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HYERS–ULAM STABILITY OF MEAN VALUE POINTS

PASC GĂVRUŢĂ[1], SOON-MO JUNG[2*] AND YONGJIN LI[3]

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ABSTRACT. We prove the Hyers–Ulam stability of the Lagrange's mean value points and the Hyers–Ulam–Rassias stability of a differential equation derived from the equation defining the Flett's mean value point.

1. INTRODUCTION

In 1940, S. M. Ulam [16] presented a wide ranging talk to the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. The question concerning the stability of group homomorphisms was among one of the presented topics:

> *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?*

D. H. Hyers [5] worked on and solved Ulam problem for the case of approximately additive functions under the assumption that G_1 and G_2 are Banach spaces. In fact, Hyers proved that each solution of the inequality $\|f(x + y) - f(x) - f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. In this case, it is said that the Cauchy additive functional equation, $f(x + y) = f(x) + f(y)$, satisfies the Hyers–Ulam stability or that the equation is stable in the sense of Hyers and Ulam.

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* Corresponding author.
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Th. M. Rassias [15] attempted to moderate the condition for the bound of the norm of the Cauchy difference as follows

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

and derived Hyers' theorem for the stability of the additive mapping as a special case. Thus Rassias obtained a proof of the generalized Hyers–Ulam stability for the linear mapping between Banach spaces in [15], while T. Aoki [1] proved a particular case of Rassias' theorem regarding the Hyers–Ulam stability of the additive mapping.

The stability concept introduced and presented by Rassias' theorem has influenced a number of mathematicians studying the stability problems of functional equations. Since then, the stability of several functional equations has been extensively investigated by several mathematicians (see for example [4, 6, 7, 9] and the references therein). The terminologies Hyers–Ulam stability and Hyers–Ulam–Rassias stability can also be applied to the case of other mathematical objects (see [10, 11, 12, 13]).

We will now introduce the Lagrange's mean value theorem:

Theorem 1.1. *If a function $f : \mathbb{R} \to \mathbb{R}$ is continuous on the finite closed inteval $[a, b]$ and differentiable on (a, b), then there exists a point $\eta \in (a, b)$ such that*

$$f'(\eta) = \frac{f(b) - f(a)}{b - a}.$$

The point η will be called a Lagrange's (mean value) point of f.

In 1958, T. M. Flett [3] proved a variant of Lagrange's mean value theorem: If a function $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and $f'(a) = f'(b)$, then there exists a point $\eta \in (a, b)$ satisfying

$$f'(\eta) = \frac{f(\eta) - f(a)}{\eta - a},$$

and the point η is called the Flett's (mean value) point.

Recently, M. Das, T. Riedel and P. K. Sahoo examined the stability problem for Flett's mean value points (see [2]). Subsequently, W. Lee, S. Xu and F. Ye [14] applied the idea from [2] to prove the Hyers–Ulam stability of Sahoo-Riedel's points. (For the exact definition of Sahoo-Riedel's points, we refer to [14].)

In Section 2 of this paper, employing the ideas from [2, 14], we prove the Hyers–Ulam stability of the Lagrange's mean value points. Moreover, in Section 3, we investigate the Hyers–Ulam–Rassias stability of the differential equation

$$f'(x) - \frac{f(x) - f(a)}{x - a} = 0 \tag{1.1}$$

which copies the equation for the definition of Flett's mean value points.

2. Hyers–Ulam stability of Lagrange's mean value points

First, we will introduce a theorem proved by Hyers and Ulam in 1954 that plays an important role in proving our main theorem (see [8]).

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be n-times differentiable in a neighborhood N of the point η. Suppose that $f^{(n)}(\eta) = 0$ and $f^{(n)}(x)$ changes sign at η. Then, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for every function $g : \mathbb{R} \to \mathbb{R}$ which is n-times differentiable in N and satisfies $|f(x) - g(x)| < \delta$ for any $x \in N$, there exists a point $\xi \in N$ with $g^{(n)}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$.*

Using Theorem 2.1 and the ideas from [2, 14], we will now prove our main theorem concerning the Hyers–Ulam stability of the Lagrange's mean value points.

Theorem 2.2. *Let a, b, η be real numbers satisfying $a < \eta < b$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function and η is the unique Lagrange's mean value point of f in an open interval (a, b) and moreover that $f''(\eta) \neq 0$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all $x \in [a, b]$, then there is a Lagrange's mean value point $\xi \in (a, b)$ of g with $|\xi - \eta| < \varepsilon$.*

Proof. First, we define an auxiliary function $H_f : \mathbb{R} \to \mathbb{R}$ by

$$H_f(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Obviously, H_f is also twice continuously differentiable and $H_f(a) = H_f(b)$. By the Rolle's theorem, there exists an $\eta^* \in (a, b)$ with

$$H_f'(\eta^*) = f'(\eta^*) - \frac{f(b) - f(a)}{b - a} = 0,$$

that is, η^* is a Lagrange's mean value point of f in (a, b), and the uniqueness of η in (a, b) implies that $\eta^* = \eta$.

Since $f''(\eta) \neq 0$ and $f''(x)$ is continuous at η, there exists a $\sigma > 0$ such that either $f''(x) > 0$ for all $x \in (\eta - \sigma, \eta + \sigma)$ or $f''(x) < 0$ for each $x \in (\eta - \sigma, \eta + \sigma)$, that is, either $f'(x)$ is strictly increasing on $(\eta - \sigma, \eta + \sigma)$ or $f'(x)$ is strictly decreasing on $(\eta - \sigma, \eta + \sigma)$. More explicitly, it holds true that either

$$H_f'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \begin{cases} < 0 & \text{for } x \in (\eta - \sigma, \eta) \\ = 0 & \text{for } x = \eta \\ > 0 & \text{for } x \in (\eta, \eta + \sigma) \end{cases}$$

or

$$H_f'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \begin{cases} > 0 & \text{for } x \in (\eta - \sigma, \eta) \\ = 0 & \text{for } x = \eta \\ < 0 & \text{for } x \in (\eta, \eta + \sigma), \end{cases}$$

that is, H_f' changes sign at η.

Now, let us define a differentiable function $H_g : \mathbb{R} \to \mathbb{R}$ by

$$H_g(x) = g(x) - \frac{g(b) - g(a)}{b - a}(x - a),$$

and assume that $|f(x) - g(x)| < \delta$ for any $x \in [a, b]$ and for some $\delta > 0$. Then, such function yields

$$\begin{aligned} & |H_f(x) - H_g(x)| \\ \leq \; & |f(x) - g(x)| + \frac{x-a}{b-a}|f(a) - g(a)| + \frac{x-a}{b-a}|f(b) - g(b)| \qquad (2.1) \\ \leq \; & |f(x) - g(x)| + |f(a) - g(a)| + |f(b) - g(b)| \\ < \; & 3\delta \end{aligned}$$

for any $x \in (a, b)$.

Assume that $\varepsilon > 0$ is given. According to Theorem 2.1 and (2.1), there exists a $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all $x \in [a, b]$, then there is a point $\xi \in (a, b)$ satisfying $|\xi - \eta| < \varepsilon$ and

$$H_g'(\xi) = g'(\xi) - \frac{g(b) - g(a)}{b - a} = 0,$$

from which it follows that ξ is a Lagrange's mean value point of g. □

Another type of Hyers–Ulam stability problem for the Lagrange's mean value points is presented in the following theorem.

Theorem 2.3. *Let a, b, ξ be real numbers satisfying $a < \xi < b$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function satisfying either $f''(x) > 0$ for all $x \in [a, b]$ or $f''(x) < 0$ for all $x \in [a, b]$. If*

$$\left| f'(\xi) - \frac{f(b) - f(a)}{b - a} \right| \leq \varepsilon \qquad (2.2)$$

for some $\varepsilon > 0$, then there exists a Lagrange's mean value point η of f on (a, b) satisfying

$$|\eta - \xi| \leq \frac{\varepsilon}{\min_{x\in[a,b]} |f''(x)|}.$$

Proof. Due to Lagrange's mean value theorem, there exists a Lagrange's mean value point $\eta \in (a, b)$ with

$$f'(\eta) = \frac{f(b) - f(a)}{b - a}.$$

Hence it follows from (2.2) that

$$|f'(\xi) - f'(\eta)| \leq \varepsilon.$$

If $\xi = \eta$ then our assertion is true. Otherwise, without loss of generality, we assume that $a < \eta < \xi < b$. Since f is twice differentiable, by Lagrange's mean value theorem again, there exists a point $\xi_0 \in (\eta, \xi)$ such that

$$|\eta - \xi||f''(\xi_0)| = |f'(\eta) - f'(\xi)|.$$

Since f'' is continuous, we further have

$$|\eta - \xi| = \frac{|f'(\eta) - f'(\xi)|}{|f''(\xi_0)|} \leq \frac{\varepsilon}{\min_{x\in[a,b]} |f''(x)|},$$

which ends the proof. □

3. Hyers–Ulam–Rassias stability of (1.1)

We will now investigate the Hyers–Ulam–Rassias stability of the differential equation (1.1) which copies the equation defining the Flett's mean value point.

Theorem 3.1. *Given $a, b \in \mathbb{R}$ with $a < b$, let $f : [a, b] \to \mathbb{C}$ be a function, which is continuous on $[a, b]$ and continuously differentiable on (a, b). Assume that $\varphi : [a, b] \to [0, \infty)$ is a function satisfying*

$$\int_a^x \frac{\varphi(\tau)}{\tau - a} d\tau < \infty \tag{3.1}$$

for any $x \in (a, b)$. If the function f satisfies

$$\left| f'(x) - \frac{f(x) - f(a)}{x - a} \right| \leq \varphi(x)$$

for all $x \in (a, b)$, then there exists a unique function $y : [a, b] \to \mathbb{C}$, which is continuously differentiable on (a, b), such that

$$y'(x) = \frac{y(x) - y(a)}{x - a}$$

and

$$|f(x) - y(x)| \leq (x - a) \int_a^x \frac{\varphi(\tau)}{\tau - a} d\tau$$

for all $x \in (a, b)$.

Proof. It is obvious that the function $\frac{-1}{x-a}$ is integrable on (c, b) for $a < c < b$. Moreover, we have

$$\int_c^x \exp\left\{ -\int_b^\tau \frac{du}{u - a} \right\} \frac{f(a)}{\tau - a} d\tau = (b - a) \left\{ \frac{f(a)}{c - a} - \frac{f(a)}{x - a} \right\} < \infty$$

for any $c, x \in (a, b)$ with $c < x$. Taking these observations and (3.1) into consideration, [12, Corollary 2] implies that there exists a unique complex number z such that

$$\begin{aligned} &\left| f(x) - \exp\left\{ \int_b^x \frac{du}{u - a} \right\} \left(z - \int_b^x \exp\left\{ -\int_b^\tau \frac{du}{u - a} \right\} \frac{f(a)}{\tau - a} d\tau \right) \right| \\ &\leq \exp\left\{ \int_b^x \frac{du}{u - a} \right\} \int_a^x \varphi(\tau) \exp\left\{ -\int_b^\tau \frac{du}{u - a} \right\} d\tau \end{aligned}$$

for any $x \in (a, b)$, that is, there is a unique function $y : [a, b] \to \mathbb{C}$ such that

$$|f(x) - y(x)| \leq (x - a) \int_a^x \frac{\varphi(\tau)}{\tau - a} d\tau$$

for all $x \in (a, b)$, where we set $y(x) = \frac{z - f(a)}{b - a} x + \frac{bf(a) - za}{b - a}$, and we know that y is continuously differentiable on (a, b) and $y(a) = f(a)$.

Moreover, we get

$$\begin{aligned} y'(x) &= \frac{z-f(a)}{b-a} \\ &= \frac{1}{x-a}\left(\frac{z-y(a)}{b-a}x - \frac{z-y(a)}{b-a}a\right) \\ &= \frac{1}{x-a}\left(\frac{z-y(a)}{b-a}x + \frac{by(a)-za}{b-a} - y(a)\right) \\ &= \frac{y(x)-y(a)}{x-a} \end{aligned}$$

for all $x \in (a,b)$. □

If we set $\varphi(x) = \varepsilon(x-a)^p$ for some $\varepsilon \geq 0$ and $p > 0$, then we obtain the following

Corollary 3.2. *Given $a, b \in \mathbb{R}$ with $a < b$, let $f : [a,b] \to \mathbb{C}$ be a function, which is continuous on $[a,b]$ and continuously differentiable on (a,b). If the function f satisfies*

$$\left| f'(x) - \frac{f(x)-f(a)}{x-a} \right| \leq \varepsilon(x-a)^p$$

for all $x \in (a,b)$ and for some $\varepsilon \geq 0$ and $p > 0$, then there exists a unique function $y : [a,b] \to \mathbb{C}$, which is continuously differentiable on (a,b), such that

$$y'(x) = \frac{y(x)-y(a)}{x-a}$$

and

$$|f(x) - y(x)| \leq \frac{\varepsilon}{p}(x-a)^{p+1}$$

for all $x \in (a,b)$.

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[1] Department of Mathematics, University "Politehnica" of Timisoara, Piata Victoriei, No. 2, 300006 Timisoara, Romania.
E-mail address: pgavruta@yahoo.com

[2] Mathematics Section, College of Science and Technology, Hongik University, 339-701 Jochiwon, Republic of Korea.
E-mail address: smjung@hongik.ac.kr

[3] Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, China.
E-mail address: stslyj@mail.sysu.edu.cn