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INTEGRAL FUNCTIONALS ON C^*-ALGEBRA OF VECTOR-VALUED REGULATED FUNCTIONS

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ABSTRACT. In this paper we deal with the notion of regulated functions with values in a C^*-algebra $\mathcal{A}$ and present examples using a special bi-dimensional C^*-algebra of triangular matrices. We consider the Dushnik integral for these functions and shows that a convenient choice of the integrator function produces an integral homomorphism on the C^*-algebra of all regulated functions $G([a,b],\mathcal{A})$. Finally we construct a family of linear integral functionals on this C^*-algebra.

1. Introduction and Preliminaries

Sometimes to describe physical events we need a model that has, besides the basic operations of linear spaces and the notion of size of their elements, an internal multiplication completely compatible with the normed linear space structure. These spaces are known as Banach algebras, subject that was treated by J. von Neumann, I. M. Gelfand and M. A. Naimark, among others, in the years 1930-60 (for details see [2]). Our interest here is to study the set of all well-behaved funtions $f : I = [a,b] \subset \mathbb{R} \to \mathcal{A}$, known as regulated functions, when $\mathcal{A}$ is a special case of Banach algebra with an involution $* : A \to A$, called C^*-algebra by I. E. Segal (see [3], [8]). On the other hand, the notion of regulated function first appears in Dieudonn's book of 1969 [1] . The space of regulated functions was approached by several authors in the last years (see for example [6], [7], [9]). The classical notation for this set of functions is $G([a,b],\mathcal{A})$ and

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it is a Banach space with the uniform convergence norm. Below we present the notions of regulated functions, Dushnik or interior integral and C^*-algebras, and we present in Section 2 the proofs of some results to guarantee that $G([a,b],\mathcal{A})$ inherits the structure of A, in other words, it is also a C^*-algebra. Finally, in Section 3, the notions and results are then applied in two special cases: first we take $\mathcal{A}=\mathcal{S}_2(\mathbb{R})$, the set of square matrices such that $a_{11}=a_{22}$ and $a_{21}=0$. This set has the structure of commutative C^*-algebra with suitable norm and involution. Secondly we consider the well-known example of the set $C(Z,\mathbb{C})$ of all continuous complex valued functions $x: Z\to\mathbb{C}$, where Z is a Hausdorff space, with the norm $\|x\|_\infty=\sup\{|x(s)| : s\in Z\}$ and involution $x^*=\bar{x}$. In both cases we will discuss the behavior of functions with values in these C^*-algebras, and of integral functionals on $G([a,b],\mathcal{S}_2(\mathbb{R}))$ and $G([a,b],C(Z,\mathbb{C}))$.

Initially, we will present the main notions and notations on C^*-algebras, regulated functions and Dushnik integral.

Let X be a Banach space. We say that $f:[a,b]\to X$ is a regulated function if for every $t\in[a,b]$ there exist both one-sided limits $f(t+)$ and $f(t-)$, with the convention $f(a-)=f(a)$ and $f(b+)=f(b)$. We denote by $G([a,b],X)$ the Banach space of all X-valued regulated functions on $[a,b]$, with the uniform convergence norm $\|f\|_\infty=\sup\{\|f(t)\|_{\mathcal{X}},\ t\in[a,b]\}$. Recall that if A a complex algebra (not necessarily commutative), a mapping $x\in A\mapsto x^*\in A$ is called an involution on A if, for all $x,y\in A$ and $\lambda\in\mathbb{C}$, it is satisfied

a. $(x+y)^*=x^*+y^*$;
b. $(\lambda\, x)^*=\overline{\lambda}\, x^*$;
c. $(x\times y)^*=y^*\times x^*$;
d. $x^{**}=x$.

Moreover, any $x\in A$ for which $x^*=x$ is called *hermitian* (or *self-adjoint*). A Banach algebra A with an involution $x\in A\mapsto x^*\in A$ that satisfies the C^*-identity

$$\|x\times x^*\|=\|x\|^2$$

for every $x\in A$, is called a C^*-algebra.

To complete we present now the notion of integral (in sense of Dushnik) that we use to describe a perfomance criterion on the Banach algebra of regulated function. This kind of Stieltjes integral, finest than the Riemann-Stieltjes integral, is a convenient choice because, when the integrand function belongs to $G([a,b],A)$ and the integrator function is of bounded semivariation, the integral there exists. The original definition of the Riemann integral has been modified in several different extensions. T. J. Stieltjes generalized the Riemann integral defining an integration of a continuous integrand with respect a bounded variation integrator, instead of the variable of integration. B. Dushnik in turn considered a integrand modification that consists in restricting integrand values only to the open segments of corresponding partitions of the interval $[a,b]$. This is a special case of the weighted refinement integral.

Let A,B be two C^*-algebras with multiplications and involutions $\times_{A}$, $\times_{B}$, $*_{A}$ and $*_{B}$ respectively, and suppose that $\alpha\in SV([a,b],\mathcal{L}(A,B))$, the Banach algebra of all bounded semivariation functions $\alpha : [a, b] \to \mathcal{L}(A, B)$, and $f \in G([a, b], A)$. Then there exists the Dushnik integral (we refer to [7] for details) defined as

$$F_\alpha(f) = \int_a^b \cdot \, d\,\alpha(t) \cdot f(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i^\cdot), \tag{1.1}$$

where $\xi_i^\cdot \in]t_{i-1}, t_i[$. Here the limit is take over the set of all partitions of the interval $[a, b]$, denoted by $\mathcal{D}_{[a,b]}$. Note that $F_\alpha : G([a, b], A) \to B$ is a linear map between the C^*-algebras $G([a, b], A)$ and B. Moreover have sense to ask about

$$\begin{aligned}
F_\alpha(f) \times_{_B} F_\alpha(g) &= \int_a^b \cdot \, d\,\alpha(t) \cdot f(t) \times_{_B} \int_a^b \cdot \, d\,\alpha(t) \cdot g(t) \in B \ , \\
F_\alpha(f \times_{_G} g) &= \int_a^b \cdot \, d\,\alpha(t) \cdot [f \times_{_G} g]\,(t) \ , \\
[F_\alpha(f^{*_G})] &= \left[\int_a^b \cdot \, d\,\alpha(t) \cdot f^{*_G}(t)\right] = \lim_{d \in D} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f^*(\xi_i^\cdot), \\
[F_\alpha(f)]^{*_B} &= \left[\int_a^b \cdot \, d\,\alpha(t) \cdot f(t)\right]^{*_B} = \left[\lim_{d \in D} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i^\cdot)\right]^* .
\end{aligned}$$

Recall that a homomorphism ϕ between the C^*-algebras $G([a, b], A)$ and B preserves the involution, that is, $\phi(f^*) = [\phi(f)]^*$. Note that, in general,

$$F_\alpha(f) \times_{_B} F_\alpha(g) \neq F_\alpha(f \times_{_G} g) \text{ and } F_\alpha(f^*) \neq [F_\alpha(f)]^*$$

we have that F_α is not a homomorphism of C^*-algebras. However will be shown that, if we choose the integrator function conveniently (1.1) becames a homomorphism.

2. Main results

We start this section with the results that establish the object of our interest. We begin by showing that multiplication and involution on X induces an internal multiplication and involution on $G([a, b], X)$.

Lemma 2.1. *Let f and g be two regulated functions on $[a, b]$ with values in a C^*-algebra A. Then the pointwise multiplication $[f \times g](t) = f(t) \times_{_A} g(t)$, $t \in [a, b]$ is a regulated function on $[a, b]$.*

Proof. A proof can be found in [4]. □

Lemma 2.2. *Let f be a regulated function on $[a, b]$ with values in a commutative semisimple C^*-algebra A. If $x \mapsto x^*$ denotes the involution on A, then the function $f^* : [a, b] \to A$ given by $f^*(t) = [f(t)]^*$, is a regulated function.*

Proof. Is well known that if A a commutative semi-simple C^*-algebra, then every involution on A is continuous, that is, if $y \to x$ then $y^* \to x^*$ (see, for example Theorem 11.16 of [8]). So, for every $\tau \in [a, b[$,

$$f(\tau) \xrightarrow{\tau\downarrow t} l \implies f^*(\tau) = [f(\tau)]^* \xrightarrow{\tau\downarrow t} l^* \ ,$$

that is,

$$\lim_{\tau\downarrow t} f^*(\tau) := \lim_{\tau\downarrow t}[f(\tau)]^* = l^* = [\lim_{\tau\downarrow t} f(\tau)]^*$$

Analogously, if $\tau \in]a, b]$,

$$f(\tau) \xrightarrow{\tau\uparrow t} l \implies f^*(\tau) = [f(\tau)]^* \xrightarrow{\tau\uparrow t} l^* \ ,$$

that is,

$$\lim_{\tau\uparrow t} f^*(\tau) := \lim_{\tau\uparrow t}[f(\tau)]^* = l^* = [\lim_{\tau\uparrow t} f(\tau)]^*$$

□

As a consequence we have that the structure of C^*-algebra is transferred to the space of regulated functions.

Theorem 2.3. *Suppose that A is a commutative semi-simple C^*-algebra with multiplication $\times$ and involution $*$. Then $G([a, b], A)$, with pointwise operations of multiplication and involution, is a C^*-algebra.*

Proof. Consider the multiplication defined as $[f \times g](t) = f(t) \times g(t)$, and involution as $f^*(t) = [f(t)]^*$. In [4] the authors proved that $G([a, b], X)$ is a Banach algebra with pointwise multiplication, with unit element $e(t) = e_X, t \in [a, b]$. We will just prove now that $f^*(t) = [f(t)]^*$, for all $t \in [a, b]$, defines an involution on $G([a, b], X)$. In fact, for all $t \in [a, b]$, we have

$$\begin{aligned}
(f + g)^*(t) &= [(f + g)(t)]^* = [f(t) + g(t)]^* = [f(t)]^* + [g(t)]^* = f^*(t) + g^*(t) \ , \\
(\lambda\ f)^*(t) &= [(\lambda\ f)(t)]^* = [\lambda\ f(t)]^* = \overline{\lambda}\ [f(t)]^* = \overline{\lambda}\ f^*(t) \ , \\
(f \times g)^*(t) &= [(f \times g)(t)]^* = [f(t) \times g(t)]^* = [g(t)]^* \times [f(t)]^* = g^*(t) \times f^*(t) \\
&= (g^* \times f^*)(t) \\
(f^{**})(t) &= ((f^*)^*)(t) = [(f^*)(t)]^* = [[f(t)]^*]^* = [f(t)]^{**} = f(t) \ .
\end{aligned}$$

and so $(f + g)^* = f^* + g^*$, $(\lambda\ f)^* = \overline{\lambda}\ f^*$, $(f \times g)^* = g^* \times f^*$ and $f^{**} = f$.

Finally, for all $t \in [a, b]$,

$$\|f(t)\times f^*(t)\|_X = \|[f(t)\times[f(t)]^*\|_X = \|[f(t)\|_X^2 \ ,$$

and then

$$\|f \times f^*\| = \sup\{\|f(t) \times f^*(t)\|_X : t \in [a, b]\} = \sup\{\|[f(t)\|_X^2 : t \in [a, b]\} = \|f\|^2.$$

□

Remark 2.4. We note that if A is commutative C^*-algebra, then $G([a, b], A)$ is also commutative.

Let $T \in \mathcal{L}(A,B)$ be a fixed linear multiplicative operator. We will denote by $\mathcal{K}^T([a,b],\mathcal{L}(A,B))$ the set of all functions $\alpha_c^T : [a,b] \to \mathcal{L}(A,B)$, $a \leq c \leq b$, defined as $\alpha_c^T = \mathcal{X}_{]c,b]}T$,

$$[\mathcal{X}_{]c,b]}T](t) = \begin{cases} 0 \text{ , } t \in [a,c] \text{ ,} \\ \\ T \text{ , } t \in]c,b] \text{ .} \end{cases} \tag{2.1}$$

where $\mathcal{X}$ is the characteristic function.

Theorem 2.5. *Let A, B be two C^*-algebras with multiplications and involutions $\times_A$, $\times_B$, $*_A$ and $*_B$ respectively. Suppose that $\alpha_c \in \mathcal{K}^T([a,b],\mathcal{L}(A,B))$, $a \leq c \leq b$, for some linear multiplicative operator $T \in \mathcal{L}(A,B)$. Then $F_\alpha : G([a,b],A) \to B$ defined as*

$$F_{\alpha_c}(f) = \int_a^b \cdot\, d\, \alpha_c(t) \cdot f(t),$$

is a homomorphism.

Proof. Of course we have $\mathcal{K}^T([a,b],\mathcal{L}(A,B)) \subset SV([a,b],\mathcal{L}(A,B))$. If $d : t_0 < t_1 < \cdots < t_n$ is a partition of $[a,b]$ with $t_{k-1} = c$, for some $1 \leq k \leq n$. Then

$$\sum_{i=1}^{|d|} [\alpha_c^T(t_i) - \alpha_c^T(t_{i-1})]\ f(\xi_i^{\cdot}) = [\underbrace{\alpha_c^T(t_k)}_{T} - \underbrace{\alpha_c^T(t_{k-1})}_{0}] \cdot f(\xi_k^{\cdot})$$

and for every partition d' finest than d we have $t_k \downarrow c$ and so $\xi_k^{\cdot} \downarrow c$. Therefore,

$$\int_a^b \cdot\, d\, \alpha_c^T(t) \cdot f(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\alpha_c^T(t_i) - \alpha_c^T(t_{i-1})] \cdot f(\xi_i^{\cdot}) = T \cdot f(c^+)$$

So for $f, g \in G([a,b],A)$,

$$\begin{aligned} F_{\alpha_c^T}(f \times_G g) &= \int_a^b \cdot\, d\, \alpha_c^T(t) \cdot [f \times_G g]\,(t) \\ &= T \cdot [f \times_G g]\,(c^+) \\ &= T \cdot \left[f(c^+) \times_A g(c^+)\right] \\ &= T \cdot \left[f(c^+)\right] \times_B T \cdot \left[g(c^+)\right] \\ &= \int_a^b \cdot\, d\, \alpha_c^T(t) \cdot f(t) \times_B \int_a^b \cdot\, d\, \alpha_c^T(t) \cdot g(t) \\ &= F_{\alpha_c^T}(f) \times_B F_{\alpha_c^T}(g) \end{aligned}$$

that is, $F_{\alpha_c^T} : G([a,b],A) \to B$ is a homomorphism of Banach algebras. If we use the notation e_G and e_B for the units of $G([a,b],A)$ and B, respectively, a simple calculation shows that $F_{\alpha_c^T}(e_G) = e_B$. $\square$

If $T \in A^{'} = \mathcal{L}(A, \mathbb{C})$ (the dual space of A) is a fixed linear multiplicative functional, we will denote by $\mathcal{K}^T([a,b], A^{'})$ the family of all functions $\alpha_c^T : [a,b] \to A^{'}$, $a \le c \le b$, defined as (2.1).

Corollary 2.6. *Suppose that $\alpha_c \in \mathcal{K}^T([a,b], A^{'})$, $a \le c \le b$, for some linear multiplicative functional $T \in A^{'}$. Then $F_\alpha : G([a,b], A) \to \mathbb{C}$ defined as*

$$F_\alpha(f) = \int_a^b \cdot\, d\ \alpha_c(t) \cdot f(t),$$

is a C^*-homomorphism.

Proof. In this case we have $\mathcal{K}^T([a,b], A^{'}) \subset BV([a,b], A^{'})$ (recall that is true because $dim\ B < \infty$, see [7], remark 1.3), and then

$$\int_a^b \cdot\, d\ \alpha_c^T(t) \cdot f(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\alpha_c^T(t_i) - \alpha_c^T(t_{i-1})] \cdot f(\xi_i) = T \cdot f(c^+) \in \mathbb{C}$$

So for $f, g \in G([a,b], A)$,

$$\begin{aligned} F_{\alpha_c}(f \times_G g) &= \int_a^b \cdot\, d\ \alpha_c^T(t) \cdot [f \times_G g]\,(t) = T \cdot [f \times_G g]\,(c^+) = T \cdot \left[f(c^+) \times_A g(c^+)\right] \\ &= T \cdot \left[f(c^+)\right] \times_B T \cdot \left[g(c^+)\right] = \int_a^b \cdot\, d\ \alpha_c(t) \cdot f(t) \times_B \int_a^b \cdot\, d\ \alpha_c(t) \cdot g(t) \\ &= F_{\alpha_c^T}(f) \times_B F_{\alpha_c^T}(g) \end{aligned}$$

that is, $F_{\alpha_c^T} : G([a,b], A) \to \mathbb{C}$ is a multiplicative linear functional on the C^*-algebra $G([a,b], A)$. Moreover it preserves the involution. In fact,

$$F_{\alpha_c^T}(f^*) = T \cdot f^*(c^+) = T \cdot [f(c^+)]^* = [T \cdot f(c^+)]^* = [F_{\alpha_c^T}(f)]^*$$

□

Another special case is when $B = G([a,b], A)$ and $F_\alpha(f) \in G([a,b], A)$,

$$[F_\alpha(f)](s) = \int_a^s \cdot\, d\ \alpha(t) \cdot f(t) \in A\ ,\ \ s \in [a,b]\ ,$$

or still $B = \mathbb{R}$ and $F_\alpha : G([a,b], C([a,b], \mathbb{R})) \in \mathbb{R}$,

$$F_\alpha(f) = \int_a^b \cdot\, d\ \alpha(t) \cdot f(t) \in \mathbb{R}\ .$$

Our first example of this situation was choosing by A the non-commutative algebra of quaternions, because of the application in modelling of 3-D rotations (see [4]). In [5] the authors considered $G([a,b], C([a,b], \mathbb{C}))$, where $C([a,b], \mathbb{C})$ is the Banach algebra of all continuous complex valued functions. The case $G([a,b], \mathcal{S}_2(\mathbb{R}))$, where $\mathcal{S}_2(\mathbb{R})$ is the C*-algebra of all triangular matrices $M = (a_{ij})_{1 \le i,j \le 2}$ with $a_{11} = a_{22}$ and $a_{21} = 0$, will be considered in the next sections, where we present examples. Another special case is when $B = G([a,b],A)$ and $F_\alpha(f) \in G([a,b],A)$,

$$[F_\alpha(f)](s) = \int_a^s \cdot\, d\,\alpha(t) \cdot f(t) \in A\ ,\ s \in [a,b]\ .$$

3. Some examples

The elements of $G([a,b],A)$ that we consider now have an important role because they are support of construction of step functions, essentials to approach regulated functions.

Example 3.1. Let $x \in A$ be a fixed element, and x^* its involution. Denote by f_c^x the function

$$f_c^x(t) = [\mathcal{X}_{[a,c[}x](t) = \begin{cases} x\ , t \in [a,c[\ , \\ 0\ , t \in [c,b]\ . \end{cases}$$

Then

$$[f_c^x]^*(t) = [f_c^x(t)]^* = \begin{cases} x^*\ , t \in [a,c[\ , \\ 0\ , t \in [c,b]\ . \end{cases}$$

So $f_c^x \in G([a,b],A)$ is self-adjoint iff x is self-adjoint. Moreover

$$\{[f_c^x]^*.f_c^x\}(t) = \begin{cases} x^*.x\ , t \in [a,c[\ , \\ 0\ , t \in [c,b]\ . \end{cases}$$

and the spectrum of $f_c^x \in G([a,b],A)$ is the set

$$\sigma(f_c^x) = \{\lambda \in \mathbb{C}\ :\ f_c^x - \lambda\, e \text{ is not invertible at } G([a,b],A)\}$$

We have that

$$f_c^x(t) - \lambda\, e(t) = [\mathcal{X}_{[a,c[}x](t) - \lambda\, e(t) = \begin{cases} x - \lambda\, e_A\ , t \in [a,c[\ , \\ -\lambda\, e_A\ \ , t \in [c,b]\ . \end{cases}$$

and so, $f_c^x(t) - \lambda\, e(t)$ is invertible iff $x - \lambda\, e_A, -\lambda\, e_A \in A$ are both invertible elements, that is, $\lambda \neq 0$ and x is invetirble. Recall that

$$\sigma(x) = \{\lambda \in \mathbb{C}\ :\ x - \lambda\, e_A \text{ is not invertible at } A\}$$

If $\lambda \in \sigma(x)$ and $\lambda \neq 0$, then $f_c^x(t) - \lambda\, e(t)$ is not invertible, and so $\lambda \in \sigma(f_c^x(t))$, i. e,

$$\sigma(x) - \{0\} \subset \sigma(f_c^x(t))$$

Remark 3.2. Analogously we have the same results for the functions g_d^y,

$$g_d^y(t) = [\mathcal{X}_{]d,b]}y](t) = \begin{cases} 0 \ , t \in [a,d] \ , \\ \\ y \ , t \in]d,b] \ . \end{cases}$$

and

$$[g_d^y]^*(t) = [g_d^y(t)]^* = \begin{cases} 0 \ , t \in [a,d] \ , \\ \\ y^* \ , t \in]d,b] \ . \end{cases}$$

3.1. **Matrices.** Here we start the application of the notions and results of the previous sections in the special case when $I = [0,T]$ and $\mathcal{A} = S_2(\mathbb{R})$. Let $\mathcal{S}_2(\mathbb{R}) \subset M_2(\mathbb{R})$ be the subset of all matrices of the form

$$T = \begin{bmatrix} p & q \\ 0 & p \end{bmatrix}$$

where $p, q \in \mathbb{R}$, where the multiplication is the usual multiplication of matrices and the norm is the induced by the norm on $M_2(\mathbb{R})$, defined as $\|M\| = \sqrt{\lambda}$, where $\lambda = \max\{\gamma : \gamma$ is a eigenvalue of $(adj\ M)\ M\}$. If $M \in \mathcal{S}_2(\mathbb{R})$ then

$$(adj\ M)\ M = \begin{bmatrix} p & -q \\ 0 & p \end{bmatrix} \begin{bmatrix} p & q \\ 0 & p \end{bmatrix} = \begin{bmatrix} p^2 & 0 \\ 0 & p^2 \end{bmatrix}$$

whose eigenvalues are $\lambda = p^2$. So $\|M\| = |p|$.

Remark 3.3. $\mathcal{S}_2(\mathbb{R})$ is the subspace generated by

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} ,$$

the sub-multiplicity condition is satisfied and then $\mathcal{S}_2(\mathbb{R})$ is a 2-dimensional commutative Banach algebra. Moreover, the function $* : M \in \mathcal{S}(\mathbb{R}) \to M^* \in \mathcal{S}_2(\mathbb{R})$, where $M^* = adj\ M$, is an involution on $\mathcal{S}_2(\mathbb{R})$, and is valid the C^*-identity

$$\|A\ A^*\| = \left\| \begin{bmatrix} p & q \\ 0 & p \end{bmatrix} \begin{bmatrix} p & -q \\ 0 & p \end{bmatrix} \right\| = \left\| \begin{bmatrix} p^2 & 0 \\ 0 & p^2 \end{bmatrix} \right\| = |p|^2 = \|A\|^2 ,$$

that is, $\mathcal{S}_2(\mathbb{R})$ is a 2-dimensional commutative C^*-algebra. So, using Theorem 2.3, the function space $G([a,b], \mathcal{S}_2(\mathbb{R}))$ is a commutative C^*-algebra.

3.2. **Integral functionals on** $G([a,b], \mathcal{S}_2(\mathbb{R}))$**.** Let $M : [a,b] \to \mathcal{S}_2(\mathbb{R})$ be a function. Then

$$M(t) = \begin{bmatrix} p(t) & q(t) \\ 0 & p(t) \end{bmatrix} ,$$

where p, q are real functions, $t \in [a,b]$. Is easy to see that

Lemma 3.4. *If $p, q \to \mathbb{R}$ are two regulated functions, then $M : [a, b] \to \mathcal{S}_2(\mathbb{R})$ defined as*

$$M(t) = \begin{bmatrix} p(t) & q(t) \\ 0 & p(t) \end{bmatrix} ,$$

is a regulated function, that is, $M \in G([a, b], \mathcal{S}_2(\mathbb{R}))$.

Proof. Is sufficient to identify $\mathcal{S}_2(\mathbb{R})$ with a 2-dimensional subspace of $\mathbb{R}^4$ (see [9]). ☐

Remark 3.5. In this case

$$\|M\| = \sup_{t\in[a,b]} \{\|M(t)\|\} = \sup_{t\in[a,b]} \{|p(t)|\}$$

Let $M : [0, T] \to S_2(\mathbb{R})$ be a function. Then, for $t \in [0, T]$,

$$M(t) = \begin{bmatrix} p(t) & q(t) \\ 0 & p(t) \end{bmatrix} ,$$

where p, q are real functions on $[0, T]$. If $\beta : [a, b] \to L(\mathcal{S}_2(\mathbb{R})), \mathbb{C})$ (or $\mathbb{C}$) a bounded variation function, then $\mathcal{F} : G([a, b], \mathcal{S}_2(\mathbb{R})) \to \mathbb{C}$ (or $\mathbb{C}$) defined as

$$\mathcal{F}(M) = \int_a^b \cdot\, d\, \beta(s) \cdot M(s)$$

is a bounded linear functional. We construct bellow a family of these kind of functionals.

Example 3.6. We know that all linear functional $T : \mathcal{S}_2(\mathbb{R}) \to \mathbb{R}$ is defined by

$$T : M(t) = \begin{bmatrix} p & q \\ 0 & p \end{bmatrix} \in \mathcal{S}_2(\mathbb{R}) \longmapsto r\ p + s\ q \in \mathbb{R}.$$

for some $r, s \in \mathbb{R}$. We denote by $T_{r,s}$ such functional, to a fixed couple of real numbers r, s, and consider $\gamma_c : [a, b] \to \mathcal{L}(\mathcal{S}_2(\mathbb{R}), \mathbb{R})$,

$$\gamma_c(t) \cdot M = \begin{cases} 0\ ,\ t \in [a, c[\ , \\ \\ T_{r,s}\ M\ ,\ t \in [c, b]\ . \end{cases} = \begin{cases} 0\ ,\ t \in [a, c[\ , \\ \\ r\ p + s\ q\ ,\ t \in [c, b]\ . \end{cases}$$

As in the proof of the Theorem 2.5, we choose the partition $d : a = t_0 < t_1 < .. < t_n = b$ such that $t_k = c$. We have then $\gamma_c(t_i) - \gamma_c(t_{i-1}) = 0$, if $i \neq k$. Then, if $M : [a, b] \to \mathcal{S}_2(\mathbb{R})$ is a regulated function,

$$\sum_{i=1}^{n} [\gamma_c(t_i) - \gamma_c(t_{i-1})] \cdot M(\xi_i^{\cdot}) = [\underbrace{\gamma_c(t_k)}_{T_{r,s}} - \underbrace{\gamma_c(t_{k-1})}_{0}] \cdot M(\xi_i^{\cdot}) = T_{r,s} \cdot M(\xi_k^{\cdot})$$

where $\xi_k^{\cdot} \in]t_{k-1}, t_k[$. Now we have that

$$\mathcal{F}_{\gamma_c}(M) = \int_a^b \cdot\, d\, \gamma_c(t) \cdot M(t) = \lim_{d\in D} \sum_{i=1}^{|d|} [\gamma_c(t_i) - \gamma_c(t_{i-1})] \cdot M(\xi_i^{\cdot})$$
$$= T_{r,s} \cdot M(c^-) = r\ p(c^-) + s\ q(c^-)$$

Remark 3.7. In particular, if $r \geq 0$, since

$$M^*(t)M(t) = \begin{bmatrix} p^2(t) & 0 \\ 0 & p^2(t) \end{bmatrix} ,$$

$$\mathcal{F}_{\gamma_c}(M^*M) = \int_a^b \cdot\, d\, \gamma_c(t) \cdot [M^*M](t) = T_{r,s} \cdot [M^*M](c^-) = r\ p^2(c^-) \geq 0 ,$$

that is, $\mathcal{F}_{\gamma_c}$ is a positive linear functional on $G([a,b], \mathcal{S}_2(\mathbb{R}))$. Moreover we have that $\|\mathcal{F}_{\gamma_c}(I_2)\| = r$,

$$|\mathcal{F}_{\gamma_c}(M)|^2 \leq \mathcal{F}_{\gamma_c}(I_2)\ \mathcal{F}_{\gamma_c}(M^*M) = [r\ p(c^-)]^2 ,$$

because I_2 is the unit element of $\mathcal{S}_2(\mathbb{R})$, $\mathcal{F}_{\gamma_c}$ is continuous and

$$\|\mathcal{F}_{\gamma_c}\| = \mathcal{F}_{\gamma_c}(I_2) = r .$$

Observe that if

$$M(t) = \begin{bmatrix} p(t) & q(t) \\ 0 & p(t) \end{bmatrix} ,\ N(t) = \begin{bmatrix} \mu(t) & \eta(t) \\ 0 & \mu(t) \end{bmatrix} ,$$

we have $\|M^*(t)M(t)\| = \|M(t)\|^2 = |p(t)|$ and

$$N^*M = \begin{bmatrix} \mu(t)\ p(t) & \mu(t)\ q(t) - \eta(t)\ p(t) \\ 0 & \mu(t)\ p(t) \end{bmatrix} ,$$

and so

$$\mathcal{F}_{\gamma_c}(N^*M) = r\ \mu(c^-)\ p(c^-) + s\ [\mu(c^-)\ q(c^-) - \eta(c^-)\ p(c^-)]$$
$$= \mu(c^-)\ [r\ p(c^-) + s\ q(c^-)] - s\ \eta(c^-)\ p(c^-).$$

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