Ann. Funct. Anal. 3 (2012), no. 2, 170–182
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

INTEGRAL OPERATORS ACTING AS VARIABLES OF THE MATRIX POLYNOMIAL: APPLICATION TO SYSTEM OF INTEGRAL EQUATIONS

NIKTA SHAYANFAR[1], MAHMOUD HADIZADEH[1*] AND AMIR AMIRASLANI[2]

Communicated by C. Cuevas

ABSTRACT. The aim of this work is to clarify a new viewpoint of connection between system of integral equations and matrix polynomials. A procedure is described for transforming a linear system of integral equations to an independent system. The latter is converted to equalities by considering its equivalent matrix polynomial equation which employs the integral operator as its variable, and admits a normal form for simplifying the system. We will show that under certain suitable conditions, an independent reduced system is obtained, which can be shown to have the same unknowns as the main system, and has only one unknown in each equation. In fact, the basic idea enables us to develop a methodology to solve general systems of linear integral equations.

1. INTRODUCTION

The development of mathematical treatment of systems of integral equations is arguably one of the greatest achievements of the recent centuries. A few key concepts have made this development possible, one of which is linear algebraic methods. References that give rise to these methods include some quiet prominent books and papers such as [3, 5, 22].

In this paper, we consider the linear system of Volterra integral equations of the form

$$\mathbf{X}(t) + \int_a^t \mathbf{K}(t,s)\mathbf{X}(s)ds = \mathbf{\Phi}(t), \qquad a \le t \le b, \tag{1.1}$$

Date: Received: 12 May 2012; Accepted: 14 June 2012.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 47G10; Secondary 45F05, 11C99.

Key words and phrases. Smith normal form, integral operator, system of integral equations, multivariate matrix polynomial.

for two particular, although fairly interesting, cases. In this system $\mathbf{K}(t,s) := [k_{ij}(t,s)]_{i,j=1}^{n} : \mathbb{C}^2[a,b] \to \mathbb{C}$ is a continuous kernel, $\mathbf{\Phi}(t) := [\varphi_i(t)]_{i=1}^{n} : \mathbb{C} \to \mathbb{C}$ is a given continuous vector function, and $\mathbf{X}(t)$ is an unknown function should be determined. The existence and uniqueness of solutions to the system is shown in [5].

Although there exist various symbolic or numeric-analytic methods for solution of (1.1), here we develop a method based on the properties of matrix polynomials, which was first introduced for the solution of differential equations [13] and proposed by the authors in [19] for systems of integral equations. An application of this approach to the Hertz contact problem confirms the efficiency of the proposed approach [20].

Matrix polynomials have been used extensively in recent years in the areas of signal processing [7], analysis of linear feedback systems [6], nonlinear eigenvalues problem [1], and rational approximation and system theory [9], etc.

Initially, we turn the given system of integral equations into a matrix polynomial equation and a simple reduced system will be obtained based on the Smith normal form of the matrix polynomial. In fact, a novel matrix polynomial equation is obtained if the integral operator is replaced by a variable of the matrix polynomial. However, various kernels in the system of integral equations lead to multivariate matrix polynomials. The definition and notion of multivariate matrix polynomial as well as its properties have been studied by several authors e.g. [2, 4, 8]. Such matrices arise in the mathematical treatment of multidimensional systems which can be considered as extensions of the ordinary differential or difference systems [18]. Particulary, among all multivariate matrix polynomials, we are interested in the case when the variables do commute.

The paper is organized as follows. Some well-known tools and results from matrix polynomial theory, which will be needed in the following sections, are reviewed in section 2. Section 3 contains the key to understanding the idea of the method for general system of linear operator equations. When attention is restricted to matrix polynomial equations, the Smith normal form of the matrix polynomial completely characterizes the system, as is also discussed. Finally, the main results of the theory for reducing systems of integral equations in particular cases are included in section 4.

2. Comments on matrix polynomials

This section describes some basic mathematical concepts that will be used throughout this paper. For simplicity, let us assume that all matrices are of normal full-rank. For detailed proofs, the reader may see standard monographs like [10, 11, 16].

Let the set of square matrix polynomials be denoted by $\mathbb{C}^{n\times n}[\mathcal{K}]$, (where $\mathbb{C}[\mathcal{K}]$ is the set of all polynomials in $\mathcal{K}$) and consists of polynomials in $\mathcal{K}$ with coefficients belonging to $\mathbb{C}^{n\times n}$, where $\mathbb{C}^{n\times n}$ is the class of $n \times n$ matrices whose elements belong to $\mathbb{C}$. Precisely, *the univariate matrix polynomial* of degree l, is defined by $A(\mathcal{K}) = \sum_{i=0}^{l} A_i\mathcal{K}^i$, with $A_i \in \mathbb{C}^{n\times n},\ i = 0, \cdots, l$.

Now, let $\mathbb{C}[\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_n]$ denotes the set of multivariate polynomials in n variables $\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_n$ with coefficients in $\mathbb{C}$, and $\mathbb{C}^{n\times n}[\mathcal{K}_1, \cdots, \mathcal{K}_n]$ be the set of multivariate polynomials with coefficients in $\mathbb{C}^{n\times n}$, called *multivariate matrix polynomials*. An obvious approach which facilitates the extension of results for univariate matrix polynomials to multivariate ones is to consider these latter matrices as univariate matrices with one principal variable.

Definition 2.1. A *unimodular matrix polynomial* is a square one with nonzero determinant in $\mathbb{C}$.

One of the best known normal forms for matrix polynomials is a diagonal matrix polynomial obtained by performing elementary operations, and called Smith decomposition.

Theorem 2.2. (From [11]) *Let $\mathbf{A}(\mathcal{K})$ be a full-rank matrix polynomial. There exist $n \times n$ unimodular matrix polynomials $\mathbf{U}(\mathcal{K})$ and $\mathbf{V}(\mathcal{K})$, such that*

$$\mathbf{U}(\mathcal{K})\mathbf{A}(\mathcal{K})\mathbf{V}(\mathcal{K}) = \mathbf{D}(\mathcal{K}), \tag{2.1}$$

where $\mathbf{D}(\mathcal{K}) = diag[d_1(\mathcal{K}), \cdots, d_n(\mathcal{K})]$ is a diagonal matrix with monic polynomials $d_i(\mathcal{K})$ over $\mathbb{C}$ such that $d_i(\mathcal{K})$ is divisible by $d_{i-1}(\mathcal{K})$.

The diagonal matrix polynomial $\mathbf{D}(\mathcal{K})$ is called the *Smith normal form* of the matrix polynomial $\mathbf{A}(\mathcal{K})$, and $\mathbf{U}(\mathcal{K})$ and $\mathbf{V}(\mathcal{K})$ are called *pre- and post-multipliers*, respectively. This decomposition can be obtained by repeated application of a finite number of elementary row and column operations to find unimodular matrices $\mathbf{U}(\mathcal{K})$ and $\mathbf{V}(\mathcal{K})$, respectively, such that $\mathbf{U}(\mathcal{K})\mathbf{A}(\mathcal{K})\mathbf{V}(\mathcal{K})$ is in the Smith normal form. (For more details see [10, 11])

The Smith form was first originated for integer matrices by Smith [21], and it was then extended to matrix polynomials. Applications of this normal form include, for example, solving systems of Diophantine equations [17], integer programming [12], and computing additional normal forms such as Frobenius and Jordan normal forms [10, 14]. This decomposition is well known theoretically, but can be difficult to compute in practice. Methods of calculating decomposition forms of matrix polynomials have been studied by many authors [15, 23, 24].

In the extension of univariate matrix polynomials to multivariate ones, it seems appropriate to consider the concept of the Smith normal form of a multivariate matrix polynomial, which is computed with respect to the principal variable. Dealing with $\mathbb{C}^{n\times n}[\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_n]$, although we are mainly concerned with situations in which there are several choices for principal variable, from now on we consider $\mathbb{C}^{n\times n}(\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_{n-1})[\mathcal{K}_n]$ or $\mathbb{C}^{n\times n}(\mathcal{K}_2, \mathcal{K}_3, \cdots, \mathcal{K}_{n-1}, \mathcal{K}_n)[\mathcal{K}_1]$. For example, take $\mathbb{C}^{n\times n}(\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_{n-1})[\mathcal{K}_n]$. In this case, the condition of Theorem 2.2 holds where $\mathcal{K} := \mathcal{K}_n$ is the principal variable, and the other variables $(\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_{n-1})$ are considered as constants. A similar procedure happens for $\mathbb{C}^{n\times n}(\mathcal{K}_2, \mathcal{K}_3, \cdots, \mathcal{K}_{n-1}, \mathcal{K}_n)[\mathcal{K}_1]$. This approach, of course, can lead to Smith forms which contain rational elements.

Throughout this paper, we will assume that the command *smith* in Maple® gives the Smith decomposition of a matrix polynomial with respect to a particular variable. Since Maple® assumes $\mathcal{K}_1\mathcal{K}_2 = \mathcal{K}_2\mathcal{K}_1$, while considering $\mathcal{K}_i$ as operators in general, this may not be true, we should emphasize that in an attempt to obtain the Smith decomposition of a multivariate matrix polynomial, the condition under which a matrix polynomial could be equivalent to its Smith form shown to be *the commutativity of the variables* of the matrix polynomial.

3. Basic idea of the method

The following Lemma is a straightforward consequence of the Smith decomposition, and will be used to form the basis of our method:

Lemma 3.1. (From [11]) *Let the matrix polynomial equation*

$$\mathbf{A}(\mathcal{K})\mathbf{X}(t) = \mathbf{\Phi}(t), \tag{3.1}$$

present a system of operator equations, where $\mathbf{A}(\mathcal{K})$ *is a matrix polynomial,* $\mathbf{\Phi}(t)$ *is a given vector function, and* $\mathbf{X}(t)$ *is the unknown vector should be determined.*

Consider the Smith decomposition (2.1)*, then the solution of the system is defined explicitly by*

$$\mathbf{X}(t) = \mathbf{V}(\mathcal{K})\mathbf{Y}(t), \tag{3.2}$$

where

$$\mathbf{D}(\mathcal{K})\mathbf{Y}(t) = \mathbf{U}(\mathcal{K})\mathbf{\Phi}(t). \tag{3.3}$$

The proof can be checked easily by substituting the Smith decomposition (2.1) in (3.1).

We extend the above Lemma to include systems of integral equations, and henceforth, the argument (t) is omitted whenever that does not cause confusion.

4. Main results

The system (1.1) can be written as a matrix polynomial equation (3.1). This is accomplished by introducing an n dimensional column vector

$$\mathbf{X}(t) = col(x_1(t), \cdots, x_n(t)).$$

Similarly, let $\mathbf{\Phi}$ be a vector valued function with components φ_i, and $\mathbf{K}(t,s)$ be an $n \times n$ kernel matrix with entries $k_{i,j}(t,s)$, in two cases:

$$\mathbf{K} = \sum_{i,j=1}^{n} k_i(t,s)\mathcal{E}_{i,j}, \tag{4.1}$$

$$\mathbf{K}' = \sum_{i,j=1}^{n} k_j(t,s)\mathcal{E}_{i,j}, \tag{4.2}$$

where $\mathcal{E}_{i,j}$ stands for a matrix with entry (i,j) equal to 1, and all other entries equal to zero.

Moreover, assume that in each system, there exists at least one invertible integral operator $\mathcal{K}_i$, with $\mathcal{K}_i^{-1}$ denoted as its inverse. Despite the fact that these assumptions may appear restricting, they still include a large class of important applications.

First, we establish an independent system, equivalent to the system of integral equations (1.1) where the kernel is given by (4.1) for $n = 2$, and then we explain the procedure for higher values of n.

Defining the integral operator

$$(\mathcal{K}_j f)(t) := \int_a^t k_j(t,s) f(s) ds, \qquad j = 1, 2, \tag{4.3}$$

we can assign the following matrix polynomial equation:

$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} = \begin{pmatrix} 1+\mathcal{K}_1 & \mathcal{K}_1 \\ \mathcal{K}_2 & 1+\mathcal{K}_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

to the system. For obtaining the decomposition (2.1), consider $\mathbb{C}^{2\times 2}(\mathcal{K}_2)[\mathcal{K}_1]$, therefore

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1+\mathcal{K}_1+\mathcal{K}_2 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 0 & {\mathcal{K}_2}^{-1} \\ -\mathcal{K}_2 & 1+\mathcal{K}_1 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 1 & -(1+\mathcal{K}_2){\mathcal{K}_2}^{-1} \\ 0 & 1 \end{pmatrix}.$$

Taking (3.3) for $\mathbf{Y}$, leads to

$$\begin{cases} y_1 = {\mathcal{K}_2}^{-1}\varphi_2, \\ (1+\mathcal{K}_1+\mathcal{K}_2)y_2 = -\mathcal{K}_2\varphi_1 + (1+\mathcal{K}_1)\varphi_2, \end{cases} \tag{4.4}$$

and equation (3.2) yields

$$\begin{cases} x_1 = y_1 - (1+\mathcal{K}_2){\mathcal{K}_2}^{-1} y_2, \\ x_2 = y_2. \end{cases} \tag{4.5}$$

Replacing second equation of (4.5) in the second equation of (4.4) shows the operator equation

$$(1+\mathcal{K}_1+\mathcal{K}_2)x_2 = -\mathcal{K}_2\varphi_1 + (1+\mathcal{K}_1)\varphi_2.$$

Then, multiplying both sides of the first equation of (4.5) by $(1+\mathcal{K}_1+\mathcal{K}_2)$ and considering (4.4), gives the following operator equation

$$(1+\mathcal{K}_1+\mathcal{K}_2)x_1 = -\mathcal{K}_1\varphi_2 + (1+\mathcal{K}_2)\varphi_1.$$

Now, expanding the operators gives two independent equations for obtaining x_1 and x_2.

It is important to mention that from now on, the invertibility of the operator $\mathcal{K}_2$ is needed in this argument. Generally, we should suppose that there exists at least one invertible integral operator, namely $\mathcal{K}_2$, in this proof. If $\mathcal{K}_2$ is not invertible, we can reorder the equations such that the second equation of the system contains the invertible integral operator $\mathcal{K}_2$.

For the general case, the system of integral equations (1.1) with (4.1) requires introducing the integral operator $\mathcal{K}_j$ in (4.3) for $j = 1, \cdots, n$, as the variable of the matrix polynomial. Thus the equation (3.1) represents the system, with the multivariate matrix polynomial

$$\mathbf{A}_{ij} = \delta_{ij} + \mathcal{K}_i. \tag{4.6}$$

In the decomposition of this matrix polynomial, we consider $\mathbb{C}^{n\times n}(\mathcal{K}_2,\cdots,\mathcal{K}_n)[\mathcal{K}_1]$, i.e. we choose $\mathcal{K}_1$ as the principal variable. The Smith decomposition (2.1) is obtained as follows.

$$\mathbf{D}=\sum_{j=1}^{n-1}\mathcal{E}_{j,j}+(1+\sum_{j=1}^{n}\mathcal{K}_j)\mathcal{E}_{n,n}.$$

The pre-multiplier is $\mathbf{U}=\sum_{i=1}^{n}\mathbf{U}_i$, where:

$$\begin{aligned}
\mathbf{U}_1 =&\mathcal{K}_2{}^{-1}\mathcal{E}_{1,2},\\
\mathbf{U}_i =&-\mathcal{K}_{i+1}\mathcal{K}_2{}^{-1}\mathcal{E}_{i,2}+\mathcal{E}_{i,i+1}, \qquad i=2,\cdots,n-1,\\
\mathbf{U}_n =&-\sum_{j=1}^{n}\mathcal{K}_2\mathcal{E}_{n,j}+(1+\sum_{j=1}^{n}\mathcal{K}_j)\mathcal{E}_{n,2},
\end{aligned}$$

and the post-multiplier matrix is $\mathbf{V}=\sum_{i=1}^{n}\mathbf{V}_i$, where:

$$\begin{aligned}
\mathbf{V}_1 =&\mathcal{E}_{1,1}-\sum_{j=2}^{n-1}\mathcal{E}_{1,j}-\mathcal{K}_2{}^{-1}(1+\sum_{j=2}^{n}\mathcal{K}_j)\mathcal{E}_{1,n},\\
\mathbf{V}_2 =&\mathcal{E}_{2,n},\\
\mathbf{V}_i =&\mathcal{E}_{i,i-1}+\mathcal{K}_i\mathcal{K}_2{}^{-1}\mathcal{E}_{i,n}, \qquad i=3,\cdots,n.
\end{aligned}$$

Our task is now to make the system (3.3), as:

$$\begin{aligned}
y_1 =&\mathcal{K}_2{}^{-1}\varphi_2,\\
y_i =&-\mathcal{K}_{i+1}\mathcal{K}_2{}^{-1}\varphi_2+\varphi_{i+1}, \qquad i=2,\cdots,n-1,\\
(1+\sum_{j=1}^{n}\mathcal{K}_j)y_n =&(1+\sum_{j=1}^{n}\mathcal{K}_j)\varphi_2-\mathcal{K}_2\sum_{j=1}^{n}\varphi_j.
\end{aligned}$$

Note that, the structures of $\mathbf{U}$ and $\mathbf{Y}$ support the claim for $n>2$, while the procedure has been discussed for $n=2$, before. Finally, (3.2) implies:

$$\begin{aligned}
x_1 =&y_1-\sum_{j=2}^{n-1}y_j-\mathcal{K}_2{}^{-1}(1+\sum_{j=2}^{n}\mathcal{K}_j)y_n,\\
x_2 =&y_n,\\
x_i =&y_{i-1}+\mathcal{K}_i\mathcal{K}_2{}^{-1}y_n, \qquad i=3,\cdots,n.
\end{aligned}$$

Initially, for $i = 3, \cdots, n$, we have

$$\begin{aligned}(1+\sum_{j=1}^{n}\mathcal{K}_j)x_i =&(1+\sum_{j=1}^{n}\mathcal{K}_j)y_{i-1}+\mathcal{K}_i{\mathcal{K}_2}^{-1}(1+\sum_{j=1}^{n}\mathcal{K}_j)y_n\\ =&(1+\sum_{j=1}^{n}\mathcal{K}_j)(-\mathcal{K}_i{\mathcal{K}_2}^{-1}\varphi_2+\varphi_i)\\ &+\mathcal{K}_i{\mathcal{K}_2}^{-1}\Big((1+\sum_{j=1}^{n}\mathcal{K}_j)\varphi_2-\mathcal{K}_2\sum_{j=1}^{n}\varphi_j\Big)\\ =&(1+\sum_{j=1}^{n}\mathcal{K}_j)\varphi_i-\mathcal{K}_i\sum_{j=1}^{n}\varphi_j.\end{aligned}$$

Furthermore, for $i = 2$, replacing x_2 in y_n, we get

$$(1+\sum_{j=1}^{n}\mathcal{K}_j)x_2 = (1+\sum_{j=1}^{n}\mathcal{K}_j)\varphi_2-\mathcal{K}_2\sum_{j=1}^{n}\varphi_j,$$

and for $i = 1$, refraining from going into the details, we can get

$$(1+\sum_{j=1}^{n}\mathcal{K}_j)x_1 = (1+\sum_{j=1}^{n}\mathcal{K}_j)\varphi_1-\mathcal{K}_1\sum_{j=1}^{n}\varphi_j.$$

This can be generally stated as

$$(1+\sum_{j=1}^{n}\mathcal{K}_j)x_i = (1+\sum_{j=1}^{n}\mathcal{K}_j)\varphi_i-\mathcal{K}_i\sum_{j=1}^{n}\varphi_j, \quad i=1,\cdots,n,$$

and applying the integral operator $\mathcal{K}_i$, presents an independent reduced system of integral equations.

What we have done in this case, leads to the following theorem:

Theorem 4.1. *Consider the following system of integral equations:*

$$x_i(t)+\sum_{j=1}^{n}\int_a^t k_i(t,s)x_j(s)ds=\varphi_i(t), \qquad i=1,\cdots,n, \tag{4.7}$$

with at least one invertible integral operator defined as (4.3), for $j = 1, \cdots, n$. The reduced equivalent system can be shown to be the independent system

$$x_i(t)+\int_a^t k(t,s)x_i(s)ds=\psi_i(t), \qquad i=1,\ldots,n, \tag{4.8}$$

where

$$\psi_i(t)=\varphi_i(t)+\int_a^t \Big[k(t,s)\varphi_i(s)-k_i(t,s)\varphi(s)\Big]ds, \qquad i=1,\cdots,n, \tag{4.9}$$

and

$$k(t,s)=\sum_{j=1}^{n}k_j(t,s), \qquad \varphi(s)=\sum_{j=1}^{n}\varphi_j(s).$$

In what follows, we shall encounter an analogous result for the system of integral equations for the second case (4.2).

Theorem 4.2. *Given the system* (1.1) *with the kernel* (4.2)*, suppose the assumptions of Theorem 4.1 hold. Then there exist*

$$k(t,s)=\sum_{j=1}^{n} k_j(t,s),$$

and

$$\psi_i(t)=\varphi_i(t)+\int_a^t k(t,s)\varphi_i(s)ds-\sum_{j=1}^{n}\int_a^t k_j(t,s)\varphi_j(s)ds, \qquad i=1,\dots,n, \quad (4.10)$$

such that the independent reduced system (4.8) *obtains the solution of the main system.*

Proof. An argument similar to the one used in the previous theorem, is needed for this proof. The system (1.1) with the condition (4.2) can be written explicitly as

$$x_i(t)+\sum_{j=1}^{n}\int_a^t k_j(t,s)x_j(s)ds=\varphi_i(t), \qquad i=1,\dots,n. \quad (4.11)$$

Our first goal is to study the system (4.11) for $n=2$. According to the definition of the integral operator (4.3), the system can be expressed in a compact form (3.1), with multivariate $\mathbf{A}'$ in $\mathbb{C}^{2\times2}[\mathcal{K}_1,\mathcal{K}_2]$ as:

$$\mathbf{A}'=\begin{pmatrix} 1+\mathcal{K}_1 & \mathcal{K}_2 \\ \mathcal{K}_1 & 1+\mathcal{K}_2 \end{pmatrix},$$

and its Smith decomposition obtained in $\mathbb{C}^{2\times2}(\mathcal{K}_1)[\mathcal{K}_2]$:

$$\mathbf{D}'=\begin{pmatrix} 1 & 0 \\ 0 & 1+\mathcal{K}_1+\mathcal{K}_2 \end{pmatrix}, \mathbf{U}'=\begin{pmatrix} 0 & {\mathcal{K}_1}^{-1} \\ -\mathcal{K}_1 & 1+\mathcal{K}_1 \end{pmatrix}, \mathbf{V}'=\begin{pmatrix} 1 & -(1+\mathcal{K}_2){\mathcal{K}_1}^{-1} \\ 0 & 1 \end{pmatrix}.$$

Taking into account our previous assumption of existing at least one invertible integral operator, we suppose that $\mathcal{K}_1$ is invertible. Now, the system (3.3) has the form

$$\begin{cases} y_1={\mathcal{K}_1}^{-1}\varphi_2, \\ (1+\mathcal{K}_1+\mathcal{K}_2)y_2=-\mathcal{K}_1\varphi_1+(1+\mathcal{K}_1)\varphi_2. \end{cases} \quad (4.12)$$

Then we return to (3.2) as:

$$\begin{cases} x_1=y_1-(1+\mathcal{K}_2){\mathcal{K}_1}^{-1}y_2, \\ x_2=y_2\,. \end{cases} \quad (4.13)$$

Through substituting the equations of (4.12) in the equations of (4.13), some direct calculation yields

$$\begin{cases} (1+\mathcal{K}_1+\mathcal{K}_2)x_1=-\mathcal{K}_2\varphi_2+(1+\mathcal{K}_2)\varphi_1, \\ (1+\mathcal{K}_1+\mathcal{K}_2)x_2=-\mathcal{K}_1\varphi_1+(1+\mathcal{K}_1)\varphi_2, \end{cases}$$

for obtaining x_1 and x_2, which is the system (4.8) with (4.10) for $n = 2$.

For higher values of n, with $\mathcal{K}_j$ for $j = 1, \ldots, n$, denoting the variable of the matrix polynomial corresponding to the given integral operator, the representation (3.1) of the system (4.11) is hold, where $\mathbf{A}$ is:

$$\mathbf{A'}_{ij} = \delta_{ij} + \mathcal{K}_j. \tag{4.14}$$

Alternatively, consider $\mathbb{C}^{n\times n}(\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_{n-1})[\mathcal{K}_n]$ for obtaining the Smith normal form of the multivariate matrix polynomial in $\mathbb{C}^{n\times n}[\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_{n-1}, \mathcal{K}_n]$. The Smith matrix polynomial is:

$$\mathbf{D'} = \sum_{j=1}^{n-1} \mathcal{E}_{j,j} + \gamma_n \mathcal{E}_{n,n}, \qquad \gamma_n = 1 + \sum_{j=1}^{n} \mathcal{K}_j.$$

Additionally, the pre-multiplier matrix polynomial is $\mathbf{U'} = \sum_{i=1}^{n} \mathbf{U'}_i$:

$$\begin{aligned}
\mathbf{U'}_1 =& \mathcal{K}_1{}^{-1} \mathcal{E}_{1,2},\\
\mathbf{U'}_i =& \mathcal{E}_{i,i} - \mathcal{E}_{i,i+1}, \qquad i = 2, \cdots, n-1,\\
\mathbf{U'}_n =& -\sum_{j=1}^{n} \mathcal{K}_j \mathcal{E}_{n,j} + \gamma_n \mathcal{E}_{n,n}.
\end{aligned}$$

Providing that, up to a replacement of the equations, at least one invertible integral operator exists, without loss of generality, we can suppose that $\mathcal{K}_1$ is the integral operator with the inverse $\mathcal{K}_1{}^{-1}$.

The system (3.3) can be obtained as:

$$\begin{aligned}
y_1 =& \mathcal{K}_1{}^{-1} \varphi_2,\\
y_i =& \varphi_i - \varphi_{i+1}, \quad i = 2, \ldots, n-1,\\
\gamma_n y_n =& \gamma_n \varphi_n - \sum_{j=1}^{n} \mathcal{K}_j \varphi_j.
\end{aligned} \tag{4.15}$$

Moreover, the post multiplier $\mathbf{V'} = \sum_{i=1}^{n} \mathbf{V'}_i$:

$$\begin{aligned}
\mathbf{V'}_1 =& \mathcal{E}_{1,1} - \sum_{j=2}^{n} \mathcal{K}_1{}^{-1}(1 + \sum_{k=2}^{j} \mathcal{K}_k) \mathcal{E}_{1,j},\\
\mathbf{V'}_i =& \sum_{j=i}^{n} \mathcal{E}_{i,j}, \qquad i = 2, \cdots, n.
\end{aligned}$$

It remains to obtain x_i from (3.2) as:

$$\begin{aligned}
x_1 =& y_1 - \sum_{j=2}^{n} \mathcal{K}_1{}^{-1}(1 + \sum_{k=2}^{j} \mathcal{K}_k) y_j,\\
x_i =& \sum_{j=i}^{n} y_j, \qquad i = 2, \cdots, n.
\end{aligned}$$

Actually, for $i = 2, \cdots, n$, we have

$$\gamma_n x_i = \gamma_n \sum_{j=i}^{n-1} (\varphi_j - \varphi_{j+1}) + (\gamma_n \varphi_n - \sum_{j=1}^{n} \mathcal{K}_j \varphi_j) = \gamma_n \varphi_i - \sum_{j=1}^{n} \mathcal{K}_j \varphi_j,$$

and to get a similar relation for $i = 1$, by replacing y_i's from (4.15) in x_1, some tedious manipulation implies:

$$\gamma_n x_1 = \gamma_n \varphi_1 - \sum_{j=1}^{n} \mathcal{K}_j \varphi_j.$$

We are now in a position to state the general form

$$\gamma_n x_i = \gamma_n \varphi_i - \sum_{j=1}^{n} \mathcal{K}_j \varphi_j, \quad i = 1, \cdots, n,$$

an defining (4.10), we can directly obtain x_i from the independent system of integral equations (4.8), and the theorem is proved. □

Finally, the essential observation is a system of integral equations as an example of the systems investigated in Theorem 4.1 as well as Theorem 4.2.

Proposition 4.3. *Let us consider the system of integral equations:*

$$x_i(t) + \sum_{j=1}^{n} \int_a^t x_j(s)ds = \varphi_i(t), \qquad i = 1, \cdots, n. \tag{4.16}$$

It can be written as an equivalent independent system of integral equations:

$$x_i(t) + n \int_a^t x_i(s)ds = \psi_i(t), \qquad i = 1, \cdots, n, \tag{4.17}$$

where

$$\psi_i(t) = \varphi_i(t) + \int_a^t \varphi(s)ds, \qquad \varphi(s) = n\varphi_i(s) - \sum_{j=1}^{n} \varphi_j(s). \tag{4.18}$$

Proof. From the definition of the operator $\mathcal{K}$ as

$$(\mathcal{K}f)(t) := \int_a^t f(s)ds, \tag{4.19}$$

the system (4.16) has the presentation (3.1) with the univariate matrix polynomial:

$$\mathbf{A}_{ij} = \delta_{ij} + \mathcal{K}. \tag{4.20}$$

We can now apply the method of reducing (3.1). First, the vector $\mathbf{Y}$ can be obtained through (3.3), with the Smith decomposition obtained with respect to the only variable, as

$$\mathbf{D} = \sum_{j=1}^{n-1} \mathcal{E}_{j,j} + (\mathcal{K} + \frac{1}{n})\mathcal{E}_{n,n},$$

and

$$\mathbf{U} = \sum_{j=1}^{n-1} \mathcal{E}_{j,j} - \sum_{j=1}^{n-1} \mathcal{E}_{j,j+1} - \frac{\mathcal{K}}{n} \sum_{j=1}^{n-1} \mathcal{E}_{n,j} + \Big(\frac{(n-1)\mathcal{K}+1}{n}\Big)\mathcal{E}_{n,n},$$

in the form

$$\begin{aligned} y_i =& \varphi_i - \varphi_{i+1}, \qquad\qquad i = 1, \cdots, n-1, \\ (\mathcal{K} + \frac{1}{n})y_n =& -\frac{\mathcal{K}}{n}\sum_{i=1}^{n} \varphi_i + (\mathcal{K} + \frac{1}{n})\varphi_n. \end{aligned}$$

Making (3.2) needs $\mathbf{V}$ as $\mathbf{V} = \sum_{i=1}^{n}\sum_{j=i}^{n} \mathcal{E}_{i,j}$, which gives $x_i = \sum_{j=i}^{n} y_j$, and

$$\begin{aligned} (\mathcal{K} + \frac{1}{n})x_i =& (\mathcal{K} + \frac{1}{n})\sum_{j=i}^{n-1}(\varphi_j - \varphi_{j+1}) + \Big(-\frac{\mathcal{K}}{n}\sum_{i=1}^{n}\varphi_i + (\mathcal{K} + \frac{1}{n})\varphi_n\Big) \\ =& (\mathcal{K} + \frac{1}{n})\varphi_i - \frac{\mathcal{K}}{n}\sum_{i=1}^{n}\varphi_i. \end{aligned}$$

Multiplying both sides by constant n, we get

$$(n\mathcal{K} + 1)x_i = (n\mathcal{K} + 1)\varphi_i - \mathcal{K}\sum_{i=1}^{n}\varphi_i,$$

applying the operator (4.19), and defining (4.18) lead to (4.17), as claimed. □

In view of the mentioned approach, it is worthy to mention that, the system (4.16) is a special case of both systems (4.7) and (4.11). However it can be presented by a univariate matrix polynomial (4.20) while the the other systems are corresponding to the multivariate ones (4.6) and (4.14), respectively. On the other hand, the obtained independent system (4.17) with (4.18), is a special case of the reduced system (4.8) with (4.9), and also (4.10), obtained in Theorem 4.1 and Theorem 4.2, respectively. Clearly the results are in accordance with each other, and this ensures the reliability of the method.

5. Conclusion

Given the wide variety of problems in modeling physical phenomena that can be posed as systems of integral equations, being able to efficiently solve them is important. The proposed procedure for the solution of the system of integral equations consists of three main steps. The requisite steps include transforming the original system into a matrix polynomial equation, by choosing an appropriate integral operator. Then, using the Smith canonical form as well as inverting the solution of the transformed equation lead to a reduced system. The present study confirms that exploiting the Smith decomposition, split the system of integral equations to an independent one. Finally, under certain suitable conditions, the desired solution of the altered system can be found by any classical method. In addition, the reformulation proposed here might be used to develop a methodology to solve the other systems of integral equations, however it contains some complications and restrictions for selecting the variables of the matrix polynomial which will be the subject of our future work.

Acknowledgement. The authors are grateful to Dr. Jon Wilkening (University of California, Berkeley) for his insightful remarks and discussion in this subject which helped to improve the manuscript.

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[1] Department of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran.

E-mail address: shayanfar@dena.kntu.ac.ir; hadizadeh@kntu.ac.ir

[2] STEM Department, University of Hawaii-Maui College, Kahului, HI 96732, USA.

E-mail address: aamirasl@hawaii.edu