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ON CERTAIN PROJECTIONS OF C^*-MATRIX ALGEBRAS

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ABSTRACT. In 1955, H. Dye defined certain projections of a C^*-matrix algebra by

$$\begin{aligned} P_{i,j}(a) &= (1+aa^*)^{-1} \otimes E_{i,i} + (1+aa^*)^{-1}a \otimes E_{i,j} \\ &+ a^*(1+aa^*)^{-1} \otimes E_{j,i} + a^*(1+aa^*)^{-1}a \otimes E_{j,j}, \end{aligned}$$

which was used to show that in the case of factors not of type I_{2n}, the unitary group determines the algebraic type of that factor. We study these projections and we show that in $\mathbb{M}_2(\mathbb{C})$, the set of such projections includes all the projections. For infinite C^*-algebra A, having a system of matrix units, we have $A \simeq \mathbb{M}_n(A)$. M. Leen proved that in a simple, purely infinite C^*-algebra A, the $*$-symmetries generate $\mathcal{U}_0(A)$. Assuming $K_1(A)$ is trivial, we revise Leen's proof and we use the same construction to show that any unitary close to the unity can be written as a product of eleven $*$-symmetries, eight of such are of the form $1 - 2P_{i,j}(\omega)$, $\omega \in \mathcal{U}(A)$. In simple, unital purely infinite C^*-algebras having trivial K_1-group, we prove that all $P_{i,j}(\omega)$ have trivial K_0-class. Consequently, we prove that every unitary of $\mathcal{O}_n$ can be written as a finite product of $*$-symmetries, of which a multiple of eight are conjugate as group elements.

1. INTRODUCTION AND PRELIMINARIES

Let A be a unital C^*-algebra. The set of projections and the group of unitaries of A are denoted by $\mathcal{P}(A)$ and $\mathcal{U}(A)$, respectively. Recall that the C^*-matrix algebra over A which is denoted by $\mathbb{M}_n(A)$ is the algebra of all $n \times n$ matrices $(a_{i,j})$ over A, with the usual addition, scalar multiplication, and multiplication of matrices and the involution (adjoint) is $(a_{i,j})^* = (a^*_{j,i})$. As in Dye's viewpoint of $\mathbb{M}_n(A)$, let $S_n(A)$ denote the direct sum of n copies of A, considered as a left

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A-module. Addition of n-tuples $\bar{x} = (x_1, x_2, \dots, x_n)$ in $S_n(A)$ is componentwise and $a \in A$ acts on $\bar{x}$ by $a(\bar{x}) = (ax_1, ax_2, \dots, ax_n)$. Then $S_n(A)$ is a Hilbert C^*-algebra module, with the inner product defined by

$$< \bar{x}, \bar{y} >= \sum_{i=1}^{n} x_i y_i^*.$$

By an A-endomorphism T of $S_n(A)$, we mean an additive mapping on $S_n(A)$ which commutes with left multiplication: $a(\bar{x}T) = (a\bar{x})T$. In a familiar way, assign to any T a uniquely determined matrix (t_{ij}) over A $(1 \le i, j \le n)$ so that $\bar{x}T = (\sum_i x_i t_{i1}, \dots, \sum_i x_i t_{in})$.

If p is a projection in $\mathbb{M}_n(A)$, then p is a mapping on $S_n(A)$ having its range as a sub-module of $S_n(A)$. Then two projections are orthogonal means their sub-module ranges are so. The C^*-algebra $\mathbb{M}_n(A)$ contains numerous projections. For each $a \in A$ and each pair of indices $i, j (i \neq j,\ 1 \le i, j \le n)$, H. Dye in [7] defined the projection $P_{i,j}(a)$ in $\mathbb{M}_n(A)$, whose range consists of all left multiples of the vector with 1 in the i^{th}-place, a in the j^{th}-place and zeros elsewhere. As a matrix, it has the form

$$P_{i,j}(a) = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & \cdots & (1+aa^*)^{-1} & \cdots & (1+aa^*)^{-1}a & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & \cdots & a^*(1+aa^*)^{-1} & \cdots & a^*(1+aa^*)^{-1}a & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Recall that (see [7], p.74) a system of matrix units of a unital C^*-algebra A is a subset $\{e^r_{i,j}\}, 1 \le i, j \le n$ and $1 \le r \le m$ of A, such that

$$e^r_{i,j} e^r_{j,k} = e^r_{i,k},\ e^r_{i,j} e^s_{k,l} = 0 \text{ if } r \neq\ s \text{ or } j \neq k,\ (e^r_{i,j})^* = e^r_{j,i},\ \sum_{i,r}^{n,m} e^r_{i,i} = 1$$

and for every i, $e^r_{i,i} \in \mathcal{P}(A)$. For the C^*-complex matrix algebra $\mathbb{M}_n(\mathbb{C})$, let $\{E_{i,j}\}^n_{i,j=1}$ denote the standard system of matrix units of the algebra, that is $E_{i,j}$ is the $n \times n$ matrix over $\mathbb{C}$ with 1 at the place $i \times j$ and zeros elsewhere. It is also known that $\mathbb{M}_n(A)$ is $*$-isomorphic to $A \otimes \mathbb{M}_n(\mathbb{C})$ (see [11]). We will see that having a system of matrix units is a necessary condition in order that a C^*-algebra A is $*$-isomorphic to a C^*-matrix algebra $\mathbb{M}_n(B)$. Using the notion of a system of matrix units, we write

$$\begin{aligned} P_{i,j}(a) &= (1+aa^*)^{-1} \otimes E_{i,i} + (1+aa^*)^{-1}a \otimes E_{i,j} \\ &+ a^*(1+aa^*)^{-1} \otimes E_{j,i} + a^*(1+aa^*)^{-1}a \otimes E_{j,j} \in \mathcal{P}(\mathbb{M}_n(A)). \end{aligned}$$

If $a = 0$, then $P_{i,j}(a)$ is the i^{th} diagonal matrix unit of $\mathbb{M}_n(A)$, which is $1 \otimes E_{i,i}$, or simply E_i.

Also in [10], M. Stone called the projection $P_{i,j}(a)$ by the characteristics matrix of a.

H. Dye used these projections as a main tool to prove that an isomorphism between the discrete unitary groups of von Neumann factors not of type I_n, is implemented by a $*$-isomorphism between the factors themselves [[7], Theorem 2]. Indeed, let us recall main parts of his proof. Let A and B be two unital C^*-algebras and let $\varphi : \mathcal{U}(A) \to \mathcal{U}(B)$ be an isomorphism. As φ preserves self-adjoint unitaries, it induces a natural bijection $\theta_\varphi : \mathcal{P}(A) \to \mathcal{P}(B)$ between the sets of projections of A and B given by

$$1 - 2\theta_\varphi(p) = \varphi(1 - 2p),\ p \in \mathcal{P}(A).$$

This mapping is called a projection orthoisomorphism, if it preserves orthogonality, i.e. $pq = 0$ iff $\theta(p)\theta(q) = 0$.

Now, let θ be an orthoisomorphism from $\mathcal{P}(\mathbb{M}_n(A))$ onto $\mathcal{P}(\mathbb{M}_n(B))$. In [[7], Lemma 8] when A and B are von Neumann algebras, Dye proved that for any unitary $u \in \mathcal{U}(A)$, $\theta(P_{i,j}(u)) = P_{i,j}(v)$, for some unitary $v \in \mathcal{U}(B)$. A similar result is proved in the case of simple, unital C^*-algebras by the author in [1]. Afterwards, Dye in [[7], Lemma 6], proved that there exists a $*$-isomorphism (or $*$-antiisomorphism) from $\mathbb{M}_n(A)$ onto $\mathbb{M}_n(B)$ which coincides with θ on the projections $P_{i,j}(a)$. In fact, he proved that θ induces the $*$-isomorphism ϕ from A onto B defined by the relation $\theta(P_{i,j}(a)) = P_{i,j}(\phi(a))$.

In this paper, we study the projections $P_{i,j}(a)$ of a C^*-matrix algebra $\mathbb{M}_n(A)$, for some C^*-algebra A, and we deduce main results concerning such projections. The paper is organized as follows: In Section 2, we show that every projection in $\mathbb{M}_2(\mathbb{C})$ is of the form $P_{1,2}(a)$, for $a \in \mathbb{C}$. In Section 3, we show that some infinite C^*-algebra A is isomorphic to its matrix algebra $\mathbb{M}_n(A)$, such as the Cuntz algebra $\mathcal{O}_n$, so the projections $P_{i,j}(a)$ can be considered as projections of A.

In a simple, unital purely infinite C^*-algebra A, M. Leen proved that self-adjoint unitaries (also called $*$-symmetries, or involutions) generate the connected component $\mathcal{U}_0(A)$ of the unitary group $\mathcal{U}(A)$. In Section 4, assuming in addition that $K_1(A)$ is trivial, we revise Leen's proof, we fix certain projections and then following the same construction, we show that every unitary which is close to the unity, can be written as a product of eleven $*$-symmetries, eight of which are of the form $1 - 2P_{i,j}(\omega),\ \omega \in \mathcal{U}(A)$.

Consequently, since every unitary in the connected component of the unity can be written as a finite product of unitaries that are close to the unity (see [11], § 4.2), we have the following result:

Theorem 1.1. *Let A be a simple, unital purely infinite C^*-algebra, such that $K_1(A) = 0$ and for $n \geq 3$, let $\{e_{i,j}\}_{i,j=1}^n$ be a system of matrix units of A, with $e_{1,1} \sim 1$. Then every unitary of A can be written as a finite product of $*$-symmetries, of which a multiple of eight have the form $1 - 2P_{i,j}(\omega)$, for some $\omega \in \mathcal{U}(A)$.*

Finally in Section 5, we compute the K_0-class of such certain projections, and we prove that in simple, unital purely infinite C^*-algebras (assuming $K_1 = 0$), all projections of the form $P_{i,j}(u),\ u \in \mathcal{U}(A)$ have trivial K_0-class. As a good application for $\mathcal{O}_n$, we have that every unitary can be written as a finite product of $*$-symmetries, of which a multiple of eight have the form $1-2P_{i,j}(\omega),\ \omega \in \mathcal{U}(\mathcal{O}_n)$. Hence using [2] (Lemma 2.1), all such involutions of the form $1-2P_{i,j}(\omega)$ are in fact conjugate, as group elements of $\mathcal{U}(\mathcal{O}_n)$.

2. The 2×2-Complex Algebra Case

Let A be a unital C^*-algebra, and let $\mathcal{P}^n_{i,j}(A)$ denote the family of all projections in $\mathbb{M}_n(A)$ of the form $P_{i,j}(a),\ 1 \le i,j \le n,\ a \in A$. Also, let $\mathcal{U}^n_{i,j}(A)$ denote the set of all self-adjoint unitaries in $\mathbb{M}_n(A)$ of the form $1-2P_{i,j}(a),\ 1 \le i,j \le n,\ a \in A$. Notice that $\mathcal{P}^n_{i,j}(A)$ contains non-trivial projections. In this small section, we show that in the case of $\mathbb{M}_2(\mathbb{C})$, the set $\mathcal{P}^2_{i,j}(\mathbb{C})$ includes all the non-trivial projections $\mathcal{P}(\mathbb{M}_2(\mathbb{C}))$, i.e. every non-trivial projection is of the form $P_{i,j}(a)$, for some complex number a.

Proposition 2.1. *If $p \in \mathcal{P}(\mathbb{M}_2(\mathbb{C}))\backslash\{0,1\}$, then $p \in \mathcal{P}^2_{i,j}(\mathbb{C})$.*

Proof. Let $p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a non-trivial projection in $\mathcal{P}(\mathbb{M}_2(\mathbb{C}))$. Then a and d are real numbers. If $b=0$, then p is either the diagonal matrix unit $E_{1,1}$ or $E_{2,2}$. Otherwise, we have $a+b=1, a=a^2+|b|^2$ and $d=d^2+|b|^2$, therefore $|b|^2 \le \frac{1}{4}$. By strightforward computations, one can deduce that p is of the form

$$P_{1,2}\left(\frac{2b}{1+\sqrt{1-4|b|^2}}\right), \text{ or } \quad P_{1,2}\left(\frac{2b}{1-\sqrt{1-4|b|^2}}\right).$$

□

Remark 2.2. The projections in $\mathcal{P}^n_{i,j}(\mathbb{C})$ are all of rank one by definition, this implies that in the case of $\mathbb{M}_3(\mathbb{C})$, the set $\mathcal{P}^3_{i,j}(\mathbb{C})$ does not cover all the non-trivial projections. Indeed, there are projections in $\mathcal{P}(\mathbb{M}_3(\mathbb{C}))$ of rank one which do not belong to $\mathcal{P}^3_{i,j}(\mathbb{C})$, since every projection in this latest family projects into a subspace of $\mathbb{C}^3$ which lies entirely in one coordinate plan.

3. Some Results for infinite C^*-algebras

Let A be a unital C^*-algebra having a system of matrix units $\{e_{i,j}\}^n_{i,j=1}$, for some $n \ge 3$. Recall that $e_{1,1}Ae_{1,1}$ is a C^*-algebra (corner algebra) which has $e_{1,1}$ as a unit. This system of matrix units implements a $*$-isomorphism between A and $\mathbb{M}_n(e_{1,1}Ae_{1,1})$. Indeed, let us define the mapping

$$\eta_1 : \mathbb{M}_n(e_{1,1}Ae_{1,1}) \to A$$

by

$$\eta_1((a_{i,j})^n) = \sum_{i,j=1}^{n} e_{i,1}a_{i,j}e_{1,j}.$$

Moreover if $e_{1,1}$ is equivalent to 1 (i.e. A is assumed to be an infinite C^*-algebra), then there exists a partial isometry v of A such that $v^*v = e_{1,1}$ and $vv^* = 1$, and this defines the $*$-isomorphism $\Delta_v : A \to e_{1,1}Ae_{1,1}$ by $\Delta_v(x) = v^*xv$. The isomorphism Δ_v can be used to decompose a projection as a sum of orthogonal equivalent projections.

Proposition 3.1. *Let A be a unital C^*-algebra having a system of matrix units $\{e_{i,j}\}_{i=1}^n$. If p is equivalent to the unity, then p can be written as a sum of orthogonal equivalent subprojections.*

Proof. As p equivalent to 1, we consider the isomorphism Δ_v, then apply it to the equality $1 = \sum_{i=1}^n e_{i,i}$, to get $p = \sum_{i=1}^n v^* e_{i,i} v$. Then $p_i = v^* e_{i,i} v$, for all $1 \le i \le n$, are equivalent subprojections of p. $\square$

Recall that, for two unital C^*-algebras A and B, if $\alpha : A \to B$ is a $*$-isomorphism, then α induces the $*$-isomorphism $\widehat{\alpha} : \mathbb{M}_n(A) \to \mathbb{M}_n(B)$, which is defined by $(a_{i,j}) \mapsto (\alpha(a_{i,j}))$. Then we have the following result.

Proposition 3.2. *Let A be an infinite unital C^*-algebra having a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$. If $e_{1,1}$ is equivalent to 1, then $\mathbb{M}_n(A)$ is $*$-isomorphic to A.*

Proof. Let $\Delta_v : A \to e_{1,1} A e_{1,1}$ and $\eta_1 : \mathbb{M}_n(e_{1,1} A e_{1,1}) \to A$ be defined as above. Then the mapping $\eta = \eta_1 \circ \widehat{\Delta_v}$ is a $*$-isomorphism from $\mathbb{M}_n(A)$ onto A. Moreover,

$$\eta(a_{i,j})^n = \sum_{i,j}^n e_{i,1} v^* a_{i,j} v e_{1,j}, \text{ and}$$

$$\eta^{-1}(x) = (v e_{1,i} x e_{j,1} v^*)_{i,j}^n.$$

$\square$

As a main example of purely infinite C^*-algebras, let us recall the Cuntz algebra $\mathcal{O}_n$; $n \ge 2$, is the universal C^*-algebra which is generated by isometries $s_1, s_2, \ldots, s_n$, such that $\sum_{i=1}^n s_i s_i^* = 1$ with $s_i^* s_j = 0$, when $i \ne j$ and $s_i^* s_i = 1$ (for more details, see [5], [[6], p.149]). Let

$$e_{i,j} = s_i s_j^*, \qquad 1 \le i, j \le n \ .$$

Then $\{e_{i,j}\}_{i,j=1}^n$ forms a system of matrix units for $\mathcal{O}_n$. As s_1^* partial isometry between $e_{1,1}$ and the unity, then Proposition 3.2 shows that the mapping

$$\eta : \mathbb{M}_n(\mathcal{O}_n) \to \mathcal{O}_n, \quad (a_{i,j})_{i,j} \mapsto \sum_{i,j=1}^n s_i a_{i,j} s_j^*$$

is a $*$-isomorphism. Moreover, for $x \in \mathcal{O}_n$, $\eta^{-1}(x) = (s_i^* x s_j)_{i,j} \in \mathbb{M}_n(\mathcal{O}_n)$.
Therefore, we have proved the following result, which is in fact known, but for sake of completeness:

Proposition 3.3. *The Cuntz algebra $\mathcal{O}_n$ is isomorphic to the C^*-algebra $\mathbb{M}_n(\mathcal{O}_n)$.*

Then for $a \in \mathcal{O}_n$, $P_{i,j}(a)$ are considered as projections of $\mathcal{O}_n$ by applying the mapping η. Therefore,

$$P_{i,j}(a) = s_i(1+aa^*)^{-1}s_i^* + s_i(1+aa^*)^{-1}as_j^* + s_j a^*(1+aa^*)^{-1}s_i^* + s_j a^*(1+aa^*)^{-1}as_j^*.$$

4. Unitary Factors in Purely Infinite C^*-Algebras

Recall that in a unital C^*-algebra A, every self-adjoint unitary u can be written as $u = 1 - 2p$, for some projection $p \in \mathcal{P}(A)$, let us say " the self-adjoint unitary u is associated to the projection p". In this section, we assume that A is purely infinite simple C^*-algebra, and we study the factorizations of unitaries of A. In order to prove our main theorem (Theorem 4.2), let us first recall the following result of M. Leen.

Theorem 4.1 ([9], Theorem 3.8)**.** *Let A be a simple, unital purely infinite C^*-algebra. Then the $*$-symmetries (self-adjoint unitaries) generate the connected component of the unity $\mathcal{U}_0(A)$.*

Now, consider a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$ of A, with $e_{1,1} \sim 1$. Let us recall the $*$-isomorphisms $\eta_1 : \mathbb{M}_n(e_{1,1}Ae_{1,1}) \to A$, and $\eta = \eta_1 \circ \widehat{\Delta_v}$ from $\mathbb{M}_n(A)$ onto A. In this section we revise Leens' proof of Theorem 3.5 in [9] and we fix some projections, then by following the same construction, we prove the following main theorem, which shows that every unitary of A which lies within a neighborhood of the unity can be factorized as a product of eleven self-adjoint unitaries moreover, eight of such factors are associated to the projections $P_{i,j}(\mu)$, for some $\mu \in \mathcal{U}(A)$.

Theorem 4.2. *Let A be a simple, unital purely infinite C^*-algebra, such that $K_1(A) = 0$ and for $n \geq 3$, let $\{e_{i,j}\}_{i,j=1}^n$ be a system of matrix units of A, with $e_{1,1} \sim 1$. Then there exists $\epsilon > 0$ such that every unitary a of A with $\|a - 1\| < \epsilon$ can be written as a product of eleven self-adjoint unitaries, of which eight have the form:*

$$1 - 2\eta(P_{1,2}(-\alpha)),\ 1 - 2\eta(P_{1,2}(-1))$$

$$1 - 2\eta(P_{1,3}(-\alpha)),\ 1 - 2\eta(P_{1,3}(-1))$$

$$1 - 2\eta(P_{1,2}(-\gamma)),\ 1 - 2\eta(P_{1,2}(-1))$$

$$1 - 2\eta(P_{1,3}(-\gamma)),\ 1 - 2\eta(P_{1,3}(-1)),$$

for some $\alpha, \gamma \in \mathcal{U}(A)$.

Consequently, as the Cuntz algebra is simple, unital purely infinite C^*-algebra with $K_1(\mathcal{O}_n) = 0$ (see [4]) and using Proposition 3.3, we have the following result.

Corollary 4.3. *Let n be given, there is a positive number ϵ such that if $u \in \mathcal{U}(\mathcal{O}_n)$ with $\|u - 1\| < \epsilon$, then*

$$\begin{aligned} u &= z_1(1 - 2P_{1,2}(-\alpha))(1 - 2P_{1,2}(-1))(1 - 2P_{1,3}(-\alpha))(1 - 2P_{1,3}(-1)) \\ &\quad (1 - 2P_{1,2}(-\gamma))(1 - 2P_{1,2}(-1))(1 - 2P_{1,3}(-\gamma))(1 - 2P_{1,3}(-1))z_2z_3, \end{aligned}$$

for some self-adjoint unitaries z_1, z_2, z_3 and $\alpha, \gamma \in \mathcal{U}(\mathcal{O}_n)$.

Let us introduce the following lemma which is used by M. Leen in his proof, and we shall use it as well.

Lemma 4.4. *Let A be a simple, unital purely infinite C^*-algebra, and let ρ be a non-trivial projections of A. There is a positive number ϵ such that if $a \in \mathcal{U}_0(A)$ with $\|a-1\| < \epsilon$, then there exist self-adjoint unitaries z_1, z_2, z_3 of A and $x \in \mathcal{U}_0(\rho A \rho)$ such that*

$$z_1 a z_2 z_3 = \begin{pmatrix} x & 0 \\ 0 & 1-\rho \end{pmatrix}.$$

Proof. Mimic the first part of the proof of Theorem 3.5 in [9], with replacing symmetries by $*$-symmetries and invertible by unitaries. □

Proof of Theorem 4.2:

Proof. Since A is a simple, unital purely infinite C^*-algebra, using [4], we have $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}_0(A)$. As $K_1(A)$ is assumed to be trivial, we have $\mathcal{U}(A) = \mathcal{U}_0(A)$.

Let $p = e_{1,1}$, as $p \sim 1$, use Proposition 3.1 and the isomorphism Δ_u ($u^*u = e_{1,1}, uu^* = 1$) to find a projection $p_1 < p$ (precisely, $p_1 = u^* e_{1,1} u$) which is equivalent to p moreover, set the partial isometry $v = u^* e_{1,1}$, and put $\rho = p - p_1$, so ρ is a non-trivial projection. Therefore applying Lemma 4.4, there is a positive number ϵ such that if $a \in \mathcal{U}(A)$ with $\|a-1\| < \epsilon$, then there exist self-adjoint unitaries z_1, z_2 and z_3 such that

$$z_1 a z_2 z_3 = \begin{pmatrix} x & 0 \\ 0 & 1-\rho \end{pmatrix},$$

where $x \in \mathcal{U}(\rho A \rho)$.

Now, we shall use Leen's approach to exhibit the desired factorization of a. Choose $q = e_{2,2}$, $r = e_{3,3}$ and put $r_1 = p + q + r$, then we have $q \sim r < 1 - p - q$. Following Leen's notations, we choose $v_1 = e_{2,1}, v_2 = e_{3,2}$ and $v_3 = e_{1,3}$, so v_1, v_2 and v_3 are partial isometries such that

$$v_1^* v_1 = p,\ v_1 v_1^* = q,\ v_2^* v_2 = q,\ v_2 v_2^* = r,\ v_3^* v_3 = r,\ \text{and } v_3 v_3^* = p.$$

Let $w = v_1 + v_2 + v v_3$. Then following the construction in Leen's proof, we get

$$\begin{aligned} z_1 a z_2 z_3 \quad = \quad & (1 - 2\eta_1(P_{1,2}(-\alpha_p)))(1 - 2\eta_1(P_{1,2}(-p))) \\ & (1 - 2\eta_1(P_{1,3}(-\alpha_p)))(1 - 2\eta_1(P_{1,3}(-p))) \\ & (1 - 2\eta_1(P_{1,2}(-\gamma_p)))(1 - 2\eta_1(P_{1,2}(-p))) \\ & (1 - 2\eta_1(P_{1,3}(-\gamma_p)))(1 - 2\eta_1(P_{1,3}(-p))) \end{aligned}$$

where α_p and γ_p are in $\mathcal{U}(pAp)$. Notice that the factors in the right hand side are self-adjoint unitaries in A. Hence using the mapping η, we then get

$$\begin{aligned} a \quad = z_1 \quad & (1 - 2\eta(P_{1,2}(-\alpha)))(1 - 2\eta(P_{1,2}(-1))) \\ & (1 - 2\eta(P_{1,3}(-\alpha)))(1 - 2\eta(P_{1,3}(-1))) \\ & (1 - 2\eta(P_{1,2}(-\gamma)))(1 - 2\eta(P_{1,2}(-1))) \\ & (1 - 2\eta(P_{1,3}(-\gamma)))(1 - 2\eta(P_{1,3}(-1))) z_3 z_2 \end{aligned}$$

where α and γ are unitaries in A, and this ends the proof. □

Finally, let us finish this section by presenting the following open question:
Q. In the Cuntz algebra $\mathcal{O}_n$, do self-adjoint unitaries of the form $\{1 - 2P_{i,j}(a)\}$ generate the unitary group $\mathcal{U}(\mathcal{O}_n)$?

5. K-Theory of Certain Projections

In this section, we study the K_0-class of the projections $P_{i,j}(u)$, where u is a unitary of some unital C^*-algebra A. In particular, if A is a simple purely infinite C^*-algebra, with $K_1(A) = 0$, or A is a von Neumann factor of type II_1, or III, then for any unitary u of A, $P_{i,j}(u)$ has trivial K_0-class. Afterwards, we present an application of Theorem 4.2, to the case of Cuntz algebras.

Proposition 5.1. *Let A be a unital C^*-algebra. If v is a unitary in A of finite order, then $[P_{i,j}(v)] = [1]$ in $K_0(A)$.*

Proof. Consider a unitary v in A, such that $v^m = 1$, for some positive integer m. For $i \neq j$, let

$$W = \frac{1}{\sqrt{2}}(v \otimes E_{i,i} + v \otimes E_{i,j} + E_{j,i} - E_{j,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k}) \ ,$$

then $W^* = \frac{1}{\sqrt{2}}(v^{m-1} \otimes E_{i,i} + E_{i,j} + v^{m-1} \otimes E_{j,i} - E_{j,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k})$, therefore $W \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover,

$$\begin{aligned} W^* P_{i,j}(v) W &= \frac{1}{4}(2v^{m-1} \otimes E_{i,i} + 2 \otimes E_{i,j})(\sqrt{2}W) \\ &= \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \text{(1 at the i-th place)} \\ &= E_{i,i}. \end{aligned}$$

This implies that the projection $P_{i,j}(v)$ is unitarily equivalent to $E_{i,i}$ in $\mathbb{M}_n(A)$, therefore we have that $[P_{i,j}(v)] = [1]$ in $K_0(A)$, hence the proposition has been checked. □

Proposition 5.2. *Let A be a unital C^*-algebra. If w_1, w_2 and v are unitaries of A such that v has order m, then $[P_{i,j}(w_1 v w_2)] = [1]$ in $K_0(A)$.*

Proof. As w_1 and w_2 are unitaries in A, then for all $i \neq j$, $W = w_1 \otimes E_{i,i} + w_2^* \otimes E_{j,j} + \sum_{k \notin \{i,j\}} E_{k,k} \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover, $W P_{i,j}(v) W^* = P_{i,j}(w_1 v w_2)$, therefore by Proposition 5.1 we have $[P_{i,j}(w_1 v w_2)] = [P_{i,j}(v)] = [1]$. □

Proposition 5.3. *Let A be a unital C^*-algebra. If u and v are self-adjoint unitaries in A, then $[P_{i,j}(uv)] = [1]$ in $K_0(A)$.*

Proof. For $i \neq j$, let

$$W = \frac{1}{\sqrt{2}}(uv \otimes E_{i,i} + uv \otimes E_{i,j} + E_{j,i} - E_{j,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k}) \ ,$$

then $W \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover,

$$\begin{aligned} W^*P_{i,j}(uv)W &= \frac{1}{4}(2uv \otimes E_{i,i} + 2 \otimes E_{i,j})(\sqrt{2}W) \\ &= E_{i,i}, \end{aligned}$$

and this implies that the projection $P_{i,j}(uv)$ is unitarily equivalent to $E_{i,i}$ in $\mathbb{M}_n(A)$, therefore we have that $[P_{i,j}(uv)] = [1]$ in $K_0(A)$, hence the proposition has been checked. □

Combining the previous results, we have the following theorem concerning the K_0-class of those projections $P_{i,j}(u)$ in $\mathcal{P}(\mathbb{M}_n(A))$, evaluated at any unitary u of A.

Theorem 5.4. *Let A be a simple, unital purely infinite C^*-algebra, such that $K_1(A)$ is the trivial group. If $u \in \mathcal{U}(A)$, then $[P_{i,j}(u)] = [1]$ in $K_0(A)$.*

Proof. Consider a unitary u of A. As $K_1(A) = 0$, and we know by [[4], p.188] that $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}_0(A)$ then using M. Leen's result (Theorem 4.1), we have that $u = \prod_{k=1}^{n} v_k$, where v_k is a self-adjoint unitary ($*$-symmetry) of A. If $n = 1$, then the result holds by using Proposition 5.1. Proposition 5.3 proves the case $n = 2$. If $n \geq 3$, then the result is done by Proposition 5.2, hence the proof is completed. □

Moreover, as M. Broise in [[3], Theorem 1] proved that in the case of von Neumann factors of either type II_1 or III, the unitaries are generated by the self-adjoint unitaries, then a similar result in the case of von Neumann factors can be deduced as follows:

Theorem 5.5. *Let A be a von Neumann factor of type II_1 or III. If $u \in \mathcal{U}(A)$, then $[P_{i,j}(u)] = [1]$ in $K_0(A)$.*

Proof. Let u be a unitary of A. By [[3], Theorem 1], u can be written as a finite product of self-adjoint unitaries of A, then mimic the proof of Theorem 5.4. □

Consequently, we have the following results concerning the K_0-class of some certain projections.

Corollary 5.6. *Let A be a unital C^*-algebra which is either:*
(1) simple, purely infinite, with $K_1(A) = 0$, or
(2) von Neumann factor of type II_1, or III.
If v is a unitary of A, and p is the projection of $\mathbb{M}_n(A)$ defined by

$$p = \frac{1}{2} \otimes E_{1,1} + \frac{v}{2} \otimes E_{1,2} + \frac{v^*}{2} \otimes E_{2,1} + \frac{1}{2} \otimes E_{2,2} + E_{3,3} + E_{4,4} \cdots + E_{m,m}$$

for some positive integer $m \leq n - 2$, then $[p] = (m - 1)[1]$, in $K_0(A)$.

Proof. As the projection p is the orthogonal sums of $P_{1,2}(v) + E_{3,3} + E_{4,4} \cdots + E_{m,m}$, then by either Theorem 5.4 or 5.5,

$$[p] = [1] + ([1] + \cdots + [1]) = (m - 1)[1].$$

□

Corollary 5.7. *Let A be a unital C^*-algebra which is either:*
(1) simple, purely infinite, with $K_1(A) = 0$, or
(2) von Neumann factor of type II_1, or III.
If $v_1, v_2 \cdots v_n$ are unitaries of A, and p is the projection of $\mathbb{M}_{2n}(A)$ defined by

$$\begin{aligned} p &= \frac{1}{2} \otimes E_{1,1} + \frac{v_1}{2} \otimes E_{1,2} + \frac{v_1^*}{2} \otimes E_{2,1} + \frac{1}{2} \otimes E_{2,2} \\ &+ \frac{1}{2} \otimes E_{3,3} + \frac{v_2}{2} \otimes E_{3,4} + \frac{v_2^*}{2} \otimes E_{4,3} + \frac{1}{2} \otimes E_{4,4} + \cdots \\ &+ \frac{1}{2} \otimes E_{2n-1,2n-1} + \frac{v_n}{2} \otimes E_{2n-1,2n} + \frac{v_n^*}{2} \otimes E_{2n,2n-1} + \frac{1}{2} \otimes E_{2n,2n}, \end{aligned}$$

then $[p] = n[1]$, in $K_0(A)$.

Proof. Using Theorem 5.4 (or Theorem 5.5), we have

$$[p] = [P_{1,2}(v_1)] + [P_{3,4}(v_2) + \cdots + [P_{2n-1,2n}(v_n)] = n[1].$$

□

Now let us prove the following lemma, which will be used in order to prove our main result in this section (Theorem 5.9), which is in fact a consequence application of Theorem 4.2, to the case of Cuntz algebras $\mathcal{O}_n$.

Lemma 5.8. *Let A be a unital, simple purely infinite C^*-algebra, with $K_1(A) = 0$, and let $\{e_{i,j}\}^n$, with $e_{1,1} \sim 1$ be a system of matrix units of A . Then for any unitary $u \in \mathcal{U}(A)$ we have $[\eta(P_{i,j}(u))] = [1]$ in $K_0(A)$.*

Proof. As we have seen in the proof of Propositions 5.1, 5.2, 5.3 and Theorem 5.4, there exists a unitary $W \in \mathcal{U}(\mathbb{M}_n(A))$, such that $W^* P_{i,j}(u) W = E_{i,i}$. Therefore,

$$\eta(W)^* \eta(P_{i,j}(u)) \eta(W) = \eta(E_{i,i}) = \eta_1 \hat{\Delta}_v(E_{i,i}) = \eta_1(e_{1,1} \otimes E_{i,i}) = e_{i,i}.$$

Then

$$\eta(P_{i,j}(u)) \sim_u e_{i,i} \sim e_{1,1} \sim 1,$$

hence $\eta(P_{i,j}(u))$ and 1 have the same class in $K_0(A)$. □

Finally, let us consider the case of the Cuntz algebra $\mathcal{O}_n$. Let u be a self-adjoint unitary (involution), so $u = 1 - 2p$, for some $p \in \mathcal{P}(\mathcal{O}_n)$. We recall the concept *type of involution* which is introduced by the author in [2], as follows: Since $K_0(\mathcal{O}_n) \simeq \mathbb{Z}_{n-1}$ (see [4]), then the type of u is defined to be the element $[p]$ in $K_0(\mathcal{O}_n)$. By ([2], Lemma 2.1), two involutions are conjugate as group elements in $\mathcal{U}(\mathcal{O}_n)$ if and only if they have the same type.

As a consequence of Theorem 4.2, and the results concerning the K_0-group of the projections $P_{i,j}(u)$, which are deduced in this section, we have the following result.

Theorem 5.9. *Let n be given. There is a positive number ϵ such that every unitary of $\mathcal{O}_n$ that lies within ϵ-neighborhood of 1 can be written as a product of eleven involutions, of which eight have the form $(1 - 2\eta P_{i,j}(\omega))$, for some $\omega \in \mathcal{U}(\mathcal{O}_n)$ and consequently, all such eight involutions are conjugate group elements of $\mathcal{U}(\mathcal{O}_n)$.*

Proof. Using [4] and [5], the Cuntz algebra $\mathcal{O}_n$ is a simple, unital purely infinite C^*-algebra with trivial K_1-group. Then by Theorem 4.2, there exists $\epsilon > 0$ such that for every $u \in \mathcal{U}(\mathcal{O}_n)$ with $\|u - 1\| < \epsilon$, then u can be written as a product of eleven involutions, of which eight have the form $(1 - 2\eta P_{i,j}(\omega))$, for some $\omega \in \mathcal{U}(\mathcal{O}_n)$. The type of the involution $(1 - 2\eta P_{i,j}(\omega))$ is $[\eta P_{i,j}(\omega))]$ and by Lemma 5.8 equals 1 in $K_0(\mathcal{O}_n)$. Hence, by [[2], Lemma 2.1], all these involutions are conjugate indeed, to the trivial involution -1. □

Consequently, and as every unitary (precisely in the connected component of unity) can be written as a finite product of unitaries that are close to the unity (see for example [11], § 4.2), we have the following:

Corollary 5.10. *Every unitary of $\mathcal{O}_n$ can be written as a finite product of involutions, of which a multiple of eight have the form $(1 - 2\eta P_{i,j}(\omega))$, for some $\omega \in \mathcal{U}(\mathcal{O}_n)$ and consequently, all such multiple of eight involutions are conjugate group elements of $\mathcal{U}(\mathcal{O}_n)$.*

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