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THE FUGLEDE–PUTNAM THEOREM AND PUTNAM'S INEQUALITY FOR QUASI-CLASS (A, k) OPERATORS

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ABSTRACT. An operator $T \in B(\mathcal{H})$ is called quasi-class (A, k) if $T^{*k}(|T^2| - |T|^2)T^k \geq 0$ for a positive integer k, which is a common generalization of class A. The famous Fuglede–Putnam's theorem is as follows: the operator equation $AX = XB$ implies $A^*X = XB^*$ when A and B are normal operators. In this paper, firstly we show that if X is a Hilbert-Schmidt operator, A is a quasi-class (A, k) operator and B^* is an invertible class A operator such that $AX = XB$, then $A^*X = XB^*$. Secondly we consider the Putnam's inequality for quasi-class (A, k) operators and we also show that quasisimilar quasi-class (A, k) operators have equal spectrum and essential spectrum.

1. Introduction

Throughout this paper let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $B(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. The spectrum, essential spectrum and numerical range of an operator $T \in B(\mathcal{H})$ are denoted by $\sigma(T)$, $\sigma_e(T)$ and $W(T)$.

Here an operator $T \in B(\mathcal{H})$ is called p-hyponormal for $0 < p \leq 1$ if $(T^*T)^p - (TT^*)^p \geq 0$, and log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. And an operator T is called paranormal if $\| Tx \|^2 \leq \| T^2x \| \| x \|$ for all $x \in \mathcal{H}$. By the celebrated Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$", every p-hyponormal operator is q-hyponormal for $p \geq q \geq 0$. And every invertible p-hyponormal operator is log-hyponormal since logt is an operator monotone

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function. We remark that $(A^p - I)/p \to$logA as $p \to +0$ for positive invertible operator $A > 0$, so that p-hyponormality of T approaches log-hyponormality of T as $p \to +0$. In this sense, log-hyponormal can be considered as $0-$hyponormal. p-hyponormal, log-hyponormal and paranormal operators were introduced by A. Aluthge [1], K. Tanahashi [28] and T. Furuta [8, 9] respectively.

In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators, T. Furuta, M. Ito and T. Yamazaki [11] introduced a very interesting class of bounded linear Hilbert space operators: class A defined by $|T^2| - |T|^2 \geq 0$, where $|T| = (T^*T)^{\frac{1}{2}}$ which is called the absolute value of T and they showed that class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators. Class A operators have been studied by many researchers, for example [5, 6, 14, 29, 30, 31, 33].

I. H. Jeon and I. H. Kim [15] introduced quasi-class A (i.e., $T^*(|T^2| - |T|^2)T \geq 0$) operators as an extension of the notion of class A operators.

Recently K. Tanahashi et al. [29] considered an extension of quasi-class A operators, similar in sprit to the extension of the notion of p-quasihyponormality to (p, k) -quasihyponormality.

Definition 1.1. $T \in B(\mathcal{H})$ is called a quasi-class (A, k) operator for a positive integer k if

$$T^{*k}(|T^2| - |T|^2)T^k \geq 0.$$

Remark: In [12], this class of operators is called k-quasi-class A. It is clear

$$\begin{aligned} p\text{-hyponormal operators } &\subseteq \text{class A operators} \\ &\subseteq \text{quasi-class A operators} \\ &\subseteq \text{quasi-class}(A, k) \text{ operators.} \end{aligned}$$

and

$$\text{quasi-class } (A, k) \text{ operators} \subseteq \text{quasi-class } (A, k+1) \text{ operators.} \tag{1.1}$$

In [12] we show that the inclusion relation (1.1) is strict by an example.

The famous Fuglede–Putnam's theorem is as follows [7, 9, 23]:

Theorem 1.2. *Let A and B be normal operator and X be an operator such that $AX = XB$, then $A^*X = XB^*$.*

The Fuglede–Putnam's theorem is very useful in operator theory thanks to its numerous applications. In fact, the Fuglede–Putnam's theorem was first proved in case $A = B$ by B. Fuglede [7] and then a proof in the general case by C. R. Putnam [23]. A lot of researchers have worked on it since the papers of Fuglede and Putuam. S. Berberian [2] proved that the Fuglede theorem was actually equivalent to that of Putnam by a nice operator matrix derivation trick. M. Rosenblum [26] gave an elegant and simple proof of Fuglede–Putnam's theorem by using

Liouville's theorem. There were various generalizations of Fuglede–Putnam's theorem to nonnormal operators, we only cite [10, 3, 24, 32]. For example, M. Radjabalipour [24] showed that Fuglede–Putnam's theorem holds for hyponormal operators; A. Uchiyama and K. Tanahashi [32] showed that Fuglede–Putnam's theorem holds for p-hyponormal and log-hyponormal operators. But further extension for class A operators remains as an open problem. If let $X \in B(\mathcal{H})$ be Hilbert-Schmidt class, S. Mecheri and A. Uchiyama [20] showed that normality in Fuglede–Putnam's theorem can be replaced by A and B^* class A operators. Recently M. H. M. Rashid and M. S. M. Noorani [25] showed that the above result of S. Mecheri and A. Uchiyama holds for A and B^* quasi-class A operators with the additional condition $\| \, |A^*| \, \| \| \, |B|^{-1} \, \| \leq 1$. In this paper, firstly we show that if X is a Hilbert-Schmidt operator, A is a quasi-class (A, k) operator and B^* is an invertible quasi-class (A, k) operator such that $AX = XB$, then $A^*X = XB^*$. Secondly we consider the Putnam's inequality for quasi-class (A, k) operators and we also show that quasisimilar quasi-class (A, k) operators have equal spectrum and essential spectrum.

2. Main results

Let $\mathcal{C}_2(\mathcal{H})$ denote the Hilbert-Schmidt class. For each pair of operator A, $B \in B(\mathcal{H})$, there is an operator $\Gamma_{A,B}$ defined on $\mathcal{C}_2(\mathcal{H})$ via the formula $\Gamma_{A,B}(X) = AXB$ in [3]. Obviously $\| \Gamma \| \leq \| A \| \| B \|$. The adjoint of Γ is given by the formula $\Gamma^*_{A,B}(X) = A^*XB^*$, see details [3].

Let $T \otimes S$ denote the tensor product on the product space $\mathcal{H} \otimes \mathcal{H}$ for non-zero T, $S \in B(\mathcal{H})$. In [12] we give a necessary and sufficient condition for $T \otimes S$ to be a quasi-class (A, k) operator.

Lemma 2.1. [12] *Let T, $S \in B(\mathcal{H})$ be non-zero operators. Then $T \otimes S$ is a quasi-class (A, k) operator if and only if one of the following assertions holds:*

(1) $T^{k+1} = 0$ *or* $S^{k+1} = 0$*;*
(2) T *and* S *are quasi-class* (A, k) *operators.*

Theorem 2.2. *Let A and $B \in B(\mathcal{H})$. Then $\Gamma_{A,B}$ is a quasi-class (A, k) operator on $\mathcal{C}_2(\mathcal{H})$ if and only if one of the following assertions holds:*

(1) $A^{k+1} = 0$ *or* $B^{k+1} = 0$*;*
(2) A *and* B^* *are quasi-class* (A, k) *operators.*

Proof. The unitary operator $U : \mathcal{C}_2(\mathcal{H}) \to \mathcal{H} \otimes \mathcal{H}$ by a map $x \otimes y^* \to x \otimes y$ induces the $*$-isomorphism $\Psi : B(\mathcal{C}_2(\mathcal{H})) \to B(\mathcal{H} \otimes \mathcal{H})$ by a map $X \to UXU^*$. Then we can obtain $\Psi(\Gamma_{A,B}) = A \otimes B^*$, see [4]. This completes the proof by Lemma 2.1. ∎

Lemma 2.3. [29, 12] *Let $T \in B(\mathcal{H})$ be a quasi-class (A, k) operator for a positive integer k. If $\lambda \neq 0$ and $(T - \lambda)x = 0$ for some $x \in \mathcal{H}$, then $(T - \lambda)^*x = 0$.*

Now we are ready to extend Fuglede–Putnam's theorem to quasi-class (A, k) operators.

Theorem 2.4. *Let A be a quasi-class (A, k) operator and B^* be an invertible class A operator. If $AX = XB$ for $X \in \mathcal{C}_2(\mathcal{H})$, then $A^*X = XB^*$.*

Proof. Let Γ be defined on $\mathcal{C}_2(\mathcal{H})$ by $\Gamma Y = AYB^{-1}$. Since B^* is an invertible class A operator, we have that $(B^*)^{-1}$ is also a class A operator by [14, Corollary 4]. Since A is a quasi-class (A, k) operator and $(B^{-1})^* = (B^*)^{-1}$ is a quasi-class (A, k) operator, we have that Γ is a quasi-class (A, k) operator on $\mathcal{C}_2(\mathcal{H})$ by Theorem 2.2. Moreover we have $\Gamma X = AXB^{-1} = X$ because of $AX = XB$. Hence X is an eigenvector of Γ. By Lemma 2.3 we have $\Gamma^* X = A^*X(B^{-1})^* = X$, that is, $A^*X = XB^*$. The proof is complete. □

Here we give out an example that if $X \in B(\mathcal{H})$, A is a quasi-class (A, k) operator and B^* is a quasi-class (A, k) or normal operator satisfying $AX = XB$, we can not get $A^*X = XB^*$. To see this, just consider the operator $A = X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = 0$, we have $AX = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = XB$, but $A^*X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $XB^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

In general, by the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$ we can not get that T is normal. For instance, [34], if $T = SB$, where S is positive and invertible, B is self-adjoint, and S and B do not commute, then $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, but T is not normal.

I. H. Sheth [27] showed that if T is a hyponormal operator and $S^{-1}TS = T^*$ satisfying $0 \notin \overline{W(S)}$, then T is self-adjoint. I. H. Kim [18] extended the result of Sheth to the class of p-hyponormal operators. In the following, we shall show that if T or T^* is a class A operator, the result of Sheth also holds. Actually if T is a class A operator, the result has already been shown in [16] by I. H. Jeon et al. But we will give a proof for the sake of completeness.

Theorem 2.5. *Let T or T^* be a class A operator and S be an operator satisfying $0 \notin \overline{W(S)}$ such that $ST = T^*S$. Then T is self-adjoint.*

To prove Theorem 2.5 the following lemmas are needful.

Lemma 2.6. [34] *Let $T \in B(\mathcal{H})$ be a operator such that $S^{-1}TS = T^*$, where S is an operator satisfying $0 \notin \overline{W(S)}$. Then $\sigma(T) \subseteq \mathcal{R}$.*

Lemma 2.7. [21] *Let $T \in B(\mathcal{H})$ be a class A operator, then the following inequality holds:*

$$\| \, |T^2| - |T|^2 \, \| \leq \| \, |\widetilde{T}_{1,1}| - |\widetilde{T}^*_{1,1}| \, \| \leq \frac{1}{\pi} meas \ \ \sigma(T),$$

where $T = U|T|$ is the polar decomposition of T, $\widetilde{T}_{1,1} = |T|U|T|$ and $meas\sigma(T)$ is the planar Lebesgue measure of the spectrum of T. Moreover, if $meas\sigma(T) = 0$, then T is normal.

Proof of Theorem 2.5. Assume that T or T^* is a class A operator. Since $0 \notin \overline{W(S)}$ and $\sigma(S) \subseteq \overline{W(S)}$, we have that S is invertible and $0 \notin \overline{W(S^{-1})}$.

Hence $(S^{-1})^{-1}TS^{-1} = T^*$ holds by $ST = T^*S$. Hence we have $\sigma(T) \subseteq \mathcal{R}$ by implying Lemma 2.6. Thus $\sigma(T^*) = \overline{\sigma(T)} \subseteq \mathcal{R}$. So we have that meas$\sigma(T) =$ meas$\sigma(T^*) = 0$ for the planar Lebesgue measure. So we have that T or T^* is normal by Lemma 2.7. Hence T is self-adjoint since $\sigma(T) = \sigma(T^*) \subseteq \mathcal{R}$.

It is well known that p-hyponormal operator with real spectrum is self-adjoint. More generally, from the proof of Theorem 2.5 we have that

Corollary 2.8. *Let T be a class A operator, and $\sigma(T) \subseteq \mathcal{R}$, then T is self-adjoint.*

We recall the following lemma which summarizes some basic properties of quasi-class (A, k) operators.

Lemma 2.9. [29, 12] *Let $T \in B(\mathcal{H})$ be a quasi-class (A, k) operator for a positive integer k and $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{ran(T^k)} \bigoplus \ker T^{*k}$ be 2×2 matrix expression. Assume that $ranT^k$ is not dense, then T_1 is a class A operator on $\overline{ran(T^k)}$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

The following theorem is about Putnam's inequality for quasi-class (A, k) operators.

Theorem 2.10. *Let T be a quasi-class (A, k) operator for a positive integer k. Then*

$$\| P(|T^2| - |T|^2)P \| \leq \frac{1}{\pi} meas \;\; \sigma(T),$$

where P is the orthogonal projection of $\mathcal{H}$ onto $\overline{ran(T^k)}$ and meas$\sigma(T)$ is the planar Lebesgue measure of the spectrum of T.

Proof. Consider the matrix representation of T with respect to the decomposition $\mathcal{H} = \overline{\mathrm{ran}(T^k)} \bigoplus \ker T^{*k}$: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$. Let P be the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathrm{ran}(T^k)}$. Then $T_1 = TP = PTP$. Since T is a quasi-class (A, k) operator, we have

$$P(|T^2| - |T|^2)P \geq 0.$$

Then

$$|T_1^2| = (PT^*PT^*TPTP)^{\frac{1}{2}} = (PT^*T^*TTP)^{\frac{1}{2}} = (P|T^2|^2P)^{\frac{1}{2}} \geq P|T^2|P$$

by Hansen's inequality [13]. On the other hand

$$|T_1|^2 = T_1^*T_1 = PT^*TP = P|T|^2P \leq P|T^2|P.$$

So we have

$$|T_1|^2 = P|T|^2P \leq P|T^2|P \leq |T_1^2|.$$

Hence

$$0 \leq P(|T^2| - |T|^2)P \leq |T_1^2| - |T_1|^2.$$

Since T_1 is a class A operator by Lemma 2.9, we have

$$\| P(|T^2| - |T|^2)P \| \leq \| |T_1^2| - |T_1|^2 \| \leq \frac{1}{\pi}\mathrm{meas} \;\; \sigma(T_1) = \frac{1}{\pi}\mathrm{meas} \;\; \sigma(T),$$

by Lemmas 2.7 and 2.9. This completes the proof. □

Theorem 2.11. *Let T be a injective quasi-class (A,k) operator for a positive integer k and S be a operator satisfying $0 \notin \overline{W(S)}$ such that $ST = T^*S$. Then T is direct sum of a self-adjoint and nilpotent operator.*

Proof. Since T is a quasi-class (A,k) operator, we have the following matrix expression by Lemma 2.9: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\operatorname{ran}(T^k)} \bigoplus \ker T^{*k}$, where T_1 is a class A operator on $\overline{\operatorname{ran}(T^k)}$ and $T_3^k = 0$. Since $ST = T^*S$ and $0 \notin \overline{W(S)}$, we have that $\sigma(T) \subseteq \mathcal{R}$ by Lemma 2.6. Hence $\sigma(T_1) \subseteq \mathcal{R}$ because $\sigma(T) = \sigma(T_1) \cup \{0\}$. So we have that T_1 is self-adjoint by Corollary 2.8 since T_1 is a class A operator on $\overline{\operatorname{ran}(T^k)}$. Let P be the orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{ran}(T^k)}$. By Hansen's inequality [13] we have

$$\begin{pmatrix} |T_1^2| & 0 \\ 0 & 0 \end{pmatrix} = (P|T^2|^2P)^{\frac{1}{2}} \geq P|T^2|P \geq P|T|^2P = PT^*TP = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since T_1 is self-adjoint, hence we can write

$$|T^2| = \begin{pmatrix} T_1^2 & A \\ A^* & B \end{pmatrix}.$$

So we have

$$\begin{aligned} \begin{pmatrix} T_1^4 & 0 \\ 0 & 0 \end{pmatrix} &= P|T^2||T^2|P \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1^2 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} T_1^2 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} T_1^4 + AA^* & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

This implies $A = 0$ and $|T^2|^2 = \begin{pmatrix} T_1^4 & 0 \\ 0 & B^2 \end{pmatrix}$. On the other hand,

$$\begin{aligned} |T^2|^2 &= T^*T^*TT \\ &= \begin{pmatrix} T_1 & 0 \\ T_2^* & T_3^* \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ T_2^* & T_3^* \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \\ &= \begin{pmatrix} T_1^4 & T_1^2(T_1T_2 + T_2T_3) \\ (T_1T_2 + T_2T_3)^*T_1^2 & |T_1T_2 + T_2T_3|^2 + |T_3^2|^2 \end{pmatrix}. \end{aligned}$$

Since T is injective and $\ker T_1 \subseteq \ker T$, we have that T_1 is injective. Hence $T_1T_2 + T_2T_3 = 0$ and $B = |T_3^2|$. Since T is a quasi-class (A,k) operator, by simple calculation we have

$$\begin{aligned} 0 &\leq T^{*k}(|T^2| - |T|^2)T^k \\ &= \begin{pmatrix} 0 & (-1)^{k+1}T_1^{2k+1}T_2 \\ (-1)^{k+1}T_2^*T_1^{2k+1} & (-1)^{k+1}T_2^*T_1^{2k}T_2 + T_3^{*k}|T_3^2|T_3^k - |T_3^{k+1}|^2 \end{pmatrix}. \end{aligned}$$

Recall that $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X, Z \geq 0$ and $Y = X^{\frac{1}{2}} W Z^{\frac{1}{2}}$ for some contraction W. Thus we have $T_2 = 0$. This completes the proof. □

Recall that an operator $X \in B(\mathcal{H})$ is called a quasiaffinity if X is injective and has dense range. For T, $S \in B(\mathcal{H})$, if there exist quasiaffinities X and $Y \in B(\mathcal{H})$ such that $TX = XS$ and $YT = SY$, then we say that T and S are quasisimilar. It is well-known that in finite dimensional spaces quasiaffinity coincides with similarity; but in infinite dimensional spaces quasiaffinity is a much weaker relation than similarity. Similarity preserves the spectrum and essential spectrum, but this is not true for quasiaffinity. Many researchers have studied what conditions can insure two quasisimilar operators have equal spectrum and essential spectrum. For instance, R. Yingbin and Y. Zikun [35] proved that quasisimilar p-hyponormal operators have equal spectrum and essential spectrum; I. H. Jeon et al. [17] proved that quasisimilar injective p-quasihyponormal operators have equal spectrum and essential spectrum; A. H. Kim [19] proved that quasisimilar (p, k)-quasihyponormal operators have equal spectrum and essential spectrum respectively. Recently, I. H. Jeon et al. [16] proved that quasisimilar quasi-class A operators have equal spectrum and essential spectrum. In the following, we point out that quasisimilar quasi-class (A, k) operators also have equal spectrum and essential spectrum.

An operator T has the (Bishop's) property (β) at $\lambda \in \mathbb{C}$ if for every open neighborhood $\mathcal{D} \subset \mathbb{C}$ of λ and every vector-valued analytic functions $f_n\colon \mathcal{D} \to \mathcal{H}$ (n=1, 2,$\cdots$) for which $(T-\mu)f_n(\mu) \to 0$ uniformly on every compact subset of $\mathcal{D}$, $f_n(\mu) \to 0$ uniformly in norm on every compact subset of $\mathcal{D}$. When T has property (β) for every $\lambda \in \mathbb{C}$, we say that T has property (β). The property (β) plays an important role in the study of spectral properties of operators.

Lemma 2.12. [29] *Let T be a quasi-class (A, k) operator. Then T has Bishop's property (β).*

Lemma 2.13. [22] *If both T and S have Bishop's property (β) and if they are quasisimilar, then $\sigma(T) = \sigma(S)$ and $\sigma_e(T) = \sigma_e(S)$ hold.*

Remark 2.14. By Lemmas 2.12 and 2.13, we have that quasisimilar quasi-class (A, k) operators also have equal spectrum and essential spectrum.

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