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A MOMENT PROBLEM ON SOME TYPES OF HYPERGROUPS

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ABSTRACT. The classical moment problem is formulated for commutative hypergroups and the uniqueness is proved for polynomial hypergroups in a single variable and for Sturm–Liouville hypergroups.

1. INTRODUCTION

The classical moment problem published in 1894 by Thomas Jan Stieltjes (see [8]) is the following: Given a sequence $s_0, s_1, \ldots$ of real numbers. Find necessary and sufficient conditions for the existence of a measure μ on $[0, \infty[$ so that

$$s_n = \int_0^\infty x^n \, d\mu(x)$$

holds for $n = 0, 1, \ldots$. Another form of the moment problem, also called "Hausdorff's moment problem" or the "little moment problem," may be stated as follows: Given a sequence of numbers $(s_n)_{n=0}^\infty$, under what conditions is it possible to determine a function α of bounded variation in the interval $]0, 1[$ such that

$$s_n = \int_0^1 x^n \, d\alpha(x)$$

for $n = 0, 1, \ldots$. Such a sequence is called a *moment sequence*, and Felix Hausdorff (see [2], [3]) was the first to obtain necessary and sufficient conditions for a sequence to be a moment sequence. In both cases the question of uniqueness of μ, resp. α arise. For a detailed discussion on classical moment problems see e.g. [1].

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Let K be a commutative hypergroup and N a nonnegative integer. We say that the continuous functions $\varphi_k : K \to \mathbb{C}$ $(k = 0, 1, \dots)$ form a *generalized moment function sequence* if the equations

$$\varphi_k(x * y) = \sum_{j=0}^{k} \binom{k}{j} \varphi_j(x)\, \varphi_{k-j}(y) \tag{1.1}$$

hold for $k = 0, 1, \dots$ and for each x, y in K (see [5]). In this case the function φ_k is called a *generalized moment function of order* k. In particular, generalized moment functions of order 0 are exactly the exponentials on K. For more about generalized moment function sequences see [9, 5, 4, 6, 7].

Let μ be a compactly supported measure on K and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence. Then for each natural number n the complex number

$$m_n = \int_K \varphi_n \, d\mu$$

is called the *n-th generalized moment of* μ with respect to the given generalized moment function sequence. In this case the sequence $(m_n)_{n=0}^{\infty}$ is called the *generalized moment sequence of the measure* μ with respect to the given generalized moment function sequence.

In this setting we can formulate the problem of existence: Let the generalized moment function sequence $(\varphi_k)_{k=0}^{\infty}$ and the sequence of complex numbers $(m_n)_{n=0}^{\infty}$ be given. Under what conditions is there a compactly supported measure μ on K such that $(m_n)_{n=0}^{\infty}$ is the generalized moment sequence of the measure μ with respect to the given generalized moment function sequence? The other basic question is about the uniqueness: if the compactly supported measures μ and ν have the same generalized moment sequences with respect to the given generalized moment function sequence, then does it follow $\mu = \nu$?

In this paper we study the problem of uniqueness and solve it in the case of polynomial hypergroups in a single variable and in the case of Sturm–Liouville hypergroups.

2. The case of polynomial hypergroups

In this section we shall use the results in [5] on the representation of generalized moment functions on polynomial hypergroups in the following form.

Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. Then there exists a sequence $(c_k)_{k=0}^{\infty}$ such that for each natural number N we have

$$\varphi_k(n) = \big(P_n \circ f\big)^{(k)}(0) \ \ (k = 0, 1, \dots, N)\,,$$

where

$$f(t) = \sum_{j=0}^{N} c_j \frac{t^j}{j!}$$

for each t in $\mathbb{R}$.

Theorem 2.1. *Let Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, μ a finitely supported measure on $\mathbb{N}$ and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. If $\varphi_1 \neq 0$ and*

$$\int_{\mathbb{N}} \varphi_k(n)\, d\mu(n) = 0$$

for $k = 0, 1, 2, \ldots,$ then $\mu = 0$.

Proof. First we remark that

$$\int_{\mathbb{N}} \varphi_k(n)\, d\mu(n) = \sum_{n=0}^{N} \varphi_k(n)\mu_n.$$

Hence, by assumption, we have the following system of equations

$$\sum_{n=0}^{N} \varphi_k(n)\mu_n = 0 \tag{2.1}$$

for $k = 0, 1, 2, \ldots, N$.

On the other hand, by the result quoted from [5] we have that

$$\varphi_k(n) = \big(P_n \circ f\big)^{(k)}(0) \tag{2.2}$$

for $k = 0, 1, 2, \ldots, N, \;\; n = 0, 1, 2, \ldots, N$, where

$$f(t) = \sum_{i=0}^{N} c_i \frac{t^i}{i!}$$

is a polynomial. Let $\lambda = f(0)$. From (2.2) we have for $k = 1$

$$\varphi_1(n) = P_n'(\lambda)\, c_1\,,$$

which implies $c_1 \neq 0$.

Let n be a fixed nonnegative integer and we let for $k = 0, 1, 2, \ldots, N$ and for each t in $\mathbb{R}$:

$$F_k(t) = \big(P_n \circ f\big)^{(k)}(t)\,.$$

We show that for $k = 0, 1, 2, \ldots, N$ and for each t in $\mathbb{R}$

$$F_k(t) = \sum_{j=0}^{k} p_{k,j}(t) P_n^{(j)}\big(f(t)\big)\,, \tag{2.3}$$

where $p_{k,j}$ is a polynomial and $p_{k,k}(t) = f'(t)^k$.

We prove (2.3) by induction on k. For $k = 0$ the statement is trivial with $p_{0,0}(t) = 1$. Supposing that (2.3) is proved we show it for $k+1$ instead of k. We have

$$F_{k+1}(t) = F_k'(t) = \sum_{j=0}^{k} p_{k,j}'(t) P_n^{(j)}\big(f(t)\big) + \sum_{j=0}^{k} p_{k,j}(t) P_n^{(j+1)}\big(f(t)\big)\, f'(t)$$

and this is the form (2.3) with $k+1$ for k. Moreover, $p_{k+1,k+1}(t) = p_{k,k}(t) \cdot f'(t) = f'(t)^{k+1}$. Then, using (2.2), we have

$$\varphi_k(n) = \sum_{j=0}^{k} c_{k,j}\, P_n^{(j)}(\lambda) \qquad\qquad k = 0, 1, 2, \ldots, N\,,$$

where $c_{k,k} = f'(0)^k \neq 0$, $c_{0,0} = 1$. By (2.1) it follows

$$\sum_{n=0}^{N} \sum_{j=0}^{k} c_{k,j}\, P_n^{(j)}(\lambda)\mu_n = 0 \tag{2.4}$$

for $k = 0, 1, 2, \ldots, N$. For $k = 0$ this means

$$\sum_{n=0}^{N} P_n(\lambda)\mu_n = 0\,. \tag{2.5}$$

Now let $k = 1$ in (2.4), then we have by (2.5)

$$\sum_{n=0}^{N} c_{1,0}\, P_n(\lambda)\mu_n + c_{1,1}\, P_n'(\lambda)\mu_n = c_{1,0} \sum_{n=0}^{N} P_n(\lambda)\mu_n + c_{1,1} \sum_{n=0}^{N} P_n'(\lambda)\mu_n =$$

$$c_{1,1} \sum_{n=0}^{N} P_n'(\lambda)\mu_n = 0\,.$$

As $c_{1,1} \neq 0$, then it follows:

$$\sum_{n=0}^{N} P_n'(\lambda)\mu_n = 0\,.$$

Continuing this process we get the system of equations

$$\sum_{n=0}^{N} P_n^{(k)}(\lambda)\mu_n = 0\,, \tag{2.6}$$

for $k = 0, 1, 2, \ldots, N$. Observe, that the degree of P_n is exactly n, hence we can rewrite (2.6) in the form

$$\sum_{n=k}^{N} P_n^{(k)}(\lambda)\mu_n = 0\,,$$

for $k = 0, 1, 2, \ldots, N$. This is a homogeneous system of linear equations for the unknowns μ_n, $n = 0, 1, \ldots, N$. The fundamental matrix of this system is an $N \times N$ upper triangular matrix with the nonzero numbers $P_k^{(k)}(\lambda)$ in the main diagonal, hence this matrix is regular, which means that the system has only trivial solution: $\mu_n = 0$ for $n = 0, 1, 2, \ldots, N$. This means $\mu = 0$ and the proof is complete. $\square$

This result obviously implies the following uniqueness theorem.

Theorem 2.2. *Let Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, μ, ν finitely supported measures on $\mathbb{N}$ and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. If $\varphi_1 \neq 0$ and the generalized moment sequences of μ and ν with respect to the given generalized moment function sequence are the same, then $\mu = \nu$.*

3. The case of Sturm–Liouville hypergroups

Following the previous ideas in this section we extend consider the same problem on Sturm–Liouville hypergroups. We shall use the results in [6] on the representation of generalized moment functions on polynomial hypergroups in the following form.

Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup, and let Φ the exponential family of the hypergroups K (see [6]). This means that for each z in $\mathbb{C}$ and x in $\mathbb{R}_+$ the function Φ satisfies

$$\partial_1^2 \Phi(x,z) + \frac{A'(x)}{A(x)}\, \partial_1 \Phi(x,z) = z\, \Phi(x,z)\,,$$

further $\Phi(0,z) = 1$ and $\partial_1 \Phi(0,z) = 0$. It follows, that the function $z \mapsto \Phi(x,z)$ is entire for each x in $\mathbb{R}_0$. Let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. Then there exists a sequence $(c_k)_{k=0}^{\infty}$ such that for each natural number N we have

$$\varphi_k(x) = \frac{d^k}{dt^k} \Phi\big(x, f(t)\big)(0)$$

for $k = 0, 1, 2, \ldots, N$, x in $\mathbb{R}_0$, t in $\mathbb{C}$, where

$$f(t) = \sum_{i=0}^{N} c_i\, \frac{t^i}{i!}$$

is a polynomial.

Theorem 3.1. *Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup, μ a compactly supported measure on $\mathbb{R}_0$ and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. If $\varphi_1 \neq 0$ and*

$$\int_{\mathbb{R}_0} \varphi_k(x)\, d\mu(x) = 0$$

for $k = 0, 1, 2, \ldots,$ then $\mu = 0$.

Proof. We show that if

$$\int_{\mathbb{R}_0} \varphi_k(x)\, d\mu(x) = 0 \tag{3.1}$$

for $k = 0, 1, 2, \ldots,$ then $\mu = 0$.

Let N be a fixed positive integer. By the result quoted above from [6] we have

$$\varphi_k(x) = \frac{d^k}{dt^k} \Phi\big(x, f(t)\big)(0) \tag{3.2}$$

for $k = 0, 1, 2, \ldots, N$ x in $\mathbb{R}_0$, t in $\mathbb{C}$, where

$$f(t) = \sum_{i=0}^{N} c_i\, \frac{t^i}{i!}$$

is a polynomial. Let $\lambda = f(0)$. From (3.2) we have for $k = 1$

$$\varphi_1(x) = \frac{d}{dt})\Phi\big((x, f(t)\big)(0) = \partial_2\Phi(x,\lambda)\, c_1\,,$$

which implies $c_1 \neq 0$.

Let x be a fixed nonnegative real number and we let for $k = 0, 1, 2, \ldots, N$ and for each t in $\mathbb{R}$:

$$F_k(t) = \frac{d^k}{dt^k}\Phi\big(x, f(t)\big)\,.$$

We show that for $k = 0, 1, 2, \ldots, N$ and for each t in $\mathbb{R}$

$$F_k(t) = \sum_{j=0}^{k} p_{k,j}(t)\partial_2^{(j)}\Phi\big(x, f(t)\big)\,, \tag{3.3}$$

where $p_{k,j}$ is a polynomial, and $p_{k,k}(t) = f'(t)^k$.

We prove (3.3) by induction on k. For $k = 0$ the statement is trivial. Supposing that (3.3) is proved we show it for $k+1$ instead of k. We have

$$F_{k+1}(t) = F_k'(t) = \sum_{j=0}^{k} p_{k,j}'(t)\partial_2^{(j)}\Phi\big(x, f(t)\big) + \sum_{j=0}^{k} p_{k,j}(t)\partial_2^{(j+1)}\Phi\big(x, f(t)\big)\, f'(t)$$

and this is the form (3.3) with $k+1$ for k. Moreover, $p_{k+1,k+1}(t) = p_{k,k}(t)\cdot f'(t) = f'(t)^k$.

Then, using (3.2), we have

$$\varphi_k(x) = \sum_{j=0}^{k} c_{k,j}\partial_2^{(j)}\Phi\big(x,\lambda\big) \qquad\qquad k = 0, 1, 2, \ldots, N\,,$$

where $c_{k,k} \neq 0$, $c_{0,0} = 1$.

By (3.1) it follows

$$\sum_{j=0}^{k} c_{k,j}\int_{\mathbb{R}_0} \partial_2^{(j)}\Phi\big(x,\lambda\big)\, d\mu(x) = 0$$

for $k = 0, 1, 2, \ldots,$. For $k = 0$ this gives

$$\int_{\mathbb{R}_0} \Phi(x,\lambda)\, d\mu(x) = 0\,. \tag{3.4}$$

For $k = 1$ we have

$$c_{1,0}\int_{\mathbb{R}_0} \Phi(x,\lambda)\, d\mu(x) + c_{1,1}\int_{\mathbb{R}_0} \partial_2\Phi\big(x,\lambda\big)\, d\mu(x) = 0\,. \tag{3.5}$$

By (3.4) and $c_{1,1} \neq 0$ this implies

$$\int_{\mathbb{R}_0} \partial_2\Phi(x,\lambda)\, d\mu(x) = 0\,.$$

Continuing this process we arrive at

$$\int_{\mathbb{R}_0} \partial_2^{(k)}\Phi(x,\lambda)\, d\mu(x) = 0 \tag{3.6}$$

for $k=0,1,2,\dots,N$. As N is arbitrary, we actually have that (3.6) holds for $k=0,1,\dots$.

We recall that the function $\hat{\mu}:\mathbb{C}\to\mathbb{C}$ defined for each complex z by

$$\hat{\mu}(z)=\int_{\mathbb{R}_0}\Phi(x,z)\,d\mu(x) \tag{3.7}$$

is the Fourier–Laplace transform of the measure μ. As μ is compactly supported, $\hat{\mu}$ is an entire function. On the other hand, as the integration in (3.7) is performed on the compact support of μ and $z\mapsto\Phi(x,z)$ is an entire function, hence the differentiation and the integration in (3.5) can be interchanged. This means that we have

$$\hat{\mu}^{(k)}(z)\frac{d^k}{dz^k}\int_{\mathbb{R}_0}\Phi(x,z)\,d\mu(x)$$

holds for $k=0,1,2,\dots$, and for all z in $\mathbb{C}$. In particular, for $z=\lambda$

$$\hat{\mu}^{(k)}(\lambda)\frac{d^k}{dz^k}\int_{\mathbb{R}_0}\Phi(x,\lambda)\,d\mu(x)=0\,.$$

As $\hat{\mu}$ is an entire function, it follows $\hat{\mu}=0$. Then $\mu=0$ and our statement is proved. □

Similarly as above, we have the corresponding uniqueness result.

Theorem 3.2. *Let $K=(\mathbb{R}_0,A)$ be a Sturm–Liouville hypergroup, μ,ν compactly supported measures on $\mathbb{R}_0$ and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K. If $\varphi_1\neq 0$ and the generalized moment sequences of μ and ν with respect to the given generalized moment function sequence are the same, then $\mu=\nu$.*

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