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DETECTION OF SCALES OF HETEROGENEITY AND PARABOLIC HOMOGENIZATION APPLYING VERY WEAK MULTISCALE CONVERGENCE

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ABSTRACT. We apply a new version of multiscale convergence named very weak multiscale convergence to find possible frequencies of oscillation in an unknown coefficient of a partial differential equation from its solution. We also use this notion to study homogenization of a certain linear parabolic problem with multiple spatial and temporal scales.

1. INTRODUCTION

Let us consider a piece $\Omega \subset \mathbb{R}^N$ of a material with a heterogeneous structure. The classical problem of homogenization deals with finding a corresponding homogeneous material with a similar over all response as the composite in question. This is well studied if the material is periodically arranged with heterogeneities on one or several scale levels. The standard example is the homogenization of the stationary heat equation

$$\begin{aligned} -\nabla \cdot \left(a\left(\frac{x}{\varepsilon}\right) \nabla u^{\varepsilon}(x) \right) &= f(x) \text{ in } \Omega, \\ u^{\varepsilon}(x) &= 0 \text{ on } \partial\Omega, \end{aligned}$$

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where a is periodic with respect to a unit cube $Y \subset \mathbb{R}^N$. Homogenization means that we study the process when ε tends to zero to find a limit equation

$$\begin{aligned} -\nabla \cdot (b\nabla u(x)) &= f(x) \text{ in } \Omega, \\ u(x) &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

such that u^ε approaches u. The difficulty in this procedure consists in finding b. For appropriate choices of test functions in the weak form of (1.1) it is possible to identify

$$b\nabla u(x) = \int_Y a(y)(\nabla u(x) + \nabla_y u_1(x,y))\, dy,$$

where u_1 is found by means of a so-called local problem defined on Y. The approach in mathematical homogenization is thus to find the homogenized matrix b when the microstructure is known. For some clear and informative texts on homogenization theory we suggest e.g. [4], [1], [10] and [12].

This paper also addresses the problem from a different angle. Applying a certain type of weak convergence, very weak two-scale convergence [5], to $\{\varepsilon^{-1}u^\varepsilon\}$ we obtain u_1 as a limit for a certain choice of test functions, but only if their frequency of oscillation is in time with those of u^ε and hence of the governing coefficient $a(\frac{x}{\varepsilon})$. This can be extended to several scales $\varepsilon_1, ..., \varepsilon_n$ of heterogeneity which are detected one at the time by studying the corresponding limit for $\left\{\varepsilon_k^{-1}u^\varepsilon\right\}, k = 1, ..., n$ by means of very weak multiscale convergence (see again [5]), which will provide us with a non-zero limit when there are heterogeneities of the frequency in question there to discover. Finally, in the last section of this paper, we return to the origin of very weak multiscale convergence, the homogenization of parabolic problems with fast oscillations in both space and time. We demonstrate how minor modifications of very weak multiscale convergence enables us to homogenize quite complicated parabolic problems with multiple spatial and temporal scales.

Homogenization results for linear parabolic problems with fast oscillations in both spatial and temporal scales applying generalizations of two-scale convergence are found in e.g. [8] and have been extended to nonlinear cases in for example [13], [6] and [21]. Such techniques for problems with three time scales are developed for linear parabolic homogenization problems in [7] and further extended to the nonlinear non-monotone case in [20].

2. Multiscale convergence

In [15] a new technique for the homogenization of partial differential equations was introduced by Nguetseng. This approach, which has become known under the name of two-scale convergence, was extended to multiple scales by Allaire and Briane in [2]. We define this more general concept below. Two-scale convergence means that there is only one scale of rapid oscillation, see also [1], [4], [9], [12] and [14].

We denote $y^n = (y_1, ..., y_n)$, $dy^n = dy_1...dy_n$, $Y_k = Y = [0,1]^N$, $Y^n = Y_1 \times ... \times Y_n$ and let $\varepsilon_k(\varepsilon), k = 1, ..., n$ be functions such that $\varepsilon_k(\varepsilon) \to 0$ when $\varepsilon \to 0$. In a similar way we write $s^m = (s_1, ..., s_m)$, $ds^m = ds_1...ds_m$, $S_k = S = (0,1)$,

$S^m = S_1 \times ... \times S_m$, $\mathcal{Y}_{n,m} = Y^n \times S^m$ and let $\varepsilon_k'(\varepsilon)$, $k = 1, ..., m$ be the corresponding temporal scales. The rest of the notations are standard for homogenization theory.

Definition 2.1. We say that a sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ $(n+1)$-scale converges to a function $u_0 \in L^2(\Omega \times Y^n)$ if

$$\int_\Omega u^\varepsilon(x) v\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_n}\right) dx \to \int_\Omega \int_{Y^n} u_0(x, y^n)\, v(x, y^n)\, dy^n dx$$

for any $v \in L^2(\Omega; C_\sharp(Y^n))$. This is denoted by

$$u^\varepsilon(x) \overset{n+1}{\rightharpoonup} u_0(x, y^n).$$

To proceed we need to distinguish some types of relationships between the scales. If

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0$$

we say that the scales are separated. Sometimes a stronger assumption is needed. When

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon_k} \left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^m = 0$$

for some positive integer m the scales are called well-separated. The compactness result below is found in Theorem 2.4 in [2].

Theorem 2.2. *If $\{u^\varepsilon\}$ is bounded in $L^2(\Omega)$ and the scales are separated there exists a subsequence such that*

$$u^\varepsilon(x) \overset{n+1}{\rightharpoonup} u_0(x, y^n),$$

where $u_0 \in L^2(\Omega \times Y^n)$.

A characterization of multiscale limits for gradients of bounded sequences in $H^1(\Omega)$ can be found in Theorem 2.6 in [2].

Theorem 2.3. *Let $\{u^\varepsilon\}$ be a bounded sequence in $H^1(\Omega)$ and assume that the scales $\varepsilon_k(\varepsilon)$, $k = 1, ..., n$, are separated. Then there exists a subsequence such that*

$$u^\varepsilon(x) \rightharpoonup u(x) \text{ in } H^1(\Omega),$$
$$u^\varepsilon(x) \overset{n+1}{\rightharpoonup} u(x)$$

and

$$\nabla u^\varepsilon(x) \overset{n+1}{\rightharpoonup} \nabla u(x) + \nabla_{y_1} u_1(x, y_1) + ... + \nabla_{y_n} u_n(x, y^n),$$

where $u \in H^1(\Omega)$, $u_1 \in L^2(\Omega; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_k \in L^2(\Omega \times Y^{k-1}; H^1_\sharp(Y_k)/\mathbb{R})$ for $k = 2, ..., n$.

Remark 2.4. In [11] reiterated homogenization of elliptic problems is treated under less restrictive assumptions which do not have to include periodicity but contain periodic homogenization as a special case. The authors use a technique named Σ-convergence, which is probably the most general concept originating from two-scale convergence of today. See also e.g. [18] , [16] or [21], where parabolic problems are studied. Another example where homogenization is performed under conditions different from the usual assumptions of periodicity is stochastic homogenization, see e.g. [3] or [17].

3. Very weak multiscale convergence

For the method of detection of scales developed in the next section and the homogenization procedure in the last section we need a different type of multiscale convergence, where a more restrictive class of test functions is used. Let $\{u^{\varepsilon}\}$ be a sequence of functions such that

$$u^{\varepsilon}(x) \rightharpoonup u(x) \text{ in } H_0^1(\Omega).$$

Then, see [8], there exists a subsequence such that

$$\int_{\Omega} \frac{u^{\varepsilon}(x)-u(x)}{\varepsilon} v(x)\varphi\left(\frac{x}{\varepsilon}\right) dx \to \int_{\Omega}\int_{Y} u_1(x,y)v(x)\varphi(y)\, dydx$$

for any $v \in D(\Omega), \varphi \in C_{\sharp}^{\infty}(Y)/\mathbb{R}$. In this setting it is easy to identify assumptions such that $\{\varepsilon^{-1}(u^{\varepsilon}-u)\}$ is bounded in $L^2(\Omega)$ when $\{u^{\varepsilon}\}$ is a sequence of solutions to the homogenization problem (1.1), see [10], and hence also a usual two-scale limit exists.

A similar result, where the term u is omitted, is found in [16] , where it is proven that any bounded sequence $\{u^{\varepsilon}\}$ in $H_0^1(\Omega)$ contains a subsequence such that

$$\int_{\Omega} \frac{u^{\varepsilon}(x)}{\varepsilon} v(x)\varphi\left(\frac{x}{\varepsilon}\right) dx \to \int_{\Omega}\int_{Y} u_1(x,y)v(x)\varphi(y)\, dydx \tag{3.1}$$

for the same test functions as above, see Remarks 3.3 and 3.4. Here $\{\varepsilon^{-1}u^{\varepsilon}\}$ is not bounded in $L^2(\Omega)$ unless $\{u^{\varepsilon}\}$ passes to zero in a quite powerful way. The corresponding result for Σ-convergence, from which (3.1) can be concluded, is found in [18].

We are now ready to define the multiscale equivalent of the concept discussed above.

Definition 3.1. Let $\{g^{\varepsilon}\}$ be a sequence in $L^1(\Omega)$ and let $g_0 \in L^1(\Omega \times Y^n)$. We say that $\{g^{\varepsilon}\}$ $(n+1)$-scale converges very weakly to g_0 if

$$\int_{\Omega} g^{\varepsilon}(x) v\left(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_{n-1}}\right)\varphi\left(\frac{x}{\varepsilon_n}\right) dx \to \int_{\Omega}\int_{Y_n} g_0(x,y^n)v(x,y^{n-1})\varphi(y_n)\, dy^n dx$$

for any $v \in D(\Omega; C_{\sharp}^{\infty}(Y^{n-1}))$ and $\varphi \in C_{\sharp}^{\infty}(Y_n)/\mathbb{R}$. We write

$$g^{\varepsilon}(x) \underset{vw}{\overset{n+1}{\rightharpoonup}} g_0(x,y^n).$$

A unique representation of the limit is provided by choosing g_0 such that

$$\int_{Y_n} g_0(x,y^n)\, dy_n = 0.$$

Below we have a compactness result for very weak multiscale convergence.

Theorem 3.2. *Let $\{u^{\varepsilon}\}$ be a bounded sequence in $H_0^1(\Omega)$ and the scales $\varepsilon_k(\varepsilon)$, $k = 1, ..., n$ well-separated. Then there exists a subsequence such that*

$$\frac{u^{\varepsilon}(x)}{\varepsilon_k} \underset{vw}{\overset{k+1}{\rightharpoonup}} u_k\left(x, y^k\right),$$

where $u_1 \in L^2(\Omega; H^1_\sharp(Y_1)/\mathbb{R})$ *and* $u_k \in L^2(\Omega \times Y^{k-1}; H^1_\sharp(Y_k)/\mathbb{R})$ *for* $k = 2, ..., n$ *are the same as in Theorem 2.3.*

Proof. See Theorem 4 in [5]. □

Remark 3.3. The compactness result in Theorem 3.2 makes it possible to detect oscillations, whose amplitude goes to zero when ε does and catch them in a limit that is not disturbed by the corresponding upscaling of the macro-level trend of the functions. Moreover, it provides us with a type of multiscale compactness for sequences which are usually not bounded in $L^2(\Omega)$. The crucial point here is the choice of test functions. For a more restrictive choice of test functions the result in Theorem 3.2 holds also when the scales are separated but not necessarily well-separated.

Remark 3.4. Originally, see [8] and [18], the concept of very weak two-scale convergence was developed to treat homogenization of parabolic partial differential equations with oscillations in both spatial and temporal scales and hence the assumptions on $\{u^\varepsilon\}$ are adapted to this context in these papers. A slight modification of Theorem 3.2 will be used in the homogenization procedure for parabolic equations with multiple spatial and temporal scales developed in Section 5. See also [6], [7], [19] , [20] and [21] for some recent results on the applications of related approaches to parabolic homogenization.

Remark 3.5. Y does not have to have the volume $|Y| = 1$ or even to be a cube. The unit cube is however the usual standard for homogenization even though other choices of repetitive units can be used without major changes in the approach. This is also the case for the investigations in this paper. See e.g. [10] or [4]. The techniques are basically the same for $N = 1$ and $N > 1$ even though some simplifications are possible in the one-dimensional case.

4. Detection of scales of heterogeneity

Consider the multiscale homogenization problem

$$\begin{aligned} -\nabla \cdot (a^\varepsilon(x)\nabla u^\varepsilon(x)) &= f(x) \text{ in } \Omega, \\ u^\varepsilon(x) &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.1}$$

Contrary to how we usually treat homogenization problems we aim at identifying properties of the coefficient from the answering solution to (4.1). Our ambition is to find ways to use the solution u^ε to (4.1) to analyze which frequencies of oscillation that appear in the coefficient. This could for example mean that u^ε is obtained from measurements and not necessarily by solving (4.1). We introduce the well-separated scales

$$\varepsilon_1(\varepsilon) = \varepsilon, \varepsilon_2(\varepsilon) = \varepsilon^{\frac{3}{2}}, \varepsilon_3(\varepsilon) = \varepsilon^{\frac{5}{3}}, \varepsilon_4(\varepsilon) = \varepsilon^2$$

and compute the results for very weak multiscale convergence of $\left\{\varepsilon_k^{-1}u^\varepsilon\right\}$ for these scales and suitable choices of test functions.

In the examples below we let $\Omega = (0, 2)$ and $\varepsilon = 0.05$ and study u^ε generated by a certain choice of a^ε given in (4.2). We will see that very weak multiscale convergence gives us significantly non-zero values when we hit the frequencies of oscillation which are found in the coefficient a^ε and hence indicates existing scales of heterogeneity. The micro-oscillations of u^ε are usually of vanishing amplitude. This is however compensated by the scaling used in Theorem 3.2 for very weak multiscale convergence.

Up to the authors' knowledge the technique introduced below is new and could mean the first step in a new direction in the study of heterogeneous media.

For $k = 1$ we choose

$$\begin{aligned} v(x) &= (1-x)(1-\cos \pi x), \\ \varphi(y_1) &= \sin 2\pi y_1 \end{aligned}$$

and obtain for $\varepsilon_1 = \varepsilon = 0.05$

$$I_1(\varepsilon) = \int_\Omega \frac{u^\varepsilon(x)}{\varepsilon_1} v(x) \varphi\left(\frac{x}{\varepsilon_1}\right) dx \approx 0.$$

From Theorem 3.2 we know that when $\varepsilon \to 0$

$$I_1(\varepsilon) \to \int_\Omega \int_{Y_1} u_1(x, y_1) v(x) \varphi(y_1)\, dy_1 dx.$$

Seemingly, there are no oscillations of the chosen frequency there to detect.

Continuing with $k = 2$ and choosing

$$\begin{aligned} v(x, y_1) &= (1-x)(1-\cos \pi x)(2 + \sin 2\pi y_1), \\ \varphi(y_2) &= \cos 2\pi y_2 \end{aligned}$$

we find that for $\varepsilon_1 = \varepsilon = 0.05$ and $\varepsilon_2 = (0.05)^{\frac{3}{2}} \approx 0.0112$

$$I_2(\varepsilon) = \int_\Omega \frac{u^\varepsilon(x)}{\varepsilon_2} v\left(x, \frac{x}{\varepsilon_1}\right) \varphi\left(\frac{x}{\varepsilon_2}\right) dx \approx 0.0416.$$

Since

$$I_2(\varepsilon) \to \int_\Omega \int_{Y^2} u_2(x, y^2) v(x, y_1) \varphi(y_2)\, dy^2 dx$$

when $\varepsilon \to 0$ we have found signs of an existing frequency of oscillation.

Next we study $k = 3$ for

$$\begin{aligned} v(x, y^2) &= (1-x)(1-\cos \pi x)(2 + \sin 2\pi y_1)\left(2 + \cos^2 2\pi y_2\right), \\ \varphi(y_3) &= \sin 2\pi y_3 \end{aligned}$$

and observe that for the scale values $\varepsilon_1 = \varepsilon = 0.05$, $\varepsilon_2 = (0.05)^{\frac{3}{2}} \approx 0.0112$ and $\varepsilon_3 = (0.05)^{\frac{5}{3}} \approx 0.0068$

$$I_3(\varepsilon) = \int_\Omega \frac{u^\varepsilon(x)}{\varepsilon_3} v\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}\right) \varphi\left(\frac{x}{\varepsilon_3}\right) dx \approx 0.1039.$$

In the same way as above we note that

$$I_3(\varepsilon) \to \int_\Omega \int_{Y^3} u_3(x, y^3) v(x, y^2) \varphi(y_3)\, dy^3 dx$$

for $\varepsilon \to 0$ and hence there seems to be a scale of heterogeneity corresponding to the frequency represented by ε_3.

Finally, we investigate $k = 4$ and choose

$$\begin{aligned} v(x, y^3) &= (1-x)(1-\cos\pi x)(2+\sin 2\pi y_1)\left(2+\cos^2 2\pi y_2\right)\left(2+\sin^3 2\pi y_3\right), \\ \varphi(y_4) &= \cos 2\pi y_4. \end{aligned}$$

When $\varepsilon_1 = \varepsilon = 0.05$, $\varepsilon_2 = (0.05)^{\frac{3}{2}} \approx 0.0112$, $\varepsilon_3 = (0.05)^{\frac{5}{3}} \approx 0.0068$ and $\varepsilon_4 = (0.05)^2 \approx 0.0025$ we obtain

$$I_4(\varepsilon) = \int_\Omega \frac{u^\varepsilon(x)}{\varepsilon_4} v\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \frac{x}{\varepsilon_3}\right) \varphi\left(\frac{x}{\varepsilon_4}\right) dx \approx 0.$$

Because

$$I_4(\varepsilon) \to \int_\Omega \int_{Y^4} u_4(x, y^4) v\left(x, y^3\right) \varphi(y_4)\, dy^4 dx$$

when $\varepsilon \to 0$ this indicates that $u_4 = 0$ and we conclude that there is no indication of heterogeneities for this scale.

Our numerical experiment leads us to believe that the coefficient is of the type

$$a^\varepsilon(x) = a\left(\frac{x}{0.0112}, \frac{x}{0.0068}\right)$$

with a periodic with respect to Y^2. Revealing the "secret" coefficient used in our computations above we see that

$$a^\varepsilon(x) = \frac{1}{3 + \sin\left(2\pi \frac{x}{\varepsilon^{\frac{3}{2}}}\right) + \cos\left(2\pi \frac{x}{\varepsilon^{\frac{5}{3}}}\right)} \tag{4.2}$$

and hence what we have seen above is the stage in the homogenization for this coefficient corresponding to $\varepsilon = 0.05$.

Remark 4.1. The results for $I_1(\varepsilon)$ and $I_4(\varepsilon)$ are $-6.2 \cdot 10^{-7}$ and $-7.9 \cdot 10^{-6}$ respectively. Some care has to be taken while choosing the functions to be used in the computations to avoid to obtain values very close to zero also when the chosen speed of oscillation coincides with a frequency that appears in the problem to be studied. A more careful study of truly unknown scales of heterogeneity would be have to be based on a the use of a larger number of test functions to be convincing.

Remark 4.2. Let us assume that u^ε is given from e.g. measurements and we are looking for signs of an underlying Y-periodic structure with multiple scales. By inspection we may first look for small oscillations of u^ε on a coarsest scale roughly indicating a suitable value of $\varepsilon_1(\varepsilon) = \varepsilon$. Let ε run through such values until we obtain significantly non-zero values of $I_1(\varepsilon)$. In a similar way we look for a finer scale and fine tune until we find clearly non-zero values of the corresponding expression for this scale in I_2 and so on. I_1 above is an example of when the chosen first micro-scale does not correspond to an existing scale of heterogeneity while the second and third micro-scales introduced in I_2 and I_3 respectively do. The scale $\varepsilon_1(\varepsilon)$ is included in the computations of I_2, I_3 and I_4 above just to illustrate that a scale that does not correspond to any level of oscillation of u^ε will not disturb the procedure. Making these investigations for practical purposes we would have chosen $\varepsilon_1 = \varepsilon = 0.0112$ in $I_1(\varepsilon)$ and the oscillations with period 0.05 would appear neither in I_1 nor in the following computations with successively larger number of scales. Clearly, for a given u^ε the method discovers scales corresponding to certain levels of heterogeneity represented by real numbers and hence the choice of well-separated scales in our example defined by (4.2) is only formal. Note that very weak multiscale convergence provides us not only with an indication of existing scales of heterogeneity but also with the correctors $u_1, ..., u_n$ for these scales.

Obviously, this approach helps us to identify the relationship between the frequencies of oscillation that appear in the coefficient in (4.1). It does however not in general provide us with the homogenized limit or the shape of the oscillations of a^ε. To distinguish for which cases such a procedure could give an accurate picture of the coefficient contains many open questions and will be attended to in forthcoming studies. Let us first briefly outline how a simplest possible example could be treated and then comment on some more complicated cases.

We study (4.1) in one dimension for

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right),$$

where a is Y-periodic and strictly positive. Let a solution to (4.1) be given and scan over different small values of ε (see Remark 4.2) to find the appropriate frequency of oscillation, i.e. when

$$\int_\Omega \int_Y u_1(x,y)\, v(x) v_1(y)\, dy dx \approx \int_\Omega \frac{u^\varepsilon(x)}{\varepsilon} v(x) v_1\left(\frac{x}{\varepsilon}\right) dx \neq 0.$$

Standard numerical techniques can be applied to find u_1. Since it is not difficult to find a good approximation of $\frac{d}{dx}u$ a separation of variables

$$u_1(x,y) = \frac{d}{dx}u(x) \cdot z(y)$$

provides us with z, the solution to the local problem

$$-\frac{d}{dy}\left(a(y)\left(1 + \frac{d}{dy}z(y)\right)\right) = 0.$$

Once we know z, simple calculations leads to the result that

$$a(y) = \frac{C}{1 + \frac{d}{dy}z(y)}. \tag{4.3}$$

Introducing (4.3) with $y = \frac{x}{\varepsilon}$ in (4.1) we find the constant C, which is identical to the homogenized limit b.

Adding more scales we arrive at more complicated situations such as the one studied in the first part of this section. This means that the underlying equation could be of the type

$$\begin{aligned} -\nabla \cdot \left(a\left(\frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n}\right) \nabla u^\varepsilon(x)\right) &= f(x) \text{ in } \Omega, \\ u^\varepsilon(x) &= 0 \text{ on } \partial\Omega \end{aligned}$$

and the first step is then to find the scales of oscillation in the same way as we already did once. The next, and technically more difficult, task would be to find ways to trace the coefficient $a(y_1, ..., y_n)$ from the correctors $u_1, ..., u_n$. Also to extend the problem from ordinary to partial differential equations will require more advanced techniques and may necessitate supplementary conditions. Moreover, non-zero very weak multiscale limits for $\left\{\varepsilon_k^{-1} u^\varepsilon\right\}$ may appear also for structures that are not perfectly periodic. A future aim in this connection would be to find an equivalent material with a periodic structure and a suitable number of scales that mimics the properties of the original material as well as possible on both macro and micro level. We will return to these questions in forthcoming papers.

5. Parabolic homogenization

The origin of very weak multiscale convergence is found in [8], where homogenization of parabolic equations is studied for different relations between the speed of oscillation in the respective fast spatial and temporal scales. Below we show how the generalization to multiple scales allows us to homogenize more complicated problems with several spatial and temporal scales. The scales are chosen to illustrate how the evolution version of very weak multiscale convergence can be applied both to reveal certain resonance phenomena and the vanishing of temporal scales that are too fast to be in resonance with an appropriate spatial scale. Without rapid temporal scales the homogenization procedure follows along the same lines as for the elliptic case worked out in [2]. With rapid oscillations in time there may appear negative exponents on ε in the weak formulations of (5.1) used to identify the local problems which necessitates the use of very weak multiscale convergence and hence the homogenization procedure is essentially different from the corresponding elliptic case. We use the notation for evolution function spaces found in e.g. Chapter 23 in [22].

We will investigate the parabolic problem

$$\begin{aligned} \partial_t u^\varepsilon(x,t) - \nabla\cdot\left(a\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2},\frac{t}{\varepsilon^4},\frac{t}{\varepsilon^5}\right)\nabla u^\varepsilon(x,t)\right) &= f(x,t) \text{ in } \Omega_T, \\ u^\varepsilon(x,t) &= 0 \text{ on } \partial\Omega\times(0,T), \\ u^\varepsilon(x,0) &= g(x) \text{ in } \Omega, \end{aligned} \tag{5.1}$$

where $\Omega_T = \Omega \times (0,T)$, $f \in L^2(\Omega_T)$ and $g \in L^2(\Omega)$. Moreover, we assume that

- (i): $a \in L^\infty_\sharp(\mathcal{Y}_{2,3})^{N\times N}$
- (ii): $a(y^2, s^3)\,\xi\cdot\xi \geq \alpha\,|\xi|^2$ for all $(y^2, s^3) \in \mathbb{R}^{2N}\times\mathbb{R}^3$, all $\xi\in\mathbb{R}^N$ and some $\alpha > 0$.

Under these structure conditions the problem (5.1) allows a unique solution u^ε contained in $W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$. Here $W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$ is the space of all functions in $L^2(0,T; H_0^1(\Omega))$ such that the time derivative belongs to $L^2(0,T; H^{-1}(\Omega))$. Moreover, for some positive constant C

$$\|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} < C$$

and

$$\|u^\varepsilon\|_{W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))} < C.$$

The concept in Definition 2.1 can be extended to involve also rapid oscillations in time. We define evolution multiscale convergence.

Definition 5.1. We say that a sequence $\{u^\varepsilon\}$ in $L^2(\Omega_T)$ $(n+1, m+1)$-scale converges to $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\int_{\Omega_T} u^\varepsilon(x,t) v\left(x, t, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon'_1}, ..., \frac{t}{\varepsilon'_m}\right) dxdt \to$$
$$\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} u_0(x,t,y^n,s^m) v\left(x,t,y^n,s^m\right) dy^n ds^m dxdt$$

for any $v \in L^2(\Omega; C_\sharp(\mathcal{Y}_{n,m}))$. This is written

$$u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x,t,y^n,s^m).$$

For $\{u^\varepsilon\}$ bounded in $W_2^1(0,T;H_0^1(\Omega), L^2(\Omega))$ there is a characterization of multiscale limits for gradients.

Theorem 5.2. *Let $\{u^\varepsilon\}$ be a bounded sequence in $W_2^1(0,T;H_0^1(\Omega), L^2(\Omega))$ and chose the scales $\varepsilon_1 = \varepsilon, \varepsilon_2 = \varepsilon^2, \varepsilon'_1 = \varepsilon^2, \varepsilon'_2 = \varepsilon^4$ and $\varepsilon'_3 = \varepsilon^5$. Then there exists a subsequence such that*

$$u^\varepsilon(x,t) \to u(x,t) \ \text{ in } L^2(\Omega_T),$$
$$u^\varepsilon(x,t) \rightharpoonup u(x,t) \ \text{ in } L^2(0,T;H_0^1(\Omega)),$$
$$u^\varepsilon(x,t) \overset{3,4}{\rightharpoonup} u(x,t)$$

and

$$\nabla u^\varepsilon(x,t) \overset{3,4}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1\left(x,t,y_1,s^3\right) + \nabla_{y_2} u_2\left(x,t,y^2,s^3\right),$$

where $u \in W_2^1(0,T;H_0^1(\Omega), L^2(\Omega))$, $u_1 \in L^2(\Omega_T \times S^3; H_\sharp^1(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,3}; H_\sharp^1(Y_2)/\mathbb{R})$.

In [6] a corresponding result is proven for the case with three spatial and two temporal scales, and the theorem above can be proven in an analogous way. To homogenize (5.1) we also need a generalization of very weak multiscale convergence.

Definition 5.3. We say that $\{g^\varepsilon\}$ in $L^1(\Omega_T)$ $(n+1, m+1)$-scale converges very weakly to $g_0 \in L^1(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\int_{\Omega_T} g^\varepsilon(x,t) v\left(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_{n-1}}\right) c\left(t, \frac{t}{\varepsilon'_1}, ..., \frac{t}{\varepsilon'_m}\right) \varphi\left(\frac{x}{\varepsilon_n}\right) dxdt \to$$
$$\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} g_0(x,t,y^n,s^m) v(x,y^{n-1}) c(t,s^m) \varphi(y_n)\, dy^n ds^m dxdt$$

for any $v \in D(\Omega, C_\sharp^\infty(Y^{n-1})), c \in D(0,T; C_\sharp^\infty(S^m))$ and $\varphi \in C_\sharp^\infty(Y_n)/\mathbb{R}$. We write

$$g^\varepsilon(x,t) \underset{vw}{\overset{n+1,m+1}{\rightharpoonup}} g_0(x,t,y^n,s^m).$$

We give the following theorem.

Theorem 5.4. *Let $\{u^\varepsilon\}$ be a bounded sequence in $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$.*

(i)**:** *Let the spatial scale be $\varepsilon_1(\varepsilon)=\varepsilon$ and the temporal scales $\varepsilon_1'(\varepsilon)=\varepsilon^2$, $\varepsilon_2'(\varepsilon)=\varepsilon^4$ and $\varepsilon_3'(\varepsilon)=\varepsilon^5$. Then there exists a subsequence such that*

$$\frac{u^\varepsilon(x,t)}{\varepsilon}\overset{2,4}{\underset{vw}{\rightharpoonup}}u_1(x,t,y_1,s^3).$$

(ii)**:** *Let the spatial scales be $\varepsilon_1(\varepsilon)=\varepsilon$ and $\varepsilon_2(\varepsilon)=\varepsilon^2$ and the temporal scales $\varepsilon_1'(\varepsilon)=\varepsilon^2$, $\varepsilon_2'(\varepsilon)=\varepsilon^4$ and $\varepsilon_3'(\varepsilon)=\varepsilon^5$. Then there exists a subsequence such that*

$$\frac{u^\varepsilon(x,t)}{\varepsilon^2}\overset{3,4}{\underset{vw}{\rightharpoonup}}u_2(x,t,y^2,s^3).$$

Here $u_1\in L^2(\Omega_T\times S^3;H_\sharp^1(Y_1)/\mathbb{R})$ and $u_2\in L^2(\Omega_T\times\mathcal{Y}_{1,3};H_\sharp^1(Y_2)/\mathbb{R})$ are the same as in Theorem 5.2.

Proof. The proof is a straightforward adaptation of the proof given in [5] if we just observe that any bounded sequence in $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ possesses a subsequence that converges strongly in $L^2(\Omega_T)$. Similar results for two fast spatial scales and one fast temporal scale are formulated in terms of Σ-convergence in [21]. □

We are now ready to prove the following homogenization result.

Theorem 5.5. *Let $\{u^\varepsilon\}$ be a sequence of solutions in $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ to (5.1). Then it holds that*

$$\begin{aligned} u^\varepsilon(x,t) &\to u(x,t) \text{ in } L^2(\Omega_T),\\ u^\varepsilon(x,t) &\rightharpoonup u(x,t) \text{ in } L^2(0,T;H_0^1(\Omega)) \end{aligned}$$

and

$$\nabla u^\varepsilon(x,t)\overset{3,4}{\rightharpoonup}\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2\left(x,t,y^2,s^2\right),$$

where $u\in W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ is the unique solution to

$$\begin{aligned} \partial_t u(x,t)-\nabla\cdot(b(x,t)\nabla u(x,t)) &= f(x,t) \text{ in } \Omega_T,\\ u(x,t) &= 0 \text{ on } \partial\Omega\times(0,T),\\ u(x,0) &= g(x) \text{ in } \Omega \end{aligned}$$

with

$$b(x,t)\nabla u(x,t)=\int_{\mathcal{Y}_{2,3}}a\left(y^2,s^3\right)(\nabla u+\nabla_{y_1}u_1+\nabla_{y_2}u_2))dy^2ds^3.$$

Here $u_1\in L^2(\Omega_T\times S_1;H_\sharp^1(Y_1)/\mathbb{R})$ and $u_2\in L^2(\Omega_T\times\mathcal{Y}_{1,2};H_\sharp^1(Y_2)/\mathbb{R})$ are the unique solutions to

$$\partial_{s_2}u_2-\nabla_{y_2}\cdot\left(\left(\int_{S_3}a\left(y^2,s^3\right)ds_3\right)(\nabla u+\nabla_{y_1}u_1+\nabla_{y_2}u_2)\right)=0$$

and

$$\partial_{s_1}u_1-\nabla_{y_1}\cdot\int_{S_2}\int_{Y_2}\left(\int_{S_3}a\left(y^2,s^3\right)ds_3\right)(\nabla u+\nabla_{y_1}u_1+\nabla_{y_2}u_2)\,dy_2ds_2=0.$$

Proof. Since $\{u^\varepsilon\}$ is bounded in $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ we can apply Theorem 5.2 and obtain that, for a suitable subsequence,

$$\begin{array}{rcl} u^\varepsilon(x,t) & \to & u(x,t) \text{ in } L^2(\Omega_T), \\ u^\varepsilon(x,t) & \rightharpoonup & u(x,t) \text{ in } L^2(0,T;H_0^1(\Omega)) \end{array}$$

and

$$\nabla u^\varepsilon(x,t) \overset{3,4}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1\left(x,t,y_1,s^3\right) + \nabla_{y_2} u_2\left(x,t,y^2,s^3\right),$$

where $u_1 \in L^2(\Omega_T \times S^3; H_\sharp^1(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,3}; H_\sharp^1(Y_2)/\mathbb{R})$.

To find the homogenized problem we use the weak form

$$\int_{\Omega_T} -u^\varepsilon(x,t)v(x)\partial_t c(t) + \int_{\Omega_T} a\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2},\frac{t}{\varepsilon^4},\frac{t}{\varepsilon^5}\right)\nabla u^\varepsilon(x,t)\cdot\nabla v(x)\,c(t)\,dxdt = \int_{\Omega_T} f(x,t)v(x)\,c(t)\,dxdt \tag{5.2}$$

of (5.1), where $v \in H_0^1(\Omega)$ and $c \in D(0,T)$, and letting $\varepsilon \to 0$ we obtain from Theorem 5.2

$$\int_{\Omega_T} -u(x,t)v(x)\partial_t c(t) + \left(\int_{\mathcal{Y}_{2,3}} a\left(y^2,s^3\right)\left(\nabla u(x,t) + \nabla_{y_1} u_1\left(x,t,y_1,s^3\right) + \nabla_{y_2} u_2\left(x,t,y^2,s^3\right)\right)\cdot\nabla v(x)\,c(t)\,dy^2 ds^3\right)dxdt = \int_{\Omega_T} f(x,t)v(x)\,c(t)\,dxdt.$$

We introduce the test functions

$$v(x) = \varepsilon^p v_1(x) v_2\left(\frac{x}{\varepsilon}\right) v_3\left(\frac{x}{\varepsilon^2}\right),\ c(t) = c_1(t)c_2\left(\frac{t}{\varepsilon^2}\right)c_3\left(\frac{t}{\varepsilon^4}\right)c_4\left(\frac{t}{\varepsilon^5}\right)$$

in (5.2), where, if nothing else is stated, we choose $v_1 \in D(\Omega)$, $v_2 \in C_\sharp^\infty(Y_1)$, $v_3 \in C_\sharp^\infty(Y_2)/\mathbb{R}$, $c_1 \in D(0,T)$, $c_2 \in C_\sharp^\infty(S_1)$, $c_3 \in C_\sharp^\infty(S_2)$ and $c_4 \in C_\sharp^\infty(S_3)$. We get

$$\int_{\Omega_T} -u^\varepsilon(x,t)\,\varepsilon^p v_1(x)v_2\left(\frac{x}{\varepsilon}\right)v_3\left(\frac{x}{\varepsilon^2}\right)\partial_t\left(c_1(t)c_2\left(\frac{t}{\varepsilon^2}\right)c_3\left(\frac{t}{\varepsilon^4}\right)c_4\left(\frac{t}{\varepsilon^5}\right)\right) + a\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2},\frac{t}{\varepsilon^4},\frac{t}{\varepsilon^5}\right)\nabla u^\varepsilon(x,t)\cdot \nabla\left(\varepsilon^p v_1(x)\,v_2\left(\frac{x}{\varepsilon}\right)v_3\left(\frac{x}{\varepsilon^2}\right)\right)c_1(t)\,c_2\left(\frac{t}{\varepsilon^2}\right)c_3\left(\frac{t}{\varepsilon^4}\right)c_4\left(\frac{t}{\varepsilon^5}\right)dxdt = \int_{\Omega_T} f(x,t)\varepsilon^p v_1(x)v_2\left(\frac{x}{\varepsilon}\right)v_3\left(\frac{x}{\varepsilon^2}\right)c_1(t)c_2\left(\frac{t}{\varepsilon^2}\right)c_3\left(\frac{t}{\varepsilon^4}\right)c_4\left(\frac{t}{\varepsilon^5}\right)dxdt$$

and differentiating we have

$$\begin{aligned}
&\int_{\Omega_T} -u^\varepsilon(x,t)\, v_1(x) v_2\left(\frac{x}{\varepsilon}\right) v_3\left(\frac{x}{\varepsilon^2}\right)\Bigg(\varepsilon^p \partial_t c_1(t) c_2\left(\frac{t}{\varepsilon^2}\right) c_3\left(\frac{t}{\varepsilon^4}\right) c_4\left(\frac{t}{\varepsilon^5}\right) + \\
&\varepsilon^{p-2} c_1(t)\partial_{s_1} c_2\left(\frac{t}{\varepsilon^2}\right) c_3\left(\frac{t}{\varepsilon^4}\right) c_4\left(\frac{t}{\varepsilon^5}\right) + \varepsilon^{p-4} c_1(t) c_2\left(\frac{t}{\varepsilon^2}\right)\partial_{s_2} c_3\left(\frac{t}{\varepsilon^4}\right) c_4\left(\frac{t}{\varepsilon^5}\right) + \\
&\varepsilon^{p-5} c_1(t) c_2\left(\frac{t}{\varepsilon^2}\right) c_3\left(\frac{t}{\varepsilon^4}\right)\partial_{s_3} c_4\left(\frac{t}{\varepsilon^5}\right)\Bigg) + \\
&a\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^2},\frac{t}{\varepsilon^4},\frac{t}{\varepsilon^5}\right)\nabla u^\varepsilon(x,t)\cdot\Bigg(\varepsilon^p \nabla v_1(x) v_2\left(\frac{x}{\varepsilon}\right) v_3\left(\frac{x}{\varepsilon^2}\right) + \\
&\varepsilon^{p-1} v_1(x)\nabla_{y_1} v_2\left(\frac{x}{\varepsilon}\right) v_3\left(\frac{x}{\varepsilon^2}\right) + \varepsilon^{p-2} v_1(x) v_2\left(\frac{x}{\varepsilon}\right)\nabla_{y_2} v_3\left(\frac{x}{\varepsilon^2}\right)\Bigg)\cdot \\
&c_1(t) c_2\left(\frac{t}{\varepsilon^2}\right) c_3\left(\frac{t}{\varepsilon^4}\right) c_4\left(\frac{t}{\varepsilon^5}\right) dxdt = \\
&\int_{\Omega_T} f(x,t)\varepsilon^p v_1(x) v_2\left(\frac{x}{\varepsilon}\right) v_3\left(\frac{x}{\varepsilon^2}\right) c_1(t) c_2\left(\frac{t}{\varepsilon^2}\right) c_3\left(\frac{t}{\varepsilon^4}\right) c_4\left(\frac{t}{\varepsilon^5}\right) dxdt.
\end{aligned} \tag{5.3}$$

We will investigate the equation (5.3) for different choices of test functions and values of p.

To begin with we let $p = 3$ and by (ii) in Theorem 5.4 we get

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,3}} -u_2(x,t,y^2,s^3) v_1(x) v_2(y_1) v_3(y_2) c_1(t) c_2(s_1) c_3(s_2)\partial_{s_3} c_4(s_3)\, dy^2 ds^3 dxdt = 0$$

and, by the variational lemma,

$$\int_{S_3} -u_2(x,t,y^2,s^3)\partial_{s_3} c_4(s_3)\, ds_3 = 0$$

a.e. in $\Omega_T \times \mathcal{Y}_{2,2}$. This means that u_2 does not depend on s_3 and hence belongs to the space $L^2(\Omega_T \times \mathcal{Y}_{1,2}; H^1_\sharp(Y_2)/\mathbb{R})$.

Next we choose $p = 4$, $v_3 = 1$ and $v_2 \in C^\infty_\sharp(Y_1)/\mathbb{R}$ in (5.3). Applying (i) in Theorem 5.4 on the third and fourth term in (5.3) we obtain

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,3}} -u_1(x,t,y_1,s^3) v_1(x) v_2(y_1) c_1(t) c_2(s_1) c_3(s_2)\partial_{s_3} c_4(s_3)\, dy^2 ds^3 dxdt = 0,$$

i.e.,

$$\int_{S_3} -u_1(x,t,y_1,s^3)\partial_{s_3} c_4(s_3)\ ds_3 = 0$$

a.e. in $\Omega_T \times \mathcal{Y}_{1,2}$ and hence u_1 is also independent of s_3. In the sequel we will use the variational lemma with respect to the appropriate sets in the corresponding way without comments.

Letting $p = 3$ and $c_4 = v_3 = 1$ and $v_2 \in C^\infty_\sharp(Y_1)/\mathbb{R}$ we get by Theorem 5.4 (i)

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,3}} -u_1(x,t,y_1,s^2) v_1(x) v_2(y_1) c_1(t) c_2(s_1)\partial_{s_2} c_3(s_2)\, dy^2 ds^3 dxdt = 0.$$

Again using the variational lemma together with the fact that u_1 is independent of s_3 we end up with

$$\int_{S_2} -u_1(x,t,y_1,s^2)\partial_{s_2}c_3\left(s_2\right)ds_2 = 0,$$

that is, u_1 is independent of s_2 as well and thus $u_1 \in L^2(\Omega_T \times S_1; H^1_\sharp(Y_1)/\mathbb{R})$.

To find the first local problem we now let $p = 2$ and $c_4 = 1$. Using Theorem 5.2 and (ii) in Theorem 5.4 we find that

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{2,3}} &-u_2(x,t,y^2,s^2)v_1(x)v_2\left(y_1\right)v_3\left(y_2\right)c_1(t)c_2\left(s_1\right)\partial_{s_2}c_3\left(s_2\right) + \\ &a\left(y^2,s^3\right)\left(\nabla u\left(x,t\right)+\nabla_{y_1}u_1\left(x,t,y_1,s_1\right)+\nabla_{y_2}u_2\left(x,t,y^2,s^2\right)\right)\cdot \\ &v_1(x)v_2\left(y_1\right)\nabla_{y_2}v_3\left(y_2\right)c_1(t)c_2\left(s_1\right)c_3\left(s_2\right)dy^2ds^3dxdt = 0\end{aligned}$$

and hence

$$\begin{aligned}&\int_{S_2}\int_{Y_2} -u_2(x,t,y^2,s^2)v_3\left(y_2\right)\partial_{s_2}c_3\left(s_2\right) + \\ &\left(\left(\int_{S_3} a\left(y^2,s^3\right)ds_3\right)\left(\nabla u\left(x,t\right)+\nabla_{y_1}u_1\left(x,t,y_1,s_1\right)+\nabla_{y_2}u_2\left(x,t,y^2,s^2\right)\right)\right)\cdot \\ &\nabla_{y_2}v_3\left(y_2\right)c_3\left(s_2\right)dy_2ds_2 = 0,\end{aligned}$$

which is the weak form of the first local problem for (5.1).

Next we choose $p = 1$, $c_3 = c_4 = v_3 = 1$ and $v_2 \in C^\infty_\sharp\left(Y_1\right)/\mathbb{R}$ and receive by Theorem 5.2 and (i) in Theorem 5.4

$$\begin{aligned}&\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} -u_1(x,t,y_1,s_1)v_1(x)v_2\left(y_1\right)c_1(t)\partial_{s_1}c_2\left(s_1\right) + \\ &\left(\int_{S_2}\int_{Y_2}\left(\int_{S_3}a\left(y^2,s^3\right)ds_3\right)\left(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2\left(x,t,y^2,s^2\right)\right)dy_2ds_2\right)\cdot \\ &v_1(x)\nabla_{y_1}v_2\left(y_1\right)c_1(t)c_2\left(s_1\right)dy_1ds_1dxdt = 0\end{aligned}$$

and we end up with

$$\begin{aligned}&\int_{\mathcal{Y}_{1,1}} -u_1(x,t,y_1,s_1)v_2\left(y_1\right)\partial_{s_1}c_2\left(s_1\right) + \\ &\left(\int_{S_2}\int_{Y_2}\left(\int_{S_3}a\left(y^2,s^3\right)ds_3\right)\left(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2\left(x,t,y^2,s^2\right)\right)dy_2ds_2\right)\cdot \\ &\nabla_{y_1}v_2\left(y_1\right)c_2\left(s_1\right)dy_1ds_1 = 0,\end{aligned}$$

the weak form of the second local problem. Since the functions u_1 and u_2 are uniquely determined by the local problems the result holds for the entire sequence and not just for a subsequence. □

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