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LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. In this paper, we prove the boundedness for some multilinear commutators generated by the pseudo-differential operator and Lipschitz functions.

1. INTRODUCTION

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [4, 5, 6, 7, 8, 9, 10]). In [4, 5, 6, 7, 8, 9, 10], the authors prove that the commutators and multilinear operators generated by the singular integral operators and BMO functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$; Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [4, 11, 12, 13, 16], the boundedness for the commutators and multilinear operators generated by the singular integral operators and Lipschitz functions on Triebel–Lizorkin and $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) spaces are obtained. In [14], the weighted boundedness for the commutators generated by the singular integral operators and BMO and Lipschitz functions on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) spaces are obtained. The purpose of this paper is to prove the weighted boundedness on Lebesgue spaces for some multilinear operators associated to the pseudo-differential operators and the weighted Lipschitz functions. To do this, we first prove a sharp function estimate for the multilinear operators. Our results are new, even in the unweighted cases.

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The purpose of this paper is to prove the boundedness on Lebesgue spaces for some multilinear commutators generated by the pseudo-differential operator and Lipschitz functions.

2. NOTATIONS AND THEOREMS

In order to state our results, we begin by introducing the relevant notions and definitions.

Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes.

The A_1 weight is defined by $A_1 = \{0 < \omega \in L^1_{loc} : \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \omega(y) dy \leq c\omega(x), a.e.\}$ (see [5]).

The Lipschitz space $Lip_\beta(R^n)$ with $0 < \beta < 1$ is the space of functions f such that

$$\|f\|_{Lip_\beta} = \sup_{\substack{x, y \in R^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

For $0 < \beta < 1, 1 < p < \infty, m > 0$, the Triebel–Lizorkin space $\tilde{F}_p^{m\beta, \infty}(R^n)$ (see [14]) is the space of functions f such that

$$\|f\|_{\tilde{F}_p^{m\beta, \infty}} = \left\| \sup_{Q \ni \bar{x}} |Q|^{-1 - \frac{m\beta}{n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p} < \infty.$$

Given some function $b_j \in Lip_\beta(R^n), 1 \leq j \leq m, 0 < \beta < 1$, we denote by C_j^m the family of all finite subsets $\gamma = \{\gamma(1), \dots, \gamma(j)\}$ of j different elements in $\{1, \dots, m\}$, where $\gamma(i) < \gamma(j)$ when $i < j$.

For $\gamma \in C_j^m$, we denote $\gamma^c = \{1, \dots, m\} \setminus \gamma$.

For $\vec{b} = \{b_1, \dots, b_m\}$ and $\gamma = \{\gamma(1), \dots, \gamma(j)\} \in C_j^m$, we denote $\vec{b}_\gamma = \{b_{\gamma(1)}, \dots, b_{\gamma(j)}\}$, and $b_\gamma = b_{\gamma(1)} \cdots b_{\gamma(j)}$, and $\|\vec{b}_\gamma\|_{Lip_\beta} = \|b_{\gamma(1)}\|_{Lip_\beta} \cdots \|b_{\gamma(j)}\|_{Lip_\beta}$.

We say $\delta(x, \xi) \in S_{\epsilon, \sigma}^m$, if for $x, \xi \in R^n$,

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} \delta(x, \xi) \right| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - \epsilon|\beta| + \sigma|\alpha|},$$

where α, β are multiindex consisting of n nonnegative integers.

The pseudo-differential operators $\psi \cdot d \cdot \circ$ with symbols $\delta(x, \xi) \in S_{\epsilon, \sigma}^m$ is given by

$$T(f)(x) = \int_{R^n} e^{2\pi i(x, \xi)} \delta(x, \xi) \hat{f}(\xi) d\xi,$$

where f is a Schwartz function and \hat{f} denotes the Fourier transform of f .

The pseudo-differential operators $\psi \cdot d \cdot \circ$ also have another expression

$$T(f)(x) = \int_{R^n} K(x, x - y) f(y) dy,$$

where $K(x, x - y) = \int_{R^n} \delta(x, \xi) e^{2\pi i(x - y, \xi)} d\xi$.

Let $b_j, 1 \leq j \leq m$ be the fixed locally integrable functions on R^n . The multi-linear commutator associated to the pseudo-differential operator is defined by

$$T_{\vec{b}}(f)(x) = \int_{R^n} K(x, x-y) \prod_{j=1}^m (b_j(x) - b_j(y)) f(y) dy.$$

Now, we state the main results as follows.

Theorem 2.1. *Let T be a $\psi \cdot d \cdot o$ with symbol $\delta(x, \xi) \in S_{1-a, \sigma}^{-na/2}, 0 \leq \sigma < 1-a, 0 < a < 1$, and $b_j \in Lip_\beta(R^n)$ for $1 \leq j \leq m, 0 < \beta < \frac{1}{m(1-a)}$, and $1 < p < \infty$. Then $T_{\vec{b}}$ is bounded from $L^p(R^n)$ to $\tilde{F}_p^{m\beta, \infty}(R^n)$.*

Theorem 2.2. *Let T be a $\psi \cdot d \cdot o$ with symbol $\delta(x, \xi) \in S_{1-a, \sigma}^{-na/2}, 0 \leq \sigma < 1-a, 0 < a < 1$, and $b_j \in Lip_\beta(R^n)$ for $1 \leq j \leq m, 0 < \beta < \frac{1}{m(1-a)}$, and $1 < p < \infty, 1/q = 1/p - \frac{m\beta}{n}$. Then $T_{\vec{b}}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

3. PRELIMINARY LEMMAS

Lemma 3.1. (see [14]) *For $0 < \beta < 1, 1 \leq p \leq \infty$,*

$$\|b\|_{Lip_\beta} \approx \sup_Q |Q|^{-1-\frac{\beta}{n}} \int_Q |b(x) - b_Q| dx \approx \sup_Q |Q|^{-\frac{\beta}{n}} \left(|Q|^{-1} \int_Q |b(x) - b_Q|^p dx \right)^{1/p},$$

where $b_Q = |Q|^{-1} \int_Q b(x) dx$.

Lemma 3.2. (see [14]) *Let $Q_1 \subset Q_2$. Then $|b_{Q_1} - b_{Q_2}| \leq C \|b\|_{Lip_\beta} |Q_2|^{\beta/n}$.*

Lemma 3.3. (see [1]) *Let $\delta(x, \xi) \in S_{1-a, \sigma}^{-na/2}, 0 \leq \sigma < 1-a, 0 < a < 1$, and $K(x, w)$ denote the inverse Fourier transformations in the ξ -variable and in the distribution sense of $\delta(x, \xi)$, that is informally $K(x, w) = \int_{R^n} \delta(x, \xi) e^{2\pi i(w, \xi)} d\xi$. Then for $|x - x_0| \leq d \leq 1/2$ and $N \geq 0$,*

$$\begin{aligned} & \left(\int_{(2^N d)^{1-a} \leq |y-x_0| \leq (2^{N+1} d)^{1-a}} |K(x, x-y) - K(x_0, x_0-y)|^2 dy \right)^{1/2} \\ & \leq C |x - x_0|^{(1-a)(h-n/2)} (2^{N+1} d)^{-h(1-a)} \end{aligned}$$

where h is an integer such that $n/2 < h < n/2 + 1/(1-a)$.

Lemma 3.4. (see [1]) *Let $\delta(x, \xi) \in S_{\epsilon, \sigma}^0, 0 < \epsilon < 1$, and, as usual, $K(x, w) = \int_{R^n} \delta(x, \xi) e^{2\pi i(w, \xi)} d\xi$. Then for $|w| \geq 1/4$ and arbitrarily large $M, |K(x, w)| \leq C_M |w|^{-2M}$.*

Lemma 3.5. (see [3]) *For $0 < \beta < 1, 1 < p < \infty, m > 0$, we have*

$$\begin{aligned} \|f\|_{\tilde{F}_p^{m\beta, \infty}} & \approx \left\| \sup_{Q \ni \vec{x}} |Q|^{-1-\frac{m\beta}{n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p} \\ & \approx \left\| \sup_{Q \ni \vec{x}} \inf_{c \in C_Q} |Q|^{-1-\frac{m\beta}{n}} \int_Q |f(x) - c| dx \right\|_{L^p}, \end{aligned}$$

where $f_Q = |Q|^{-1} \int_Q f(x) dx$.

Lemma 3.6. (see [15, Section 1.9 of Chapter 5]) $\chi_Q \in A_1$ for any cube Q .

Lemma 3.7. (see [1]) Let $\delta(x, \xi) \in S_{1-a, \sigma}^{-na/2}$, $0 \leq \sigma < 1 - a$, $0 < a < 1$, $T(f)(x) = \int_{R^n} e^{2\pi i(x, \xi)} \delta(x, \xi) \hat{f}(\xi) d\xi$, and $\omega \in A_{p/2}$ with $2 \leq p < \infty$, $\|T(f)\|_{p, \omega} = (\int_{R^n} |T(f)(x)|^p \omega(x) dx)^{\frac{1}{p}}$. Then $\|T(f)\|_{p, \omega} \leq C_p \|f\|_{p, \omega}$ for $f \in C_0^\infty(R^n)$.

Lemma 3.8. (see [15, Section 1.1 of Chapter 2]) Suppose $1 < p < \infty$, $1 < r < \infty$, $M_r(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(y)|^r dy \right)^{\frac{1}{r}}$. Then

$$\left(\int_{R^n} M_r(f)(x)^p dx \right)^{1/p} \leq C \left(\int_{R^n} |f(x)|^p dx \right)^{1/p}.$$

Lemma 3.9. (see [2]) Let $M_{r, m\beta}(f)(x) = \sup_{x \in Q} \left(|Q|^{-1 + \frac{rm\beta}{n}} \int_Q |f(y)|^r dy \right)^{1/r}$ for $1 < r < \infty$, $\beta > 0$, $m > 0$, and $r < p < m\beta/n$, $1/q = 1/p - m\beta/n$. Then

$$\|M_{r, m\beta}(f)(x)\|_{L^q} \leq C \|f\|_{L^p}.$$

4. PROOF OF THEOREM 1

Proof. Fix a cube $Q = Q(\tilde{x}, d)$ with the sidelength d . we first prove that there exists some constant C and $1 < r < \infty$ such that, for $f \in L^p(R^n)$,

$$\sup_{Q \ni \tilde{x}} \inf_{c \in C_Q} |Q|^{-1 - \frac{m\beta}{n}} \int_Q |T_{\vec{b}}(x) - c| dx \leq C \|\vec{b}\|_{Lip_\beta} (M_r(T(f))(\tilde{x}) + M_r(f)(\tilde{x})).$$

We consider the case $m = 1$ and $d \leq 1$.

Let $f(x) = f_1(x) + f_2(x)$ with $f_1(x) = f(x)\chi_J(x)$ and $f_2(x) = f(x)\chi_{J^c}(x)$, and let $(b_1)_J = |J|^{-1} \int_J b_1(y) dy$, where J is a cube concentric with Q of side-length d^{1-a} .

$$\begin{aligned} T_{\vec{b}_1}^-(f)(x) &= \int_{R^n} K(x, x-y) ((b_1(x) - (b_1)_J) - (b_1(y) - (b_1)_J)) f(y) dy \\ &= (b_1(x) - (b_1)_J) \int_{R^n} K(x, x-y) f(y) dy - \int_{R^n} K(x, x-y) (b_1(y) - (b_1)_J) f_1(y) dy \\ &\quad - \int_{R^n} K(x, x-y) (b_1(y) - (b_1)_J) f_2(y) dy \\ &= (b_1(x) - (b_1)_J) T(f)(x) - T((b_1 - (b_1)_J) f_1)(x) - T((b_1 - (b_1)_J) f_2)(x), \end{aligned}$$

thus

$$\begin{aligned} &|Q|^{-1 - \frac{\beta}{n}} \int_Q |T_{\vec{b}_1}^-(f)(x) - T((b_1)_J - b_1) f_2)(\tilde{x})| dx \\ &\leq |Q|^{-1 - \frac{\beta}{n}} \int_Q |b_1(x) - (b_1)_J| |T(f)(x)| dx + |Q|^{-1 - \frac{\beta}{n}} \int_Q |T((b_1 - (b_1)_J) f_1)(x)| dx \\ &\quad + |Q|^{-1 - \frac{\beta}{n}} \int_Q |T((b_1 - (b_1)_J) f_2)(x) - T((b_1 - (b_1)_J) f_2)(\tilde{x})| dx \\ &= A_{1,1} + A_{1,2} + A_{1,3}. \end{aligned}$$

For $A_{1,1}$, by Hölder's inequality with exponent $1/r + 1/r' = 1$ and Lemma 1, we have

$$\begin{aligned}
A_{1,1} &\leq C|Q|^{-1-\frac{\beta}{n}}|Q| \left(|Q|^{-1} \int_Q |b_1(x) - (b_1)_J|^{r'} dx \right)^{1/r'} \left(|Q|^{-1} \int_Q |T(f)(x)|^r dx \right)^{1/r} \\
&\leq C|J|^{-\frac{\beta}{n}} \left(|J|^{-1} \int_J |b_1(x) - (b_1)_J|^{r'} dx \right)^{1/r'} \left(|Q|^{-1} \int_Q |T(f)(x)|^r dx \right)^{1/r} \\
&\leq C\|b_1\|_{Lip_\beta} M_r(T(f))(\tilde{x}).
\end{aligned}$$

For $A_{1,2}$, by Hölder's inequality with exponent $1/s + 1/s' = 1$ and $1/t + 1/t' = 1$, such that $st = r$, and Lemma 7, 1, we have

$$\begin{aligned}
A_{1,2} &\leq C|Q|^{-1-\frac{\beta}{n}} \left(\int_{R^n} |T((b_1 - (b_1)_J)f_1)(x)|^s dx \right)^{1/s} |Q|^{1/s'} \\
&\leq C|Q|^{-1-\frac{\beta}{n}} \left(\int_{R^n} |b_1(x) - (b_1)_J|^s |f_1(x)|^s dx \right)^{1/s} |Q|^{1/s'} \\
&= C|Q|^{-1-\frac{\beta}{n}} \left(\int_J |b_1(x) - (b_1)_J|^s |f(x)|^s dx \right)^{1/s} |Q|^{1/s'} \\
&\leq C|Q|^{-1-\frac{\beta}{n}} |Q|^{1/s} \left(|J|^{-1} \int_J |b_1(x) - (b_1)_J|^{st'} dx \right)^{1/st'} \\
&\quad \times \left(|Q|^{-1} \int_Q |f(x)|^{st} dx \right)^{1/st} |Q|^{1/s'} \\
&\leq C|J|^{-\frac{\beta}{n}} \left(|J|^{-1} \int_J |b_1(x) - (b_1)_J|^{st'} dx \right)^{1/st'} \left(|Q|^{-1} \int_Q |f(x)|^{st} dx \right)^{1/st} \\
&\leq C\|b_1\|_{Lip_\beta} M_r(f)(\tilde{x}).
\end{aligned}$$

For $A_{1,3}$, choose h such that $n/2 < h < n/2 + 1/(1-a)$ and $n/2 - h + \beta < 0$. By Hölder's inequality with exponent $1/r + 1/s + 1/2 = 1$ and Lemma 2,3,1, we have

$$\begin{aligned}
&|T((b_1 - (b_1)_J)f_2)(x) - T((b_1 - (b_1)_J)f_2)(\tilde{x})| \\
&= \left| \int_{R^n} (K(x, x-y) - K(\tilde{x}, \tilde{x}-y))(b_1(y) - (b_1)_J)f_2(y) dy \right| \\
&\leq \int_{|y-x_0| > d^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)| |b_1(y) - (b_1)_J| |f(y)| dy \\
&= \sum_{N=0}^{\infty} \int_{(2^N d)^{1-a} \leq |y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)| |b_1(y) - (b_1)_J| |f(y)| dy
\end{aligned}$$

$$\begin{aligned}
&= \sum_{N=0}^{\infty} \int_{(2^N d)^{1-a} \leq |y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)| \\
&\times |b_1(y) - (b_1)_{2^{(N+1)(1-a)}J}| |f(y)| dy \\
&+ \sum_{N=0}^{\infty} \int_{(2^N d)^{1-a} \leq |y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)| \\
&\times |(b_1)_{2^{(N+1)(1-a)}J} - (b_1)_J| |f(y)| dy \\
&\leq C \sum_{N=0}^{\infty} \left(\int_{(2^N d)^{1-a} \leq |y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)|^2 dy \right)^{1/2} \\
&\times \left(\int_{|y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |b_1(y) - (b_1)_{2^{(N+1)(1-a)}J}|^s dy \right)^{1/s} \left(\int_{|y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |f(y)|^r dy \right)^{1/r} \\
&+ C \sum_{N=0}^{\infty} \left(\int_{(2^N d)^{1-a} \leq |y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)|^2 dy \right)^{1/2} \\
&\times \|\vec{b}_1\|_{Lip_\beta} (2^{N+1} d)^{\beta(1-a)} (2^{N+1} d)^{n(1-a)/s} \left(\int_{|y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |f(y)|^r dy \right)^{1/r} \\
&\leq C \sum_{N=0}^{\infty} |x - \tilde{x}|^{(1-a)(h-n/2)} (2^{N+1} d)^{-h(1-a)+n(1-a)/2} (2^{N+1} d)^{\beta(1-a)} \\
&\times (2^{N+1} d)^{-\beta(1-a)} \left((2^{N+1} d)^{-n(1-a)} \int_{|y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |b_1(y) - (b_1)_{2^{(N+1)(1-a)}J}|^s dy \right)^{1/s} \\
&\times \left((2^{N+1} d)^{-n(1-a)} \int_{|y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |f(y)|^r dy \right)^{1/r} \\
&+ C \sum_{N=0}^{\infty} |x - \tilde{x}|^{(1-a)(h-n/2)} (2^{N+1} d)^{-h(1-a)+n(1-a)/2} \\
&\times \|b_1\|_{Lip_\beta} (2^{N+1} d)^{\beta(1-a)} \left((2^{N+1} d)^{-n(1-a)} \int_{|y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |f(y)|^r dy \right)^{1/r} \\
&\leq C \sum_{N=0}^{\infty} d^{(1-a)(h-n/2)} (2^{N+1} d)^{(1-a)(n/2-h)} (2^{N+1} d)^{\beta(1-a)} \|b_1\|_{Lip_\beta} M_r(f)(\tilde{x}) \\
&+ C \sum_{N=0}^{\infty} d^{(1-a)(h-n/2)} (2^{N+1} d)^{(1-a)(n/2-h)} (2^{N+1} d)^{\beta(1-a)} \|b_1\|_{Lip_\beta} M_r(f)(\tilde{x}) \\
&\leq C \sum_{N=0}^{\infty} 2^{(N+1)(1-a)(n/2-h+\beta)} d^{\beta(1-a)} \|b_1\|_{Lip_\beta} M_r(f)(\tilde{x}) \\
&\leq C |J|_n^{\beta} \|b_1\|_{Lip_\beta} M_r(f)(\tilde{x})
\end{aligned}$$

thus

$$A_{1,3} \leq C \|b_1\|_{Lip_\beta} M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m = 1$ and $d \leq 1$.

In case $m = 1$ and $d > 1$, we proceed the case as follows.

Let $f(x) = f_1(x) + f_2(x)$ with $f_1(x) = f(x)\chi_{2J}(x)$ and $f_2(x) = f(x)\chi_{(2Q)^c}(x)$, and let $(b_1)_{2Q} = |2Q|^{-1} \int_{2Q} b_1(y)dy$. We have

$$\begin{aligned}
T_{b_1}^-(f)(x) &= \int_{R^n} K(x, x-y)((b_1(x) - (b_1)_{2Q}) - (b_1(y) - (b_1)_{2Q}))f(y)dy \\
&= (b_1(x) - (b_1)_{2Q}) \int_{R^n} K(x, x-y)f(y)dy \\
&\quad - \int_{R^n} K(x, x-y)(b_1(y) - (b_1)_{2Q})f_1(y)dy \\
&\quad - \int_{R^n} K(x, x-y)(b_1(y) - (b_1)_{2Q})f_2(y)dy \\
&= (b_1(x) - (b_1)_{2Q})T(f)(x) - T((b_1 - (b_1)_{2Q})f_1)(x) - T((b_1 - (b_1)_{2Q})f_2)(x),
\end{aligned}$$

thus

$$\begin{aligned}
&|Q|^{-1-\frac{\beta}{n}} \int_Q |T_{b_1}^-(f)(x)|dx \\
&\leq |Q|^{-1-\frac{\beta}{n}} \int_Q |b_1(x) - (b_1)_{2Q}||T(f)(x)|dx \\
&\quad + |Q|^{-1-\frac{\beta}{n}} \int_Q |T((b_1 - (b_1)_{2Q})f_1)(x)|dx \\
&\quad + |Q|^{-1-\frac{\beta}{n}} \int_Q |T((b_1 - (b_1)_{2Q})f_2)(x)|dx \\
&= A_{2,1} + A_{2,2} + A_{2,3}.
\end{aligned}$$

Similar to $A_{1,1}$, $A_{2,1} \leq C\|b_1\|_{Lip_\beta} M_r(T(f))(\tilde{x})$.

Similar to $A_{1,2}$, $A_{2,1} \leq C\|b_1\|_{Lip_\beta} M_r(f)(\tilde{x})$.

For $A_{2,3}$, by Hölder's inequality with exponent $1/r + 1/r' = 1$ and Lemma 4, 2, 1, we have

$$\begin{aligned}
|T((b_1 - (b_1)_{2Q})f_2)(x)| &= \left| \int_{R^n} K(x, x-y)(b_1(y) - (b_1)_{2Q})f_2(y)dy \right| \\
&\leq \int_{|y-\tilde{x}|>2d} |K(x, x-y)||b_1(y) - (b_1)_{2Q}||f(y)|dy \\
&\leq C \int_{|y-\tilde{x}|>2d} |x-y|^{-2n} |b_1(y) - (b_1)_{2Q}||f(y)|dy
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{N=1}^{\infty} \int_{2^N d \leq |y-\tilde{x}| \leq 2^{N+1} d} |x-y|^{-2n} |b_1(y) - (b_1)_{2^{N+1}Q}| |f(y)| dy \\
&+ C \sum_{N=1}^{\infty} \int_{2^N d \leq |y-\tilde{x}| \leq 2^{N+1} d} |x-y|^{-2n} |(b_1)_{2^{N+1}Q} - (b_1)_{2Q}| |f(y)| dy \\
&\leq C \sum_{N=1}^{\infty} (2^{N+1}d)^{-2n+n} (2^{N+1}d)^{\beta} (2^{N+1}d)^{-\beta} \left((2^{N+1}d)^{-n} \int_{|y-\tilde{x}| \leq 2^{N+1}d} |f(y)|^r dy \right)^{1/r} \\
&\times \left((2^{N+1}d)^{-n} \int_{|y-\tilde{x}| \leq 2^{N+1}d} |b_1(y) - (b_1)_{2^{N+1}Q}|^{r'} dy \right)^{1/r'} \\
&+ C \sum_{N=1}^{\infty} (2^{N+1}d)^{-2n+n} |(b_1)_{2^{N+1}Q} - (b_1)_{2Q}| \left((2^{N+1}d)^{-n} \int_{|y-\tilde{x}| \leq 2^{N+1}d} |f(y)|^r dy \right)^{1/r} \\
&\leq C \sum_{N=1}^{\infty} (2^{N+1}d)^{-n} (2^{N+1}d)^{\beta} \|b_1\|_{Lip_{\beta}} M_r(f)(\tilde{x}) \\
&+ C \sum_{N=1}^{\infty} (2^{N+1}d)^{-n} (2^{N+1}d)^{\beta} \|b_1\|_{Lip_{\beta}} M_r(f)(\tilde{x}) \\
&\leq C \sum_{N=1}^{\infty} 2^{(N+1)(-n+\beta)} d^{-n+\beta} \|b_1\|_{Lip_{\beta}} M_r(f)(\tilde{x}) \\
&\leq C \sum_{N=1}^{\infty} 2^{(N+1)(-n+\beta)} d^{\beta} \|b_1\|_{Lip_{\beta}} M_r(f)(\tilde{x}) \\
&\leq C |Q|^{\frac{\beta}{n}} \|b_1\|_{Lip_{\beta}} M_r(f)(\tilde{x}),
\end{aligned}$$

thus

$$A_{2,3} \leq C \|b_1\|_{Lip_{\beta}} M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m = 1$ and $d > 1$.

Now, we consider the case $m \geq 2$ and $d \leq 1$.

Let $f(x) = f_1(x) + f_2(x)$ with $f_1(x) = f(x)\chi_J(x)$ and $f_2(x) = f(x)\chi_{J^c}(x)$, and let $(b_j)_J = |J|^{-1} \int_J b_j(y) dy$ for $1 \leq j \leq m$, where J is a cube concentric with Q of side-length d^{1-a} .

$$\begin{aligned}
T_{\vec{b}}(f)(x) &= \int_{R^n} K(x, x-y) \prod_{j=1}^m ((b_j(x) - (b_j)_J) - (b_j(y) - (b_j)_J)) f(y) dy \\
&= \sum_{j=0}^m \sum_{\gamma \in C_j^m} (-1)^{m-j} (b(x) - b_J)_{\gamma} \int_{R^n} K(x, x-y) (b(y) - b_J)_{\gamma^c} f(y) dy
\end{aligned}$$

$$\begin{aligned}
&= \prod_{j=1}^m (b_j(x) - (b_j)_J) \int_{R^n} K(x, x-y) f(y) dy \\
&\quad + \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} (-1)^{m-j} (b(x) - b_J)_\gamma \int_{R^n} K(x, x-y) (b(y) - b_J)_{\gamma^c} f(y) dy \\
&\quad + (-1)^m \int_{R^n} K(x, x-y) \prod_{j=1}^m (b_j(y) - (b_j)_J) f_1(y) dy \\
&\quad + (-1)^m \int_{R^n} K(x, x-y) \prod_{j=1}^m (b_j(y) - (b_j)_J) f_2(y) dy \\
&= \prod_{j=1}^m (b_j(x) - (b_j)_J) T(f)(x) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} (b(x) - b_J)_\gamma T((b - b_J)_{\gamma^c} f)(x) \\
&\quad + (-1)^m T\left(\prod_{j=1}^m (b_j - (b_j)_J) f_1\right)(x) \\
&\quad + (-1)^m T\left(\prod_{j=1}^m (b_j - (b_j)_J) f_2\right)(x),
\end{aligned}$$

thus

$$\begin{aligned}
&|Q|^{-1-\frac{m\beta}{n}} \int_Q |T_b(f)(x) - T\left(\prod_{j=1}^m ((b_j)_J - b_j) f_2\right)(\tilde{x})| dx \\
&\leq |Q|^{-1-\frac{m\beta}{n}} \int_Q \prod_{j=1}^m |b_j(x) - (b_j)_J| |T(f)(x)| dx \\
&\quad + \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} |Q|^{-1-\frac{m\beta}{n}} \int_Q |(b(x) - b_J)_\gamma| |T((b - b_J)_{\gamma^c} f)(x)| dx \\
&\quad + |Q|^{-1-\frac{m\beta}{n}} \int_Q |T\left(\prod_{j=1}^m (b_j - (b_j)_J) f_1\right)(x)| dx \\
&\quad + |Q|^{-1-\frac{m\beta}{n}} \int_Q |T\left(\prod_{j=1}^m (b_j - (b_j)_J) f_2\right)(x) - T\left(\prod_{j=1}^m (b_j - (b_j)_J) f_2\right)(\tilde{x})| dx \\
&= A_{3,1} + A_{3,2} + A_{3,3} + A_{3,4}.
\end{aligned}$$

For $A_{3,1}$, by Hölder's inequality with exponent $1/r_1 + \cdots + 1/r_m + 1/r = 1$ and Lemma 1, we have

$$\begin{aligned}
 A_{3,1} &\leq C|Q|^{-\frac{m\beta}{n}} \prod_{j=1}^m \left(|J|^{-1} \int_J |b_j(x) - (b_j)_J|^{r_j} dx \right)^{1/r_j} \left(|Q|^{-1} \int_Q |T(f)(x)|^r dx \right)^{1/r} \\
 &\leq C \prod_{j=1}^m \left(|J|^{-\beta/n} (|J|^{-1} \int_J |b_j(x) - (b_j)_J|^{r_j} dx)^{1/r_j} \right) \left(|Q|^{-1} \int_Q |T(f)(x)|^r dx \right)^{1/r} \\
 &\leq C \prod_{j=1}^m \|b_j\|_{Lip_\beta} M_r(T(f))(\tilde{x}) \\
 &\leq C \|\vec{b}\|_{Lip_\beta} M_r(T(f))(\tilde{x}).
 \end{aligned}$$

For $A_{3,2}$, by Hölder's inequality with exponent $1/s + 1/s' = 1$ and $1/t + 1/t' = 1$, such that $st = r$, and Lemma 1, 6, 7, we have

$$\begin{aligned}
 A_{3,2} &\leq C \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} |Q|^{-\frac{m\beta}{n}} \left(|J|^{-1} \int_J |(b(x) - b_J)_\gamma|^{s'} dx \right)^{1/s'} \\
 &\quad \times \left(|Q|^{-1} \int_{R^n} |T((b - b_J)_{\gamma^c} f)(x)|^s \chi_Q(x) dx \right)^{1/s} \\
 &\leq C \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} |Q|^{-\frac{m\beta}{n}} \left(|J|^{-1} \int_J |(b(x) - b_J)_\gamma|^{s'} dx \right)^{1/s'} \\
 &\quad \times \left(|Q|^{-1} \int_{R^n} |(b(x) - b_J)_{\gamma^c}|^s |f(x)|^s \chi_Q(x) dx \right)^{1/s} \\
 &\leq C \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} |Q|^{-\frac{m\beta}{n}} \left(|J|^{-1} \int_J |(b(x) - b_J)_\gamma|^{s'} dx \right)^{1/s'} \\
 &\quad \times \left(|Q|^{-1} \int_Q |(b(x) - b_J)_{\gamma^c}|^s |f(x)|^s dx \right)^{1/s} \\
 &\leq C \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} |J|^{-\frac{j\beta}{n}} \left(|J|^{-1} \int_J |(b(x) - b_J)_\gamma|^{s'} dx \right)^{1/s'} \\
 &\quad \times |J|^{-\frac{(m-j)\beta}{n}} \left(|J|^{-1} \int_J |(b(x) - b_J)_{\gamma^c}|^{st'} dx \right)^{1/st'} \left(|Q|^{-1} \int_Q |f(x)|^{st} dx \right)^{1/st} \\
 &\leq C \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} \|\vec{b}_\gamma\|_{Lip_\beta} \|\vec{b}_{\gamma^c}\|_{Lip_\beta} M_r(f)(\tilde{x}) \\
 &\leq C \|\vec{b}\|_{Lip_\beta} M_r(f)(\tilde{x}).
 \end{aligned}$$

For $A_{3,3}$, by Hölder's inequality with exponent $1/s + 1/s' = 1$ and $1/t_1 + \dots + 1/t_m + 1/t = 1$, such that $st = r$, and Lemma 1, 7, we have

$$\begin{aligned}
A_{3,3} &\leq C|Q|^{-1-\frac{m\beta}{n}} \left(\int_{R^n} |T(\prod_{j=1}^m (b_j - (b_j)_J) f_1)(x)|^s dx \right)^{1/s} |Q|^{1/s'} \\
&\leq C|Q|^{-1-\frac{m\beta}{n}} \left(\int_{R^n} \prod_{j=1}^m |b_j(x) - (b_j)_J|^s |f_1(x)|^s dx \right)^{1/s} |Q|^{1/s'} \\
&\leq C|Q|^{-1-\frac{m\beta}{n}} \left(\int_J \prod_{j=1}^m |b_j(x) - (b_j)_J|^s |f(x)|^s dx \right)^{1/s} |Q|^{1/s'} \\
&\leq C|Q|^{-1-\frac{m\beta}{n}} |Q|^{1/s} \prod_{j=1}^m \left(|J|^{-1} \int_J |b_j(x) - (b_j)_J|^{st_j} dx \right)^{1/st_j} \\
&\quad \times \left(|Q|^{-1} \int_Q |f(x)|^{st} dx \right)^{1/st} |Q|^{1/s'} \\
&\leq C \prod_{j=1}^m \left(|J|^{-\frac{\beta}{n}} (|J|^{-1} \int_J |b_j(x) - (b_j)_J|^{st_j} dx)^{1/st_j} \right) \\
&\quad \times \left(|Q|^{-1} \int_Q |f(x)|^{st} dx \right)^{1/st} \\
&\leq C \prod_{j=1}^m \|b_j\|_{Lip_\beta} M_r(f)(\tilde{x}) \\
&\leq C \|\vec{b}\|_{Lip_\beta} M_r(f)(\tilde{x}).
\end{aligned}$$

For $A_{3,4}$, choose h such that $n/2 < h < n/2 + 1/(1-a)$ and $n/2 - h + m\beta < 0$. By Hölder's inequality with exponent $1/s + 1/r + 1/2 = 1$ and Lemma 1, 2, 3, we have

$$\begin{aligned}
&\left| T\left(\prod_{j=1}^m (b_j - (b_j)_J) f_2\right)(x) - T\left(\prod_{j=1}^m (b_j - (b_j)_J) f_2\right)(\tilde{x}) \right| \\
&= \left| \int_{R^n} (K(x, x-y) - K(\tilde{x}, \tilde{x}-y)) \prod_{j=1}^m (b_j(y) - (b_j)_J) f_2(y) dy \right| \\
&\leq \int_{|y-\tilde{x}| > d^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)| \prod_{j=1}^m |b_j(y) - (b_j)_J| |f(y)| dy \\
&\leq \sum_{N=0}^{\infty} \int_{(2^N d)^{1-a} \leq |y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)| \\
&\quad \times \prod_{j=1}^m |b_j(y) - (b_j)_J| |f(y)| dy
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{N=0}^{\infty} \int_{(2^N d)^{1-a} \leq |y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)| \\
&\quad \times \prod_{j=1}^m (|b_j(y) - (b_j)_{2^{(N+1)(1-a)J}}| + |(b_j)_{2^{(N+1)(1-a)J}} - (b_j)_J|) |f(y)| dy \\
&\leq \sum_{N=0}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} |(b_{2^{(N+1)(1-a)J}} - b_J)_{\gamma}| \\
&\quad \times \int_{(2^N d)^{1-a} \leq |y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)| \\
&\quad \times |(b(y) - b_{2^{(N+1)(1-a)J}})_{\gamma^c}| |f(y)| dy \\
&\leq C \sum_{N=0}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} \|\vec{b}_{\gamma}\|_{Lip_{\beta}} (2^{N+1} d)^{j(1-a)\beta} \\
&\quad \times \left(\int_{(2^N d)^{1-a} \leq |y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |K(x, x-y) - K(\tilde{x}, \tilde{x}-y)|^2 dy \right)^{1/2} \\
&\quad \times \left(\int_{|y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |(b(y) - b_{2^{(N+1)(1-a)J}})_{\gamma^c}|^s dy \right)^{1/s} \\
&\quad \times \left(\int_{|y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |f(y)|^r dy \right)^{1/r} \\
&\leq C \sum_{N=0}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} \|\vec{b}_{\gamma}\|_{Lip_{\beta}} (2^{N+1} d)^{j(1-a)\beta} \\
&\quad \times |x - \tilde{x}|^{(1-a)(h-n/2)} (2^{N+1} d)^{-h(1-a)+n(1-a)/2} (2^{N+1} d)^{(m-j)(1-a)\beta} \\
&\quad \times (2^{N+1} d)^{-(m-j)(1-a)\beta} \\
&\quad \times \left((2^{N+1} d)^{-n(1-a)} \int_{|y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |(b(y) - b_{2^{(N+1)(1-a)J}})_{\gamma^c}|^s dy \right)^{1/s} \\
&\quad \times \left((2^{N+1} d)^{-n(1-a)} \int_{|y-\tilde{x}| \leq (2^{N+1} d)^{1-a}} |f(y)|^r dy \right)^{1/r} \\
&\leq C \sum_{N=0}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} \|\vec{b}_{\gamma}\|_{Lip_{\beta}} (2^{N+1} d)^{j(1-a)\beta} d^{(1-a)(h-n/2)} (2^{N+1} d)^{(1-a)(n/2-h)} \\
&\quad \times (2^{N+1} d)^{(m-j)(1-a)\beta} \|\vec{b}_{\gamma^c}\|_{Lip_{\beta}} M_r(f)(\tilde{x}) \\
&\leq C \sum_{N=0}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} 2^{(N+1)(1-a)(n/2-h+m\beta)} d^{m(1-a)\beta} \|\vec{b}\|_{Lip_{\beta}} M_r(f)(\tilde{x}) \\
&\leq C |J|^{\frac{m\beta}{n}} \|\vec{b}\|_{Lip_{\beta}} M_r(f)(\tilde{x}),
\end{aligned}$$

thus

$$A_{3,4} \leq C \|\vec{b}\|_{Lip_{\beta}} M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m \geq 2$ and $d \leq 1$.

In case $m \geq 2$ and $d > 1$, we proceed the case as follows.

Let $f(x) = f_1(x) + f_2(x)$, with $f_1(x) = f(x)\chi_{2Q}(x)$ and $f_2(x) = f(x)\chi_{(2Q)^c}(x)$, and let $(b_j)_{2Q} = |2Q|^{-1} \int_{2Q} b_j(y) dy$ for $1 \leq j \leq m$.

$$\begin{aligned} T_{\vec{b}}(f)(x) &= \prod_{j=1}^m (b_j(x) - (b_j)_{2Q}) T(f)(x) \\ &\quad + \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} (b(x) - b_{2Q})_\gamma T((b - b_{2Q})_\gamma f)(x) \\ &\quad + (-1)^m T\left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_1\right)(x) \\ &\quad + (-1)^m T\left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2\right)(x), \end{aligned}$$

thus

$$\begin{aligned} &|Q|^{-1-\frac{m\beta}{n}} \int_Q |T_{\vec{b}}(f)(x)| dx \\ &\leq |Q|^{-1-\frac{m\beta}{n}} \int_Q \prod_{j=1}^m |b_j(x) - (b_j)_{2Q}| |T(f)(x)| dx \\ &\quad + \sum_{j=1}^{m-1} \sum_{\gamma \in C_j^m} |Q|^{-1-\frac{m\beta}{n}} \int_Q |(b(x) - b_{2Q})_\gamma| |T((b - b_{2Q})_\gamma f)(x)| dx \\ &\quad + |Q|^{-1-\frac{m\beta}{n}} \int_Q |T\left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_1\right)(x)| dx \\ &\quad + |Q|^{-1-\frac{m\beta}{n}} \int_Q |T\left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2\right)(x)| dx \\ &= A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4}. \end{aligned}$$

Similar to $A_{3,1}$, $A_{4,1} \leq C \|\vec{b}\|_{Lip_\beta} M_r(T(f))(\tilde{x})$.

Similar to $A_{3,2}$, $A_{4,2} \leq C \|\vec{b}\|_{Lip_\beta} M_r(f)(\tilde{x})$.

Similar to $A_{3,3}$, $A_{4,3} \leq C \|\vec{b}\|_{Lip_\beta} M_r(f)(\tilde{x})$.

For $A_{4,4}$, by Hölder's inequality with exponent $1/r + 1/r' = 1$ and Lemma 1, 2, 4, we have

$$\left| T\left(\prod_{j=1}^m (b_j - (b_j)_{2Q}) f_2\right)(x) \right|$$

$$\begin{aligned}
&= \left| \int_{R^n} K(x, x-y) \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) f_2(y) dy \right| \\
&\leq \int_{|y-\tilde{x}|>2d} |K(x, x-y)| \prod_{j=1}^m |b_j(y) - (b_j)_{2Q}| |f(y)| dy \\
&\leq C \int_{|y-\tilde{x}|>2d} |x-y|^{-2n} \prod_{j=1}^m |b_j(y) - (b_j)_{2Q}| |f(y)| dy \\
&\leq C \sum_{N=1}^{\infty} \int_{2^N d \leq |y-\tilde{x}| \leq 2^{N+1} d} |x-y|^{-2n} \prod_{j=1}^m |b_j(y) - (b_j)_{2Q}| |f(y)| dy \\
&\leq C \sum_{N=1}^{\infty} (2^{N+1} d)^{-2n} \\
&\quad \times \int_{2^N d \leq |y-\tilde{x}| \leq 2^{N+1} d} \prod_{j=1}^m (|b_j(y) - (b_j)_{2^{N+1}Q}| + |(b_j)_{2^{N+1}Q} - (b_j)_{2Q}|) |f(y)| dy \\
&\leq C \sum_{N=1}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} (2^{N+1} d)^{-2n} |(b_{2^{N+1}Q} - b_{2Q})_{\gamma}| \\
&\quad \times \int_{|y-\tilde{x}| \leq 2^{N+1} d} |(b(y) - b_{2^{N+1}Q})_{\gamma^c}| |f(y)| dy \\
&\leq C \sum_{N=1}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} (2^{N+1} d)^{-2n+n} \|\vec{b}_{\gamma}\|_{Lip_{\beta}} (2^{N+1} d)^{j\beta} (2^{N+1} d)^{(m-j)\beta} \\
&\quad \times (2^{N+1} d)^{-(m-j)\beta} \left((2^{N+1} d)^{-n} \int_{|y-\tilde{x}| \leq 2^{N+1} d} |(b(y) - b_{2^{N+1}Q})_{\gamma^c}|^{r'} dy \right)^{1/r'} \\
&\quad \times \left((2^{N+1} d)^{-n} \int_{|y-\tilde{x}| \leq 2^{N+1} d} |f(x)|^r dy \right)^{1/r} \\
&\leq C \sum_{N=1}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} (2^{N+1} d)^{-n} \|\vec{b}_{\gamma}\|_{Lip_{\beta}} (2^{N+1} d)^{m\beta} \|\vec{b}_{\gamma^c}\|_{Lip_{\beta}} M_r(f)(\tilde{x}) \\
&\leq C \sum_{N=1}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} 2^{(N+1)(-n+m\beta)} d^{-n+m\beta} \|\vec{b}\|_{Lip_{\beta}} M_r(f)(\tilde{x}) \\
&\leq C \sum_{N=1}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} 2^{(N+1)(-n+m\beta)} d^{m\beta} \|\vec{b}\|_{Lip_{\beta}} M_r(f)(\tilde{x}) \\
&\leq C \sum_{N=1}^{\infty} \sum_{j=0}^m \sum_{\gamma \in C_j^m} 2^{(N+1)(-n+m\beta)} |Q|^{\frac{m\beta}{n}} \|\vec{b}\|_{Lip_{\beta}} M_r(f)(\tilde{x}) \\
&\leq C |Q|^{\frac{m\beta}{n}} \|\vec{b}\|_{Lip_{\beta}} M_r(f)(\tilde{x}),
\end{aligned}$$

thus

$$A_{4,4} \leq C \|\vec{b}\|_{Lip_\beta} M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m \geq 2$ and $d > 1$.

So, for any cube Q and $c \in C_Q$ and $1 < r < \infty$ and $f \in L^p$,

$$|Q|^{-1-\frac{m\beta}{n}} \int_Q |T_{\vec{b}}(x) - c| dx \leq C \|\vec{b}\|_{Lip_\beta} (M_r(T(f))(\tilde{x}) + M_r(f)(\tilde{x})),$$

thus

$$\sup_{Q \ni \tilde{x}} \inf_{c \in C_Q} |Q|^{-1-\frac{m\beta}{n}} \int_Q |T_{\vec{b}}(x) - c| dx \leq C \|\vec{b}\|_{Lip_\beta} (M_r(T(f))(\tilde{x}) + M_r(f)(\tilde{x})).$$

By Minkowski's inequality and Lemma 7, 8, we have

$$\begin{aligned} \|T_{\vec{b}}(f)\|_{\tilde{F}_p^{m\beta,\infty}} &\approx \left\| \sup_{Q \ni \tilde{x}} \inf_{c \in C_Q} |Q|^{-1-\frac{m\beta}{n}} \int_Q |T_{\vec{b}}(f)(x) - c| dx \right\|_{L^p} \\ &\leq C \|\vec{b}\|_{Lip_\beta} (\|M_r(T(f))(\tilde{x})\|_{L^p} + C \|M_r(f)(\tilde{x})\|_{L^p}) \\ &\leq C \|\vec{b}\|_{Lip_\beta} (\|T(f)\|_{L^p} + \|f\|_{L^p}) \\ &\leq C \|\vec{b}\|_{Lip_\beta} \|f\|_{L^p}. \end{aligned}$$

This completes the proof of Theorem 1. \square

5. PROOF OF THEOREM 2

Proof. Similar to Theorem 1, we can prove that there exists some constant c and $1 < r < \infty$, such that for $f \in L^p$,

$$|Q|^{-1} \int_Q |T_{\vec{b}}(x) - c| dx \leq C \|\vec{b}\|_{Lip_\beta} (M_{r,m\beta}(T(f))(\tilde{x}) + M_{r,m\beta}(f)(\tilde{x})).$$

Further, we have

$$(T_{\vec{b}}(f))^\#(\tilde{x}) \leq C \|\vec{b}\|_{Lip_\beta} (M_{r,m\beta}(T(f))(\tilde{x}) + M_{r,m\beta}(f)(\tilde{x})).$$

By Minkowski's inequality and Lemma 7, 9, we have

$$\begin{aligned} \|T_{\vec{b}}(f)\|_{L^q} &\leq C \|M(T_{\vec{b}}(f))(\tilde{x})\|_{L^q} \\ &\leq C \|(T_{\vec{b}}(f))^\#(\tilde{x})\|_{L^q} \\ &\leq C \|\vec{b}\|_{Lip_\beta} (\|M_{r,m\beta}(T(f))(\tilde{x})\|_{L^q} + \|M_{r,m\beta}(f)(\tilde{x})\|_{L^q}) \\ &\leq C \|\vec{b}\|_{Lip_\beta} (\|T(f)\|_{L^p} + \|f\|_{L^p}) \\ &\leq C \|\vec{b}\|_{Lip_\beta} \|f\|_{L^p}. \end{aligned}$$

This completes the proof of Theorem 2. \square

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