pa + ap - pap = p(a - ap) + ap \in p\mathcal{A} + \mathcal{A}p = I$. Therefore, $\ker T_p \subseteq I$ implies $I = \ker T_p$. Thus, I is closed in $\mathcal{A}$. Consequently, $\Sigma(\mathcal{A}; I)$ splits strongly.

(ii) By the Wederburn Structure Theorem (see [2, Theorem 1.5.9, p. 72] or [5, Theorem 8.11, p. 658].) it is known that every non-zero finite-dimensional semisimple complex algebra has an identity e. Hence, we can take $p = e$ in part (i). Then the conditions of the part (i) are fulfilled and the claim follows from the part (i). □

Let $\Sigma(\mathcal{A}; I)$ be an algebraic extension of a ring R and let J be a two-sided ideal of $\mathcal{A}$ with $J \subseteq I$. Then it is easy to see that

$$\Sigma(\mathcal{A}/J; I/J) : \theta_{I/J} \to I/J \xrightarrow{\iota_J} \mathcal{A}/J \xrightarrow{\pi_J} R \to \theta_R,$$

with $\iota_J([x]) := [\iota(x)]$ for every $x \in I$ and $\pi_J([a]) := [\pi(a)]$ for every $a \in \mathcal{A}$, is also an algebraic extension of R. In case R is a topological ring, $\Sigma(\mathcal{A}; I)$ is a topological extension of R and J is a closed two-sided ideal of $\mathcal{A}$, then $\Sigma(\mathcal{A}/J; I/J)$ is a topological extension of R.

Proposition 2.2. *Let $\Sigma(\mathcal{A}; I) : \theta_I \to I \xrightarrow{\iota} \mathcal{A} \xrightarrow{\pi} R \to \theta_R$ be a topological extension of a topological ring R and let J be a closed two-sided ideal of $\mathcal{A}$ with $J \subseteq I$.*

(i) *Suppose that every topological extension of R by J splits strongly and that the topological extension $\Sigma(\mathcal{A}/J; I/J)$ of R splits strongly. Then $\Sigma(\mathcal{A}; I)$ splits strongly, as well.*

(ii) *Suppose that every topological extension of R by J splits algebraically and that the topological extension $\Sigma(\mathcal{A}/J; I/J)$ of R splits strongly. Then $\Sigma(\mathcal{A}; I)$ splits algebraically, as well.*

Proof. Since the topological extension

$$\Sigma(\mathcal{A}/J; I/J) : \theta_{I/R} \to I/J \xrightarrow{\iota_J} \mathcal{A}/J \xrightarrow{\pi_J} R \to \theta_R$$

of R splits strongly, then there exists a closed subring $\mathcal{D}$ of $\mathcal{A}/J$ with $\mathcal{A}/J = \mathcal{D} + (I/J)$ and $\mathcal{D} \cap (I/J) = \{\theta_{I/J}\}$. Using the quotient map $q : \mathcal{A} \to \mathcal{A}/J$, we define $\mathcal{C} := q^{-1}(\mathcal{D})$. Then $\mathcal{C}$ (as an original of a closed set by a continuous linear map) is a closed subring of $\mathcal{A}$. Moreover, $J \subset \mathcal{C}$ and $\mathcal{C}/J = \mathcal{D}$.

Consider the following topological extension of R by J:

$$\Sigma(\mathcal{C}; J) : \theta_J \to J \xrightarrow{\iota|_J} \mathcal{C} \xrightarrow{\pi|_{\mathcal{C}}} R \to \theta_R.$$

It is really a topological extension of R by J because J and $\mathcal{C}$ are topological rings and $\iota|_J\,(J) = \ker\,\pi|_J$. (Clearly $\iota|_J\,(J) = \iota(J) = J \in \ker\,\pi \cap \mathcal{C} = \ker\,\pi|_{\mathcal{C}}$. Let $c \in \ker\,\pi|_{\mathcal{C}} \subseteq \ker\,\pi = I$. Then $c \in \mathcal{C} \cap I$. Hence,

$$q(c) \in q(\mathcal{C} \cap I) \subseteq q(\mathcal{C}) \cap q(I) = \mathcal{D} \cap I/J = \{\theta_{I/J}\}.$$

Thus, $c \in J$ and $\ker\,\pi|_{\mathcal{C}} \subset J = \iota|_J\,(J)$.)

By assumptions, this extension splits strongly in case (i) and algebraically in case (ii). Hence, there exists a (closed, in case (i)) subring $\mathcal{B}$ of $\mathcal{C}$ such that $\mathcal{C} = \mathcal{B} + J$ and $\mathcal{B} \cap J = \{\theta_I\}$. Since $\mathcal{C}$ is a closed subring of $\mathcal{A}$, then $\mathcal{C}$ is also a (closed, in case (i)) subring of $\mathcal{A}$.

Since $\mathcal{D} \cap (I/J) = \{\theta_{I/J}\}$, then $\mathcal{A} = q^{-1}(\mathcal{A}/J) = q^{-1}(\mathcal{D} + (I/J)) = \mathcal{C} + I$. Hence, $\mathcal{A} = \mathcal{C} + I = (\mathcal{B} + J) + I = \mathcal{B} + (J + I) = \mathcal{B} + I$.

Let $i \in \mathcal{B} \cap I \subseteq \mathcal{C} \cap I$. Then $q(i) \in q(\mathcal{C}) \cap q(I) = \mathcal{D} \cap (I/J) = \{\theta_{I/J}\}$. Therefore, $i \in J$ and hence, $i \in \mathcal{B} \cap J$. Thus, $\mathcal{B} \cap I = \mathcal{B} \cap J = \{\theta_I\}$. Hence, $\Sigma(\mathcal{A}; I)$ splits strongly in case (i) and algebraically in case (ii). □

In the next Corollary we suppose that we are already in the situation where something similar to the Wedderburn Principal Theorem holds.

Corollary 2.3. *Let $\Sigma(\mathcal{A}; I)$ be a topological extension of a topological ring R. Suppose that $\mathcal{A}$ is a Hausdorff space and $I/\mathrm{rad}\ I$ is a finite-dimensional complex algebra.*

(i) *If every topological extension of R by* rad I *splits strongly, then $\Sigma(\mathcal{A}; I)$ splits strongly, as well.*
(ii) *If every topological extension of R by* rad I *splits algebraically, then $\Sigma(\mathcal{A}; I)$ splits algebraically, as well.*

Proof. Take $J := \mathrm{rad}\ I = I \cap \mathrm{rad}\ \mathcal{A}$ (The last equation holds, for example, by [3, Theorem 1.2.5, p. 16].). Since $\Sigma(\mathcal{A}; I)$ splits strongly in both cases, then J is closed in $\mathcal{A}$. Clearly, J is a two-sided ideal in $\mathcal{A}$ (see, for example, [3, pp. 8–9].). It is known (see [3, Theorem 1.2.4, p. 15]) that I/J is semisimple. Hence, we are in the situation of Proposition 2.1, part (ii). Thus, every topological extension of R by I/J splits strongly by Proposition 2.1, part (ii). Now, the claim follows from the Proposition 2.2. □

Corollary 2.4. *Let R be a topological ring and let $m \in \mathbb{N}$. Suppose that every nilpotent topological extension of dimension at most m of R splits strongly (respectively, algebraically). If I is an Artinian topological ring with dim $I \leq m$ and $I/$ rad I is a complex algebra, then every topological extension $\Sigma(\mathcal{A}; I)$ of R, where $\mathcal{A}$ is a Hausdorff space, splits strongly (respectively, algebraically), as well.*

Proof. Let $\Sigma(\mathcal{A}; I)$ be any topological extension of R such that I is an Artinian ring and $\dim I \leq m$. Then rad I is nilpotent by [3, Theorem 1.3.1, p. 20]. Since $\dim I \leq m$, then also $\dim(\mathrm{rad} I) \leq m$. Clearly, $I/\mathrm{rad}\ I$ is finite-dimensional. The claim follows directly from Corollary 2.3. □

For the next Proposition we need the following Lemmas.

Lemma 2.5. *Let R be a topological ring and $k, m \in \mathbb{N}$. If the nets $(x_{ij}^{\alpha^{ij}})_{\alpha^{ij}\in A_{ij}}$ with*

$$x_{ij} = \lim_{\alpha^{ij}}(x_{ij}^{\alpha^{ij}})$$

consist of elements of R for every $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k\}$, then

$$\sum_{i=1}^{m} x_{1i}\ldots x_{ki} = \sum_{i=1}^{m}(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\ldots(\lim_{\alpha^{ki}} x_{ki}^{\alpha^{ki}}) = \lim_{\alpha^{km}}\ldots\lim_{\alpha^{11}}\sum_{i=1}^{m} x_{1i}^{\alpha^{1i}}\ldots x_{ki}^{\alpha^{ki}}.$$

Proof. Let the multiplication in R be separately continuous. It means that for any $p \in R$ the maps $l_p, r_p : R \to R$, defined by $l_p(q) := pq$ and $r_p(q) := qp$ for any $q \in R$, are continuous. Hence, for any $r, s \in R$ and a convergent net $(t_\alpha)_{\alpha\in A}$ of elements of R we have

$$(\lim_{\alpha} t_\alpha)r = r_r(\lim_{\alpha} t_\alpha) = \lim_{\alpha}(r_r(t_\alpha)) = \lim_{\alpha}(t_\alpha r) \tag{2.1}$$

and

$$s(\lim_{\alpha} t_\alpha) = l_s(\lim_{\alpha} t_\alpha) = \lim_{\alpha}(l_s(t_\alpha)) = \lim_{\alpha}(st_\alpha). \tag{2.2}$$

First, for any $i \in \{1, \ldots, m\}$ we have by (2.1),

$$x_{1i}x_{2i}\ldots x_{ki} = (\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})(x_{2i}\ldots x_{ki}) =$$

$$= r_{x_{2i}\ldots x_{ki}}(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}}) = \lim_{\alpha^{1i}}(r_{x_{2i}\ldots x_{ki}}(x_{1i}^{\alpha^{1i}})) = \lim_{\alpha^{1i}}(x_{1i}^{\alpha^{1i}} x_{2i}\ldots x_{ki}).$$

Now, suppose that for some $j \geq 2$ we have

$$(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\ldots\Big(\lim_{\alpha^{(j-1)i}} x_{(j-1)i}^{\alpha^{(j-1)i}}\Big)x_{ji}\ldots x_{ki} =$$

$$= \lim_{\alpha^{(j-1)i}}\Big(\lim_{\alpha^{(j-2)i}}\ldots(\lim_{\alpha^{1i}}(x_{1i}^{\alpha^{1i}} x_{2i}^{\alpha^{2i}}\ldots x_{(j-1)i}^{\alpha^{(j-1)i}} x_{ji}\ldots x_{ki}))\ldots\Big).$$

Certainly it is true for $j = 2$. Then, using (2.1) and (2.2), we have

$$b := (\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\ldots\Big(\lim_{\alpha^{(j-1)i}} x_{(j-1)i}^{\alpha^{(j-1)i}}\Big)(\lim_{\alpha^{ji}} x_{ji}^{\alpha^{ji}})x_{(j+1)i}\ldots x_{ki} =$$

$$= \Big[(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\ldots(\lim_{\alpha^{(j-1)i}} x_{(j-1)i}^{\alpha^{(j-1)i}})\Big]\Big[(\lim_{\alpha^{ji}} x_{ji}^{\alpha^{ji}})x_{(j+1)i}\ldots x_{ki}\Big] =$$

$$= l_{(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\ldots(\lim_{\alpha^{(j-1)i}} x_{(j-1)i}^{\alpha^{(j-1)i}})}(r_{x_{(j+1)i}\ldots x_{ki}}(\lim_{\alpha^{ji}} x_{ji}^{\alpha^{ji}})) =$$

$$= l_{(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\ldots(\lim_{\alpha^{(j-1)i}} x_{(j-1)i}^{\alpha^{(j-1)i}})}(\lim_{\alpha^{ji}}(r_{x_{(j+1)i}\ldots x_{ki}}(x_{ji}^{\alpha^{ji}}))) =$$

$$= \lim_{\alpha^{ji}}(l_{(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\ldots(\lim_{\alpha^{(j-1)i}} x_{(j-1)i}^{\alpha^{(j-1)i}})}(r_{x_{(j+1)i}\ldots x_{ki}}(x_{ji}^{\alpha^{ji}}))) =$$

$$= \lim_{\alpha^{ji}}\Big(\Big[(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\ldots\Big(\lim_{\alpha^{(j-1)i}} x_{(j-1)i}^{\alpha^{(j-1)i}}\Big)\Big][x_{ji}^{\alpha^{ji}} x_{(j+1)i}\ldots x_{ki}]\Big)$$

In case $j = k$, there will not be any $x_{(j+1)i}\ldots x_{ki}$. Otherwise everything will remain the same. By our assumption,

$$\Big[(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\ldots\Big(\lim_{\alpha^{(j-1)i}} x_{(j-1)i}^{\alpha^{(j-1)i}}\Big)\Big][x_{ji}^{\alpha^{ji}} x_{(j+1)i}\ldots x_{ki}] =$$

$$= \lim_{\alpha^{(j-1)i}}(\lim_{\alpha^{(j-2)i}}\ldots(\lim_{\alpha^{1i}}(x_{1i}^{\alpha^{1i}} x_{2i}^{\alpha^{2i}}\ldots x_{(j-1)i}^{\alpha^{(j-1)i}} x_{ji}^{\alpha^{ji}} x_{(j+1)i}\ldots x_{ki})\ldots).$$

Hence,

$$b = \lim_{\alpha^{ji}}(\lim_{\alpha^{(j-1)i}}(\lim_{\alpha^{(j-2)i}}\dots(\lim_{\alpha^{1i}}(x_{1i}^{\alpha^{1i}}x_{2i}^{\alpha^{2i}}\dots x_{(j-1)i}^{\alpha^{(j-1)i}}x_{ji}^{\alpha^{ji}}x_{(j+1)i}\dots x_{ki}))\dots).$$

Since it is true for any $j \in \{1,\dots,k\}$, then

$$x_{1i}x_{2i}\dots x_{ki} = \lim_{\alpha^{ki}}(\lim_{\alpha^{(k-1)i}}\dots(\lim_{\alpha^{1i}}(x_{1i}^{\alpha^{1i}}x_{2i}^{\alpha^{2i}}\dots x_{(k-1)i}^{\alpha^{(k-1)i}}x_{ki}^{\alpha^{ki}}))\dots)$$

for any $i \in \{1,\dots,m\}$. Since the addition in R is continuous, then

$$\sum_{i=1}^{m} x_{1i}\dots x_{ki} = \sum_{i=1}^{m}(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\dots(\lim_{\alpha^{ki}} x_{ki}^{\alpha^{ki}}) = \lim_{\alpha^{km}}\dots\lim_{\alpha^{11}}\sum_{i=1}^{m} x_{1i}^{\alpha^{1i}}\dots x_{ki}^{\alpha^{ki}}.$$

□

Lemma 2.6. *Let R be a topological ring, J a two-sided ideal of R and $k \in \mathbb{N}$ with $k \geq 2$. Then $(\mathrm{cl}_R(J))^k \subseteq \mathrm{cl}_R(J^k)$ (Here $cl_R(J)$ denotes the closure of J in R.).*

Proof. Take an element $x \in (\mathrm{cl}_R(J))^k$. Then there exists $m \in \mathbb{N},\ n_1,\dots,n_m \in \mathbb{Z}$ and $x_{11}, x_{12},\dots,x_{1m},x_{21},\dots,x_{km} \in \mathrm{cl}_R(J)$ such that

$$x = \sum_{i=1}^{m} n_i x_{1i}\dots x_{ki}.$$

Then for all $i \in \{1,\dots,k\}$ and all $j \in \{1,\dots,m\}$ there exist nets $(x_{ij}^{\alpha^{ij}})_{\alpha^{ij}\in A_{ij}}$ in J such that

$$x_{ij} = \lim_{\alpha^{ij}} x_{ij}^{\alpha^{ij}}.$$

Now, using Lemma 2.5, we have

$$x = \sum_{i=1}^{m} n_i x_{1i}\dots x_{ki} = \sum_{i=1}^{m} n_i(\lim_{\alpha^{1i}} x_{1i}^{\alpha^{1i}})\dots(\lim_{\alpha^{ki}} x_{ki}^{\alpha^{ki}}) = \lim_{\alpha^{km}}\dots\lim_{\alpha^{11}}\sum_{i=1}^{m} n_i x_{1i}^{\alpha^{1i}}\dots x_{ki}^{\alpha^{ki}}.$$

Since

$$\sum_{i=1}^{m} n_i x_{1i}^{\alpha^{1i}}\dots x_{ki}^{\alpha^{ki}} \in J^k,$$

then $x \in \mathrm{cl}_R(J^k)$. Therefore, $(\mathrm{cl}_R(J))^k \subseteq \mathrm{cl}_R(J^k)$. □

Now we are ready for the next result.

Proposition 2.7. *Let R be a topological ring and $m \in \mathbb{N}$.*

(i) *Suppose that every singular topological extension of dimension at most m of R splits strongly. Then every nilpotent topological extension $\Sigma(\mathcal{A}; I)$ of dimension at most m of R, where $\mathcal{A}$ is a Hausdorff space, splits strongly, as well.*

(ii) *Suppose that every singular topological extension of dimension at most m of R splits strongly. Then every topological extension $\Sigma(\mathcal{A}; I)$ of dimension at most m of R, where $\mathcal{A}$ is a Hausdorff space and I is an Artinian topological ring for which $I/\mathrm{rad}\ I$ is a complex algebra, splits strongly, as well.*

Proof. (i) We will prove it by induction on the minimum index $n \in \mathbb{N} \setminus \{1\}$ such that $I^n = \{\theta_I\}$. In case $I^2 = \{\theta_I\}$, I is singular and the claim is true.

Suppose that $\Sigma(\mathcal{A}; I)$ is a topological extension of dimension at most m of R such that I has the minimum index $k \geqslant 3$ and $\mathcal{A}$ is a Hausdorff space. By the induction, we can assume, that the claim holds for all topological extensions $\Sigma(\mathcal{A}; J)$ of R which have the dimension at most m and minimum index at most $k-1$. Take $J = \mathrm{cl}_{\mathcal{A}}(I^2)$. Then J is a closed two-sided ideal of $\mathcal{A}$. Using Lemma 2.6, we get

$$J^{k-1} = (\mathrm{cl}_{\mathcal{A}}(I^2))^{k-1} \subseteq \mathrm{cl}_{\mathcal{A}}(I^{2k-2}) = \mathrm{cl}_{\mathcal{A}}(I^k I^{k-2}) = \mathrm{cl}_{\mathcal{A}}(\{\theta_I\}) = \{\theta_I\},$$

because $k-2 \geqslant 1$ and $I^k = \{\theta_I\}$. Since $\mathrm{cl}_{\mathcal{A}}(I^2) \subseteq \mathrm{cl}_{\mathcal{A}}(I) = I$ then the dimension of J is at most m. Hence, every topological extension of R by J splits strongly by the assumption of the induction.

Consider the set $I/J = I/\mathrm{cl}_{\mathcal{A}}(I^2)$. It is clear that the dimension of I/J is at most m. Take any elements $[i_1], \ldots, [i_{k-1}] \in I/J$. Since $k-1 \geqslant 2$ and J is an ideal of $\mathcal{A}$, then $i_1 \cdot \cdots \cdot i_{k-1} \in I^2 \subseteq J$. Hence, $[i_1] \cdot \cdots \cdot [i_{k-1}] = [i_1 \cdot \cdots \cdot i_{k-1}] = [\theta_I]$. Thus, $(I/J)^{k-1} = \{\theta_I\}$. Notice, that $\mathcal{A}/J$ is a Hausdorff space, because J is a closed two-sided ideal of $\mathcal{A}$. Again, by the assumption of the induction, we get that every topological extension of R by I/J splits strongly.

Now the claim follows from Proposition 2.2, part (i).

(ii) Suppose that I is as in the part (i) of the proof and that I is also Artinian ring. Then, by Corollary 2.4, the claim (ii) follows from the part (i) of the Proposition 2.7. □

Last result is of purely algebraic nature and holds for arbitrary rings.

Theorem 2.8. *Let R be a ring and let $\Sigma(\mathcal{A}; I)$ be an algebraic extension of R. Suppose that there exist such two-sided ideals J and K of $\mathcal{A}$ that $I = J + K$ and $J \cap K = \{\theta_{\mathcal{A}}\}$. If the algebraic extensions $\Sigma(\mathcal{A}/J; I/J)$ and $\Sigma(\mathcal{A}/K; I/K)$ both split algebraically, then $\Sigma(\mathcal{A}; I)$ splits algebraically, as well.*

Proof. Since $\Sigma(\mathcal{A}/J; I/J)$ and $\Sigma(\mathcal{A}/K; I/K)$ both split algebraically, there exist subrings $\mathcal{C}_J$ and $\mathcal{C}_K$ of $\mathcal{A}/J$ and $\mathcal{A}/K$, respectively, such that $\mathcal{A}/J = \mathcal{C}_J + (I/J)$, $\mathcal{A}/K = \mathcal{C}_K + (I/K)$, with $\mathcal{C}_J \cap (I/J) = \{\theta_{\mathcal{A}/J}\}$ and $\mathcal{C}_K \cap (I/K) = \{\theta_{\mathcal{A}/K}\}$.

Define $\mathcal{B}_J := \rho_J^{-1}(\mathcal{C}_J) = \{a \in \mathcal{A} : \rho_J(a) \in \mathcal{C}_J\}$ and $\mathcal{B}_K := \rho_K^{-1}(\mathcal{C}_K) = \{a \in \mathcal{A} : \rho_K(a) \in \mathcal{C}_K\}$, where $\rho_J : \mathcal{A} \to \mathcal{A}/J$ and $\rho_K : \mathcal{A} \to \mathcal{A}/K$ are the quotient maps. Set $\mathcal{B} := \mathcal{B}_J \cap \mathcal{B}_K$. Then it is easy to check that $\mathcal{B}_J, \mathcal{B}_K$ and $\mathcal{B}$ are subrings of $\mathcal{A}$.

Take any $a \in \mathcal{A}$. Then $\rho_J(a) \in \mathcal{A}/J$ and $\rho_K(a) \in \mathcal{A}/K$. Since

$$\rho_J(\mathcal{B}_J + I) = \rho_J(\mathcal{B}_J) + \rho_j(I) = \mathcal{C}_J + (I/J) = \mathcal{A}/J,$$

then $a \in (\mathcal{B}_J + I) + J \subseteq \mathcal{B}_J + I$. Hence, there exist $b_1 \in \mathcal{B}_J$ and $i_1 \in I$ such that $a = b_1 + i_1$. Similarly from $\rho_K(\mathcal{B}_K + I) = \mathcal{A}/K$ it follows that there exist $b_2 \in \mathcal{B}_K$ and $i_2 \in I$ such that $a = b_2 + i_2$. Hence, $b_1 - b_2 = i_2 - i_1 \in I = J + K$ and there exist $j \in J$ and $k \in K$ such that $b_1 - b_2 = j + k$.

Now, $b_1 - j = b_2 + k$. Notice, that

$$\rho_J(b_1 - j) \in \rho_J(b_1 + J) \subset \rho_J(b_1) + \rho_J(J) = \rho_J(b_1) + \theta_{\mathcal{A}/J} = \rho_J(b_1) \in \mathcal{C}_J.$$

Hence, $b_1 - j \in \mathcal{B}_J$. Similarly we get that

$$\rho_K(b_1 - j) = \rho_K(b_2 + k) \in \rho_K(b_2 + K) \in \mathcal{C}_K$$

and $b_1 - j \in \mathcal{B}_K$. Hence, $b_1 - j \in \mathcal{B}_J \cap \mathcal{B}_K = \mathcal{B}$. Therefore, $b_1 = (b_1 - j) + j \in \mathcal{B} + J$. Thus, $a = b_1 + i_1 \in (\mathcal{B} + J) + I \subset \mathcal{B} + I$. Since a was an arbitrary element in $\mathcal{A}$ and $\mathcal{B} + I \subseteq \mathcal{A}$, we have $\mathcal{A} = \mathcal{B} + I$.

Take any $a \in \mathcal{B} \cap I$. Then $a \in \mathcal{B}_J$ implies $\rho_J(a) \in \mathcal{C}_J$. From $a \in I$ follows that $\rho_J(a) \in I/J$. Hence, $\rho_J(a) \in \mathcal{C}_J \cap (I/J) = \{\theta_{\mathcal{A}/J}\}$. Thus, $a \in J$. Similarly we get from $a \in \mathcal{B}_K$ that $\rho_K(a) \in \mathcal{C}_K \cap (I/K) = \{\theta_{\mathcal{A}/K}\}$. Hence, $a \in K$, as well. Since $J \cap K = \theta_{\mathcal{A}}$, we have $\mathcal{B} \cap I = \{\theta_{\mathcal{A}}\}$. Together with $\mathcal{A} = \mathcal{B} + I$ it implies that $\Sigma(\mathcal{A}; I)$ splits algebraically. □

Remark 2.9. Proposition 2.1 is a generalization of [1, Proposition 1.4, pp. 10–11]; Proposition 2.2 is a generalization of [1, Proposition 1.5, p. 11]; Corollary 2.3 is a generalization of [1, Theorem 1.6, p.12]; Corollary 2.4 is a generalization of [1, Theorem 1.7, p. 11]; Proposition 2.7 is a generalization of [1, Theorem 1.8, p. 13] and Theorem 2.8 is a generalization of [1, Theorem 1.9, p.13]. The proofs of the aforementioned results follow the ideas of proofs from [1], although in several places the proofs of [1] had to be modified or carried out in more details than in [1].

Acknowledgements. The author wishes to thank the Professors Mati Abel and Wieslaw Żelazko for suggestions and comments during the process of writing the present paper. This research was partly supported by Estonian Science Foundation Grant 7320 and partly supported by Estonian Targeted Financing Project SF0180039s08.

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