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TENSOR NORMS RELATED TO THE SPACE OF COHEN p-NUCLEAR MULTILINEAR MAPPINGS

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ABSTRACT. In this paper we consider the ideal of Cohen p-nuclear multilinear mappings, which is a natural multilinear extension of the ideal of p-nuclear linear operators. The space of Cohen p-nuclear m-linear mappings is characterized by means of a suitable tensor norm up to an isometric isomorphism.

1. INTRODUCTION AND NOTATION

The theory of operator ideals, as it was introduced by A. Pietsch in the linear case, is well established, as the reader can see in the excellent monographs [6, 13]. In [12], Pietsch sketched an m-linear approach to the theory of absolutely summing operators and since then a large number of papers has followed this line, e.g., [1, 2, 9, 10]. The multi-ideals $\mathcal{N}_p^m$ of Cohen p-nuclear multilinear mappings between Banach spaces were defined by Achour and Alouani in [1] as a natural multilinear extension of the classical ideal of p-nuclear linear operators [5]. These multi-ideals has many good properties, e.g. the Banach multi-ideal, Pietsch Domination Theorem, Kwapie's factorization, etc. Also, we refer to [1] for the relation between Cohen p-nuclear multilinear mappings and other classes of p-summing multilinear mappings, such as p-semi-integral, dominated, multiple (or, fully), strongly and absolutely summing mappings. For more details concerning the nonlinear theory of absolutely summing operators we refer to [1]-[4], [7]-[11] and references therein. The aim of this paper is to obtain characterizations of $\mathcal{N}_p^m(X_1, \cdots, X_m; Y)$ of Cohen p-nuclear multilinear mappings from $X_1 \times \cdots \times X_m$ into Y.

Our paper is organized as follows:

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In the first section, we recall important results and definitions to be used later. In section 2, by using the full general Pietsch Domination Theorem recently presented by Pellegrino et al in [11], we prove the Pietsch-type theorem for Cohen p-nuclear multilinear mappings. As a conesequence of our Theorem, we show that the space of Cohen p-nuclear $(1 \leq p < \infty)$ m-linear forms is coincide withe the space of p-semi-integral multilinear forms.

Finally, in section 3, we introduce a reasonable crossnorm $\widetilde{\omega}_{(p;p^*)}$ such that the space $\mathcal{N}_p^m(X_1, \cdots, X_m; Y^*)$ of Cohen p-nuclear m-linear mappings is isometric to the dual of $X_1 \otimes \cdots \otimes X_m \otimes Y$ endowed $\widetilde{\omega}_{(p;p^*)}$.

Now, we fix the notation used in this paper. Let $\mathbb{N}$ denote the set of natural numbers, $X, X_1, \cdots, X_m$ and Y denote Banach spaces over $\mathbb{K}$ (real or complex scalars field), we denote by $\mathcal{L}(X_1, \cdots, X_m; Y)$ the Banach space of all continuous m-linear mappings from $X_1 \times \cdots \times X_m$ to Y, under the norm $\|T\| = \sup_{x_k \in B_{X_k}} \|T(x_1, \cdots, x_m)\|$, where B_{X_k} denotes the closed unit ball of X_k $(1 \leq k \leq m)$. If $Y = \mathbb{K}$, we write $\mathcal{L}(X_1, \cdots, X_m)$. In the case $X_1 = \cdots = X_n = X$, we will simply write $\mathcal{L}(^m X; Y)$. The symbol $X_1 \otimes \cdots \otimes X_m$ denotes the algebraic tensor product of the Banach spaces $X_1, \cdots, X_m$. By $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m$ we denote the completed projective tensor product of $X_1, \cdots, X_m$. Recall that every m-linear operator $T \in \mathcal{L}(X_1, \cdots, X_m; Y)$ has an associated linear operator $T_L \in \mathcal{L}(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m; Y)$ given by

$$T_L(x^1 \otimes \cdots \otimes x^m) = T\left(x^1, \cdots, x^m\right)$$

for all $x^j \in X_j$ $(1 \leq j \leq m)$.
For $1 \leq p < \infty$, let $l_p(X)$ be the Banach space of all absolutely p-summable sequences $(x_i)_{i=1}^\infty$ in X with the norm $\|(x_i)_{i=1}^\infty\|_p = (\sum_{i=1}^{\infty} \|x_i\|^p)^{\frac{1}{p}}$. We denote by $l_p^\omega(X)$ the Banach space of all weakly p-summable sequences $(x_i)_{i=1}^\infty$ in X with the norm $\|(x_i)_{i=1}^\infty\|_{p,\omega} = \sup_{\varphi \in B_{X^*}} \|(\varphi(x_i))_{i=1}^\infty\|_p$, where X^* denotes the topological dual of X. When the sequences are finite (with n terms) we write l_p^n and $l_p^{n,\omega}$ instead of l_p and l_p^ω, respectively. When $p = \infty$ we let $\|(x_i)_{i=1}^\infty\|_{\infty,\omega} := \sup \|x_i\|$, i.e., $l_\infty^\omega(X) = l_\infty(X)$. We know (see [6]) that $l_p(X) = l_p^\omega(X)$ for some $1 \leq p < \infty$ if, and only if, $dim(X)$ is finite.
Let $1 \leq p < \infty$ and $(y_i^*)_{1 \leq i \leq n} \in l_{p,\omega}^n(Y^*)$. We know (see [10, Lemma 2.1]) that

$$\left\|(y_i^*)_{1 \leq i \leq n}\right\|_{p,\omega} = \sup_{\beta \in B_{Y^{**}}} (\sum_{i=1}^{n} |\beta(y_i^*)|^p)^{\frac{1}{p}} = \sup_{y \in B_Y} \left\|(y_i^*(y))_{1 \leq i \leq n}\right\|_p. \tag{1.1}$$

From now on, if $1 \leq p < \infty$, the symbol p^* represents the conjugate of p. It will be convenient to adopt that $\frac{p}{\infty} = 0$ for any $p > 0$.

The concept of p-nuclear (or Cohen p-nuclear) linear operators was introduced by Cohen [5].
Let $1 \leq p \leq \infty$. A continuous linear operator $u : X \to Y$ is Cohen p-nuclear (notation: $u \in \mathcal{N}_p(X; Y)$) if there is a positive constant C such that for all $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$, we have

$$\left\|\left(\left\langle u\left(x_i\right), y_i^*\right\rangle\right)_{1\leq i\leq n}\right\|_1 \leq C\left\|\left(x_i\right)_{1\leq i\leq n}\right\|_{p,\omega} \sup_{y\in B_Y}\left\|\left(y_i^*(y)\right)\right\|_{p^*} \tag{1.2}$$

(or by (1.1): $\left\|\left(\left\langle u\left(x_i\right), y_i^*\right\rangle\right)_{1\leq i\leq n}\right\|_1 \leq C\left\|\left(x_i\right)_{1\leq i\leq n}\right\|_{p,\omega}\left\|\left(y_i^*\right)_{1\leq i\leq n}\right\|_{p^*,\omega}$).
The infimum of the C defines a norm n_p on $\mathcal{N}_p\left(X;Y\right)$.

We say that $T\in\mathcal{L}\left(X_1,\cdots,X_m;Y\right)$ is p-semi-integral if there exists a constant $C\geq 0$ and a regular probability measure μ on the Borel $\sigma-$algebra $B(B_{X_1^*}\times\cdots\times B_{X_m^*})$ of $B_{X_1^*}\times\cdots\times B_{X_m^*}$ endowed with the weak star topologies $\sigma(X_j^*,X_j), 1\leq j\leq m$, such that

$$\left\|T\left(x^1,\cdots,x^m\right)\right\|\leq C(\int_{B_{X_1^*}\times\cdots\times B_{X_m^*}}\left|\varphi_1(x^1)\cdots\varphi_m(x^m)\right|^p d\mu(\varphi_1,\cdots,\varphi_m)^{\frac{1}{p}} \tag{1.3}$$

for every $x^j\in X_j$ and $j=1,\cdots,m$. Notation: $T\in\mathcal{L}_{si,p}\left(X_1,\cdots,X_m;Y\right)$. The infimum of the C defines a norm $\|.\|_{si,p}$ on the space of p-semi-integral mappings.

It is well known [4, Theorem 1] that $T\in\mathcal{L}_{si,p}\left(X_1,\cdots,X_m;Y\right)$ if and only if there exists $C\geq 0$ such that

$$(\sum_{i=1}^n\left\|T\left(x_i^1,\cdots,x_i^m\right)\right\|^p)^{\frac{1}{p}}\leq C\sup_{\phi^j\in B_{X_j^*},j=1,\cdots,m}(\sum_{i=1}^n\left|\phi^1\left(x_i^1\right)\cdots\phi^m(x_i^m)\right|^p)^{\frac{1}{p}}.$$

2. Cohen p-nuclear multilinear forms

The main results of this section are a Domination Theorem and a characterizations of the space of Cohen p-nuclear m-linear forms $(\mathcal{N}_p^m\left(X_1,\cdots,X_m\right), n_p^m)$.

The definition of Cohen p-nuclear m-linear is due to Achour–Alouani [1].

Definition 2.1. An m-linear operator $T: X_1\times\cdots\times X_m\longrightarrow Y$ is Cohen p-nuclear ($1<p\leq\infty$) if, and only if, there is a constant $C>0$ such that for any $x_1^j,\cdots,x_n^j\in X_j$, ($1\leq j\leq m$), and any $y_1^*,\cdots,y_n^*\in Y^*$, we have

$$\left|\sum_{i=1}^n\left\langle T\left(x_i^1,\cdots,x_i^m\right), y_i^*\right\rangle\right| \leq (\sup_{\substack{\varphi^j\in B_{X_j^*}\\ 1\leq j\leq m}}(\sum_{i=1}^n\left|\varphi_1(x_i^1)\cdots\varphi_m(x_i^m)\right|^p)^{\frac{1}{p}}\left\|\left(y_i^*\right)_{1\leq i\leq n}\right\|_{p^*,\omega}. \tag{2.1}$$

Again the class of all Cohen p-nuclear m-linear operators from $X_1\times\cdots\times X_m$ into Y, which is denoted by $\mathcal{N}_p^m\left(X_1,\cdots,X_m;Y\right)$, is a Banach space with the norm $n_p^m\left(T\right)$, which is the smallest constant C such that the inequality (2.1) holds.
For $p=1$, we have $\mathcal{N}_1^m\left(X_1,\cdots,X_m;Y\right)=\mathcal{L}_{si,1}\left(X_1,\cdots,X_m;Y\right)$.

The m-linear version of the Grothendieck–Pietsch Domination Theorem is the following result.

Theorem 2.2. *(Achour–Alouani) An m-linear operator $T : X_1 \times \cdots \times X_m \longrightarrow Y$ is Cohen p-nuclear $(1 < q \leq \infty)$ if and only if there is a constant $C > 0$ and there are probability measures μ_j on $K_j, j = 1, \cdots, m+1$(with $K_j = B_{X_j^*}$ for $j = 1, \cdots, m$ and $K_{m+1} = B_{Y^{**}}$), so that for all $(x^1, \cdots, x^m, y^*) \in X_1 \times \cdots \times X_m \times Y^*$ the inequality*

$$\left|\left\langle T\left(x^1, \cdots, x^m\right), y^*\right\rangle\right| \leq$$

$$C\prod_{j=1}^{m}(\int_{B_{X_j^*}} \left|\left\langle x^j, \varphi^j\right\rangle\right|^p d\mu_j(\varphi^j))^{\frac{1}{p}}(\int_{B_{Y^{**}}} \left|\left\langle y^*, \varphi\right\rangle\right|^{p^*} d\mu_{m+1}(\varphi))^{\frac{1}{p^*}}$$

is valid.

We also have the following result:

Theorem 2.3. *Let $1 < p \leq \infty$. A mapping $T \in \mathcal{L}(X_1, \cdots, X_m; Y)$ is Cohen p-nuclear if and only if there exist a constant $C > 0$, and Borel probability measures μ_1 on $B_{X_1^*} \times \cdots \times B_{X_m^*}, \mu_2$ on $B_{Y^{**}}$, such that*

$$\begin{aligned} &\left|\left\langle T\left(x^1, \cdots, x^m\right), y^*\right\rangle\right| \\ &\leq C(\int_{B_{X_1^*} \times \cdots \times B_{X_m^*}} \left|\left\langle x^1, \varphi^1\right\rangle \cdots \left\langle x^m, \varphi^m\right\rangle\right|^p d\mu_1(\varphi^1, \cdots, \varphi^m))^{\frac{1}{p}} \\ &\quad \times (\int_{B_{Y^{**}}} \left|\left\langle y^*, \varphi\right\rangle\right|^{p^*} d\mu_2(\varphi))^{\frac{1}{p^*}} \end{aligned} \tag{2.2}$$

for all $(x^1, \cdots, x^m, y^*) \in X_1 \times \cdots \times X_m \times Y^*$.

For the proof of the Domination Theorem we use the full general Pietsch Domination Theorem recently presented by Pellegrino et al in [11].

Let $X_1, \cdots, X_m, Y$ and $E_1, \cdots, E_k$ be (arbitrary) non-void sets, $\mathcal{H}$ be a family of mappings from $X_1 \times \cdots \times X_m$ to Y. Let also $K_1, .., K_t$ be compact Hausdorff topological spaces, $G_1, \cdots, G_t$ be Banach spaces and suppose that the maps

$$\begin{cases} R_j : K_j \times E_1 \times \cdots \times E_k \times G_j \rightarrow [0, +\infty), j = 1, \cdots, t \\ S : \mathcal{H} \times E_1 \times \cdots \times E_k \times G_1 \times \cdots \times G_t \rightarrow [0, +\infty) \end{cases}$$

satisfy:

(1) For each $x^l \in E_l$ and $b \in G_j$, with $(j, l) \in \{1, \cdots, t\} \times \{1, \cdots, k\}$ the mapping $(R_j)_{x^1, \cdots, x^k, b} : K_j \rightarrow [0, +\infty)$ defined by $(R_j)_{x^1, \cdots, x^k, b}(\varphi_j) = R_j(\varphi_j, x^1, \cdots, x^k, b)$ is continuous.

(2) The following inequalities hold:

$$\begin{cases} R_j(\varphi_j, x^1, \cdots, x^k, \eta_j b^j) \leq \eta_j R_j(\varphi_j, x^1, \cdots, x^k, b^j) \\ S(f, x^1, \cdots, x^k, \alpha_1 b^1, \cdots, \alpha_t b^t) \geq \alpha_1 \cdots \alpha_t S(f, x^1, \cdots, x^k, b^1, \cdots, b^t) \end{cases}$$

for every $\varphi_j \in K_j, x^l \in E_l$ (with $l \in \{1, \cdots, k\}$), $0 \leq \eta_j, \alpha_j \leq 1, b^j \in G_j$ with $j = 1, \cdots, t$ and $f \in \mathcal{H}$.

Definition 2.4. If $0 < p_1, \cdots, p_t, q < \infty$, with $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_t}$, a mapping $f : X_1 \times \cdots \times X_m \longrightarrow Y$ in H is said to be $R_1, \cdots, R_t$-S-abstract $(p_1, \cdots, p_t)$-summing if there is a constant $C > 0$ so that

$$\left(\sum_{i=1}^{n} S(f, x_i^1, \cdots, x_i^k, b_i^1, \cdots, b_i^t)^q\right)^{\frac{1}{q}} \leq C \prod_{j=1}^{t} \sup_{\varphi \in K_j} \left(\sum_{i=1}^{n} R_j(\varphi_j, x_i^1, \cdots, x_i^k, b_i^j)^{p_j}\right)^{\frac{1}{p_j}}$$

for all $x_1^s, \cdots, x_n^s \in E_s, b_1^j, \cdots, b_n^j \in G_j, n \in N$ and $(s, j) \in \{1, \cdots, k\} \times \{1, \cdots, t\}$.

Theorem 2.5. [11] *A map $f \in \mathcal{H}$ is $R_1, \cdots, R_t$-S-abstract $(p_1, \cdots, p_t)$-summing if and only if there is a constant $C > 0$ and Borel probability measures μ_j on K_j such that*

$$S(f, x^1, \cdots, x^k, b^1, \cdots, b^t) \leq C \prod_{j=1}^{t} \left(\int_{K_j} R_j(\varphi_j, x^1, \cdots, x^k, b^j)^{p_j} d\mu_j\right)^{\frac{1}{p_j}}$$

for all $x^l \in E_l$, $l \in \{1, \cdots, k\}$ and $b^j \in G_j$ with $j = 1, \cdots, t$.

Proof of Theorem 2.3. By choosing the parameters

$$\begin{cases} t = 2, r = m \\ E_r = X_r, r = 1, \cdots, m \\ K_1 = B_{X_1^*} \times \cdots \times B_{X_m^*} \text{ and } K_2 = B_{Y^{**}} \\ G_1 = \mathbb{K} \text{ and } G_2 = Y^* \\ \mathcal{H} = \mathcal{L}\left(X_1, \cdots, X_m; Y\right) \\ q = 1, p_1 = p \text{ and } p_2 = p^* \\ S(T, x^1, \cdots, x^m, b, y^*) = |b| \left|\left\langle T\left(x^1, \cdots, x^m\right), y^*\right\rangle\right| \\ R_1((\varphi^1, \cdots, \varphi^m), (x^1, \cdots, x^m), b) = |b| \left|\langle x^1, \varphi^1\rangle \cdots \langle x^m, \varphi^m\rangle\right| \\ R_2(\varphi, (x^1, \cdots, x^m), y^*) = |\langle y^*, \varphi\rangle| \end{cases}$$

we can easily conclude that $T : X_1 \times \cdots \times X_m \longrightarrow Y$ is Cohen p-nuclear if and only if T is $R_1, \cdots, R_{m+1} - S$ abstract (p, p^*)-summing. Theorem 2.5 tells us that T is $R_1, \cdots, R_{m+1}$-S abstract (p, p^*)-summing if and only if there is a $C > 0$ and there are probability measures μ_j *on* $K_j, j = 1, 2$, such that

$$S(T, x^1, \cdots, x^m, b, y^*) \leq$$
$$C \left(\int_{K_1} R_1(\varphi, x^1, \cdots, x^m, b)^p d\mu_1\right)^{\frac{1}{p}} \left(\int_{K_2} R_2(\varphi, x^1, \cdots, x^m, y^*)^{p^*} d\mu_2\right)^{\frac{1}{p^*}},$$

i.e;

$$\left|\left\langle T\left(x^1, \cdots, x^m\right), y^*\right\rangle\right|$$
$$\leq C \left(\int_{B_{X_1^*} \times \cdots \times B_{X_m^*}} \left|\langle x^1, \varphi^1\rangle \cdots \langle x^m, \varphi^m\rangle\right|^p d\mu_1\right)^{\frac{1}{p}} \left(\int_{B_{Y^{**}}} |\langle y^*, \varphi\rangle|^{p^*} d\mu_2\right)^{\frac{1}{p^*}}$$

and we recover (2.2).

We can deduce the following corollary, which is a straightforward consequence of (1.3) and Theorem 2.3.

Corollary 2.6. *In the previous theorem if we take $Y = \mathbb{K}$, then*

$$\mathcal{N}_p^m\left(X_1, \cdots, X_m\right) = \mathcal{L}_{si,p}\left(X_1, \cdots, X_m\right).$$

We need to introduce the property concerning the class of p-semi-integral$m$-linear operators.

Proposition 2.7. *a)* [3, Proposition 2]. *Let $p \geq 1$ and let*

$$\sigma_p(u) = \inf \left\|(\lambda_i)_{1\leq i\leq n}\right\|_{p^*} (\sup_{\substack{\varphi^j \in B_{X_j^*} \\ 1\leq j\leq m}} \sum_{i=1}^{n} \prod_{j=1}^{m} \left|\left\langle x_i^j, \varphi^j \right\rangle\right|^p)^{\frac{1}{p}},$$

where the infimum is taken over all representations of $u \in X_1 \otimes \cdots \otimes X_m$ in the form

$$u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes \cdots \otimes x_i^m$$

with $n, m \in \mathbb{N}, x_i^j \in X_j, \lambda_i \in \mathbb{K}, i = 1, \cdots, n, j = 1, \cdots, m$.
Then the function σ_p is a reasonable crossnorm on $X_1 \otimes \cdots \otimes X_m$.
b) [3, Proposition 7]. *The space of p-semi-integral m-linear forms $(\mathcal{L}_{si,p}(X_1, \cdots, X_m), \|.\|_{si,p})$ is isometrically isomorphic to $(X_1 \otimes \cdots \otimes X_m, \sigma_p(u))$ through the mapping $T \to \psi_T$, where $\psi_T(x^1 \otimes \cdots \otimes x^m) = T(x^1, \cdots, x^m)$ for every, $x^j \in X_j, j = 1, \cdots, m$.*

On the other hand, by Corollary 2.6 and Proposition 2.7, alternatively, we obtain the representation of the space of Cohen p-nuclear m-linear forms as the dual of the tensor product endowed with the σ_p-norm.

Corollary 2.8. *The Cohen p-nuclear m-linear forms $(\mathcal{N}_p^m(X_1, \cdots, X_m), n_p^m)$ is isometrically isomorphic to $(X_1 \otimes \cdots \otimes X_m, \sigma_p(u))^*$ through the mapping $T \to \psi_T$, where $\psi_T(x^1 \otimes \cdots \otimes x^m) = T(x^1, \cdots, x^m)$ for every, $x^j \in X_j, j = 1, \cdots, m$.*

3. Tensor norms related to Cohen p-nuclear mappings

We introduce a reasonable crossnorm on $X_1 \otimes \cdots \otimes X_m \otimes Y$ so that the topological dual of the resulting space is isometric to $(\mathcal{N}_p^m(X_1, \cdots, X_m; Y^*), n_p^m)$.

For $1 \leq p \leq \infty, 1 = \frac{1}{p} + \frac{1}{p^*}$ and $u \in X_1 \otimes \cdots \otimes X_m \otimes Y$, we consider

$$\widetilde{\omega}_{(p;p^*)}(u) = \inf \left\|(\lambda_i)_{1\leq i\leq n}\right\|_{\infty} (\sup_{\substack{\varphi^j \in B_{X_j^*} \\ 1\leq j\leq m}} \sum_{i=1}^{n} \prod_{j=1}^{m} \left|\left\langle x_i^j, \varphi^j \right\rangle\right|^p)^{\frac{1}{p}} \left\|(y_i)_{1\leq i\leq n}\right\|_{p^*,\omega},$$

where the infimum is taken over all representations of u of the form

$$u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i$$

with $x_i^j \in X_j$, $y_i \in Y$, $\lambda_i \in \mathbb{K}$, $i = 1, \cdots, n, j = 1, \cdots, m$ and $n, m \in \mathbb{N}$.

Proposition 3.1. *$\widetilde{\omega}_{(p;p^*)}$ is a reasonable crossnorm on $X_1 \otimes \cdots \otimes X_m \otimes Y$ and $\epsilon \leq \widetilde{\omega}_{(p;p^*)}$, where ϵ denotes the injective tensor norm on $X_1 \otimes \cdots \otimes X_m \otimes Y$.*

Proof. If

$$u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i$$

we have

$$\begin{aligned}
\epsilon(u) &= \sup\left\{\left|\sum_{i=1}^{n}\lambda_i\varphi^1(x_i^1)\cdots\varphi^m(x_i^m)\psi(y_i)\right|;\varphi^j\in B_{X_j^*},\psi\in B_{Y^*}\right\}\\
&\leq \left\|(\lambda_i)_{1\leq i\leq n}\right\|_\infty\left\|\sup_{\varphi^j\in B_{X_j^*},\psi\in B_{Y^*}}\sum_{i=1}^{n}\left|\varphi^1(x_i^1)\cdots\varphi^m(x_i^m)\psi(y_i)\right|\right\|\\
&\leq \left\|(\lambda_i)_{1\leq i\leq n}\right\|_\infty(\sup_{\substack{\varphi^j\in B_{X_j^*}\\ 1\leq j\leq m}}\sum_{i=1}^{n}\prod_{j=1}^{m}\left|\left\langle x_i^j,\varphi^j\right\rangle\right|^p)^{\frac{1}{p}}\sup_{\psi\in B_{Y^*}}\|(\psi(y_i))_{1\leq i\leq n}\|_{p^*}\\
&= \left\|(\lambda_i)_{1\leq i\leq n}\right\|_\infty(\sup_{\substack{\varphi^j\in B_{X_j^*}\\ 1\leq j\leq m}}\sum_{i=1}^{n}\prod_{j=1}^{m}\left|\left\langle x_i^j,\varphi^j\right\rangle\right|^p)^{\frac{1}{p}}\left\|(y_i)_{1\leq i\leq n}\right\|_{p^*,\omega}.
\end{aligned}$$

Hence $\epsilon(u)\leq\widetilde{\omega}_{(p;p^*)}(u)$.

Given $u,v\in X_1\otimes\cdots\otimes X_m\otimes Y$, for any $\delta>0$, we can find a representations of u and v of the form

$$u=\sum_{i=1}^{n}\lambda_i x_i^1\otimes\cdots\otimes x_i^m\otimes y_i,\quad v=\sum_{i=1}^{n}\alpha_i z_i^1\otimes\cdots\otimes z_i^m\otimes b_i$$

such that

$$\begin{gathered}
\left\|(\lambda_i)_{1\leq i\leq n}\right\|_\infty=1,\left\|(\alpha_i)_{1\leq i\leq n}\right\|_\infty=1\\
\sup_{\varphi^j\in B_{X_j^*},j=1,\cdots,m}(\sum_{i=1}^{n}\left|\varphi^1\left(x_i^1\right)\cdots\varphi^m(x_i^m)\right|^p)\leq(\widetilde{\omega}_{(p;p^*)}+\delta),\\
\left\|(y_i)_{1\leq i\leq n}\right\|_{p^*,\omega}\leq(\widetilde{\omega}_{(p;p^*)}+\delta)^{\frac{1}{p^*}}\\
\sup_{\varphi^j\in B_{X_j^*},j=1,\cdots,m}(\sum_{i=1}^{n}\left|\varphi^1\left(z_i^1\right)\cdots\varphi^m(z_i^m)\right|^p)\leq(\widetilde{\omega}_{(p;p^*)}+\delta),\\
\left\|(b_i)_{1\leq i\leq n}\right\|_{p^*,\omega}\leq(\widetilde{\omega}_{(p;p^*)}+\delta)^{\frac{1}{p^*}}.
\end{gathered}$$

Then it follows that

$$\begin{aligned}
\widetilde{\omega}_{(p;p^*)}(u+v)&\leq(\widetilde{\omega}_{(p;p^*)}(u)+\widetilde{\omega}_{(p;p^*)}(v)+2\delta)^{\frac{1}{p}}(\widetilde{\omega}_{(p;p^*)}(u)+\widetilde{\omega}_{(p;p^*)}(v)+2\delta)^{\frac{1}{p^*}}\\
&=(\widetilde{\omega}_{(p;p^*)}(u)+\widetilde{\omega}_{(p;p^*)}(v)+2\delta)\underset{\delta\to0}{\longrightarrow}\left(\widetilde{\omega}_{(p;p^*)}(u)+\widetilde{\omega}_{(p;p^*)}(v)\right),
\end{aligned}$$

which shows the triangular inequality. The other conditions are easily verified. Hence $\widetilde{\omega}_{(p;p^*)}$ is a norm on $X_1\otimes\cdots\otimes X_m\otimes Y$.
It is easily seen that $\widetilde{\omega}_{(p;p^*)}(x^1\otimes\cdots\otimes x^m\otimes y)\leq\|x^1\|\cdots\|x^m\|.\|y\|$ for every $x^j\in X_j,j=1,\cdots,m$ and $y\in Y$. To show that

$$\|\phi_1\otimes\cdots\otimes\phi_m\otimes\psi(u)\|\leq\|\phi_1\|\cdots\|\phi_m\|\,\|\psi\|.$$

Let $x^j\in X_j,y\in Y,j=1,\cdots,m$. Now if $\phi_j\neq0,\phi_j\in X_j^*,\psi\in Y^*$ and $u=\sum_{i=1}^{n}\lambda_i x_i^1\otimes\cdots\otimes x_i^m\otimes y_i$. Then by Hölder's inequality we get

$$
\begin{aligned}
&|\phi_1 \otimes \cdots \otimes \phi_m \otimes \psi(u)| \\
&= \left|\phi_1 \otimes \cdots \otimes \phi \otimes \psi(\sum_{i=1}^{n} \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i)\right| \\
&\leq \sum_{i=1}^{n} \left|\lambda_i \phi_1(x_i^1) \cdots \phi_m(x_i^m)\psi(y_i)\right| \\
&\leq \left\|(\lambda_i)_{1\leq i\leq n}\right\|_\infty \|\phi_1\| \cdots \|\phi_m\| \, \|\psi\| \left\| \left(\frac{\phi_1(x_i^1)}{\|\phi_1\|} \cdots \frac{\phi_m(x_i^m)}{\|\phi_m\|} \frac{(\psi(y_i)}{\|\psi\|} \right)_{1\leq i\leq n} \right\|_1 \\
&\leq \|\phi_1\| \cdots \|\phi_m\| \, \|\psi\| \left\|(\lambda_i)_{1\leq i\leq n}\right\|_\infty (\sup_{\substack{\varphi^j \in B_{X_j^*} \\ 1\leq j\leq m}} \sum_{i=1}^{n} \prod_{j=1}^{m} \left|\left\langle x_i^j, \varphi^j \right\rangle\right|^p)^{\frac{1}{p}} \left\|(y_i)_{1\leq i\leq n}\right\|_{p^*,\omega} .
\end{aligned}
$$

Therefore we obtain that

$$|\phi_1 \otimes \cdots \otimes \phi_m \otimes \psi(u)| \leq \|\phi_1\| \cdots \|\phi_m\| \, \|\psi\| \, \widetilde{\omega}_{(p;p^*)}(u),$$

and we have shown that $\widetilde{\omega}_{(p;p^*)}$ is a reasonable crossnorm. □

Remark 3.2. 1) Note that when $m = 1$, in particular, the norm $\widetilde{\omega}_{(p;p^*)}$ is reduced to the Cohen norm ω_p on $X_1 \otimes Y$ (see [5, page180]).
2) In the previous proposition if we take $Y = \mathbb{K}$, then we identify$X_1 \otimes \cdots \otimes X_m \otimes \mathbb{K}$ with $X_1 \otimes \cdots \otimes X_m$, and in this case the corresponding reasonable crossnorm will be denoted by $\omega_p(u)$, which is described as follows:

$$\omega_p(u) = \inf \left\|(\lambda_i)_{1\leq i\leq n}\right\|_\infty (\sup_{\substack{\varphi^j \in B_{X_j^*} \\ 1\leq j\leq m}} \sum_{i=1}^{n} \prod_{j=1}^{m} \left|\left\langle x_i^j, \varphi^j \right\rangle\right|^p)^{\frac{1}{p}} ,$$

where the infimum is taken over all representations of $u \in X_1 \otimes \cdots \otimes X_m$ in the form

$$u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes \cdots \otimes x_i^m.$$

3) It follows from the definitions of $\omega_p(u)$ and $\sigma_p(u)$ that $\omega_p(u) = \sigma_p(u)$ for every $u \in X_1 \otimes \cdots \otimes X_m$.

Follows the ideas if [2, Theorem 4.8] and [3, Proposition 2] we prove the following result. This result characterizes the space of Cohen p-nuclear mappings as the topological dual of the space of the tensor product $(X_1 \otimes \cdots \otimes X_m \otimes Y, \widetilde{\omega}_{(p;p^*)})$ up to an isometric isomorphism.

Theorem 3.3. *Let $X_1, \cdots, X_m$ be Banach spaces. Then, for every Banach space Y, the space $(\mathcal{N}_p^m(X_1, \cdots, X_m; Y^*), n_p^m)$ is isometrically isomorphic to $(X_1 \otimes \cdots \otimes X_m \otimes Y, \widetilde{\omega}_{(p;p^*)})^*$ through the mapping $T \to \psi_T$, where $\psi_T(x^1 \otimes \cdots \otimes x^m \otimes y) = T(x^1, \cdots, x^m)(y)$ for every, $x^j \in X_j, j = 1, \cdots, m$ and $y \in Y$.*

Proof. It is easy to see that the correspondence

$$T \in \mathcal{N}_p^m(X_1, \cdots, X_m; Y^*) \rightarrow \psi_T \in (X_1 \otimes \cdots \otimes X_m \otimes Y, \widetilde{\omega}_{(p;p^*)})^*$$

defined by

$$\psi_T(x^1 \otimes \cdots \otimes x^m \otimes y) = T(x^1, \cdots, x^m)(y)$$

for every $, x^j \in X_j, j = 1, \cdots, m$ and $y \in Y$,
is linear and injective. To show the surjectivity let $\psi \in (X_1 \otimes \cdots \otimes X_m \otimes Y, \widetilde{\omega}_{(p;p^*)})^*$ and consider the corresponding m-linear mapping $T_\psi \in \mathcal{L}(X_1, \cdots, X_m; Y^*)$, defined by $T_\psi(x^1, \cdots, x^m)(y) = \psi(x^1 \otimes \cdots \otimes x^m \otimes y)$, for $x^j \in X_j, j = 1, \cdots, m$ and $y \in Y$.
Let us consider $x_i^j \in X_j, j = 1, \cdots, m, i = 1, \cdots, n$ and $y_i \in Y^{**}$. For appropriate $\lambda_i \in \mathbb{K}$, with $|\lambda_i| = 1, i = 1, \cdots, n$, we have that

$$\begin{aligned}
&\Big| \sum_{i=1}^n \left\langle T_\psi\left(x_i^1, \cdots, x_i^m\right), y_i \right\rangle \Big| \\
&\quad = \left| \sum_{i=1}^n \lambda_i \psi(x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i) \right| \\
&\quad = \left| \psi(\sum_{i=1}^n \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i) \right| \\
&\quad \leq \|\psi\| \, \widetilde{\omega}_{(p;p^*)} \left(\sum_{i=1}^n \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i \right) \\
&\quad \leq \|\psi\| \left\| (\lambda_i)_{1 \leq i \leq n} \right\|_\infty \Big(\sup_{\substack{\varphi^j \in B_{X_j^*} \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m \left| \left\langle x_i^j, \varphi^j \right\rangle \right|^p \Big)^{\frac{1}{p}} \left\| (y_i)_{1 \leq i \leq n} \right\|_{p^*, \omega} \\
&\quad \leq \|\psi\| \Big(\sup_{\substack{\varphi^j \in B_{X_j^*} \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m \left| \left\langle x_i^j, \varphi^j \right\rangle \right|^p \Big)^{\frac{1}{p}} \left\| (y_i)_{1 \leq i \leq n} \right\|_{p^*, \omega},
\end{aligned}$$

which shows that $T_\psi \in \mathcal{N}_p^m(X_1, \cdots, X_m; Y^*)$ and $n_p^m(T_\psi) \leq \|\psi\|$.
Conversely, if T is Cohen p-nuclear from $X_1 \times \cdots \times X_m$ into Y^*, we define a linear functional on $X_1 \otimes \cdots \otimes X_m \otimes Y$ by

$$\psi_T(u) = \sum_{i=1}^n \lambda_i T(x_i^1, \cdots, x_i^m)(y_i),$$

for $u = \sum_{i=1}^n \lambda_i x_i^1 \otimes \cdots \otimes x_i^m \otimes y_i$, where $m \in \mathbb{N}, \lambda_i \in \mathbb{K}, x_i^j \in X_j, y_i \in Y, i = 1, \cdots, n, j = 1, \cdots, m$. Hence, by Holder's inequality it follows that

$$\begin{aligned}
|\psi_T(u)| &\leq \left\| (\lambda_i)_{1 \leq i \leq n} \right\|_\infty \sum_{i=1}^n \left| \left\langle T\left(x_i^1, \cdots, x_i^m\right), y_i \right\rangle \right| \\
&\leq n_p^m(T) \left\| (\lambda_i)_{1 \leq i \leq n} \right\|_\infty \Big(\sup_{\substack{\varphi^j \in B_{X_j^*} \\ 1 \leq j \leq m}} \sum_{i=1}^n \prod_{j=1}^m \left| \left\langle x_i^j, \varphi^j \right\rangle \right|^p \Big)^{\frac{1}{p}} \left\| (y_i)_{1 \leq i \leq n} \right\|_{p^*, \omega}.
\end{aligned}$$

This shows that ψ_T is $\widetilde{\omega}_{(p;p^*)}$-continuous and $\|\psi_T\| \leq n_p^m(T)$. □

Corollary 3.4. *The Cohen p-nuclear m-linear forms* $\left(\mathcal{N}_p^m\left(X_1,\cdots,X_m\right),n_p^m\right)$ *is isometrically isomorphic to* $(X_1\otimes\cdots\otimes X_m,\omega_p(u))^*$ *through the mapping* $T\rightarrow\psi_T$, *where* $\psi_T(x^1\otimes\cdots\otimes x^m)=T(x^1,\cdots,x^m)$ *for every,* $x^j\in X_j, j=1,\cdots,m$.

Proposition 3.5. *Let* $X_1,\cdots,X_m$ *and* Y *be Banach spaces. Then, a multilinear mapping* $T:X_1\times\cdots\times X_m\rightarrow Y$ *is Cohen p-nuclear if its associated linear mapping* $\widetilde{T}:X_1\otimes\cdots\otimes X_m\rightarrow Y$, *given by* $\widetilde{T}(x^1\otimes\cdots\otimes x^m)=T(x^1,\cdots,x^m)$ *for every* $x^j\in X_j, j=1,\cdots,m$, *is* ω_p*-continuous and p-nuclear. In this case we have:*

$\|T\|\leq n_p^m(T)\leq n_p(\widetilde{T})$.

Conversely, if $T\in\mathcal{N}_p^m(X_1,\cdots,X_m;Y)$, *then the associated linear mapping* $\widetilde{T}$ *is* ω_p*-continuous, that is,* $\widetilde{T}\in\mathcal{L}((X_1\otimes\cdots\otimes X_m,\omega_p);Y)$. *In this case we have:*

$\|T\|\leq\left\|\widetilde{T}\right\|_{\mathcal{L}((X_1\otimes\cdots\otimes X_m,\omega_p);Y)}\leq n_p^m(T)$.

Proof. Suppose that $\widetilde{T}\in\mathcal{L}((X_1\otimes\cdots\otimes X_m,\omega_p);Y)$. Then, by (1.2) and Corollary 3.4, it follows that

$$\begin{aligned}
&\left\|\left(\left\langle T\left(x_i^1,\cdots,x_i^m\right),y_i^*\right\rangle\right)_{1\leq i\leq n}\right\|_1\\
&=\sum_{i=1}^n\left|\left\langle\widetilde{T}(x_i^1\otimes\cdots\otimes x_i^m),y_i^*\right\rangle\right|\\
&\leq n_p(\widetilde{T})\sup_{\varphi\in B_{(X_1\otimes\cdots\otimes X_m,\omega_p)^*}}(\sum_{i=1}^n\left|\varphi\left(x_i^1\otimes\cdots\otimes x_i^m\right)\right|^p)^{\frac{1}{p}}\left\|(y_i^*)_{1\leq i\leq n}\right\|_{p^*,\omega}\\
&=n_p(\widetilde{T})\sup_{S\in B_{(\mathcal{N}_p^m(X_1,\cdots,X_m),n_p^m)}}(\sum_{i=1}^n\left|S\left(x_i^1,\cdots,x_i^m\right)\right|^p)^{\frac{1}{p}}\left\|(y_i^*)_{1\leq i\leq n}\right\|_{p^*,\omega},\\
&\leq n_p(\widetilde{T})\sup_{\varphi^j\in B_{X_j^*},1\leq j\leq m}(\sum_{i=1}^n\left|\varphi^1(x_i^1)\cdots\varphi^m(x_i^m)\right|^p)^{\frac{1}{p}}\left\|(y_i^*)_{1\leq i\leq n}\right\|_{p^*,\omega},
\end{aligned}$$

which shows that $T\in\mathcal{N}_p^m(X_1,\cdots,X_m;Y)$ with $n_p^m(T)\leq n_p(\widetilde{T})$. The fact that $\|T\|\leq n_p^m(T)$ follows from Definition 2.1.

To show the converse, suppose now $T\in\mathcal{N}_p^m(X_1,\cdots,X_m;Y)$. By Corollary 4.5(i) in [1], T is p-semi-integral and $\|T\|_{si,p}\leq n_p^m(T)$; hence Proposition 8(b) in [3] and Remark 3.2 shows that

$\left\|\widetilde{T}(u)\right\|\leq\|T\|_{si,p}\,\sigma_p(u)\leq n_p^m(T)\omega_p(u)$,

and so $\widetilde{T}$ is ω_p-continuous with $\left\|\widetilde{T}\right\|_{\mathcal{L}((X_1\otimes\cdots\otimes X_m,\omega_p);Y)}\leq n_p^m(T)$. Finally, since ω_p is a reasonable crossnorm, it readily follows that $\|T\|\leq\left\|\widetilde{T}\right\|_{\mathcal{L}((X_1\otimes\cdots\otimes X_m,\omega_p);Y)}$, which completes the proof.

□

We do not know if, in general, $\widetilde{T}\in\mathcal{L}((X_1\otimes\cdots\otimes X_m,\omega_p);Y)$ whenever $T\in\mathcal{N}_p^m(X_1,\cdots,X_m;Y)$.

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