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EIGENVALUE PROBLEM FOR A CLASS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study eigenvalue problem for a class of nonlinear fractional differential equations

$$D_{0+}^{\alpha}u(t) = \lambda f(u(t)), \quad 0 < t < 1,$$
$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where $3 < \alpha \leq 4$ is a real number, D_{0+}^{α} is the Riemann–Liouville fractional derivative, λ is a positive parameter and $f : (0, +\infty) \to (0, +\infty)$ is continuous. By the properties of the Green function and Guo–Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established. As an application, some examples are presented to illustrate the main results.

1. INTRODUCTION

The theory of fractional derivatives goes back to Leibniz's note in his list to L'Hospital, dated 30 September 1695, in which the meaning of the derivative of order 1/2 is discussed Leibniz's note led to the appearance of the theory of derivatives and integrals of arbitrary order, which by the end of nineteenth century took more or less finished form due primarily to Liouville, Grünwald, Letnikov and Riemann. Recently, there have been several books on the subject of fractional derivatives and fractional integrals, see [1]–[2].

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For three centuries the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics useful only for mathematicians. However, in the last few decades many authors pointed out that fractional derivatives and fractional integrals are very suitable for the description of properties of various real problems.

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in selfsimilar and porous structures, fluid flows, electrical networks, viscoelasticity, chemical physics, and many other branches of science. There have appeared lots of works, in which fractional derivatives are used for a better description of considered material properties, mathematical modelling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to such equations.

It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear fractional differential equations. Recently, there are some papers dealing with the existence of solutions (or positive solutions) of nonlinear initial (or boundary) fractional differential equation (or system) by the use of techniques of nonlinear analysis (fixed-point theorems, Leray–Schauder theory, Adomian decomposition method, etc.), see [3]–[5]. In fact, there has the same requirements for boundary conditions, see [4]–[5].

Xu et al. [5] considered the existence of positive solutions for the following problem

$$D^{\alpha}_{0+}u(t) = f(t, u(t)), \quad 0 < t < 1,$$
$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where $3 < \alpha \leq 4$ is a real number, D^{α}_{0+} is the Riemann–Liouville fractional derivative. By using the properties of the Green function, they gave some multiple positive solutions for singular and nonsingular boundary value problems, and also they gave uniqueness of solution for singular problem by means of Leray–Schauder nonlinear alternative, Guo–Krasnosel'skii fixed point theorem on cones and a mixed monotone method.

To the best of our knowledge, there is very little known about eigenvalue problem for a class of nonlinear fractional differential equations involving Riemann–Liouville fractional derivative

$$D^{\alpha}_{0+}u(t) = \lambda f(u(t)), \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0, \tag{1.2}$$

where $3 < \alpha \leq 4$ is a real number, D^{α}_{0+} is the Riemann–Liouville fractional derivative, λ is a positive parameter and $f : (0, +\infty) \to (0, +\infty)$ is continuous.

In mathematics, an eigenfunction of a linear operator A defined on some function space is any non-zero function f in that space that returns from the operator exactly as is, except for a multiplicative scaling factor.The existence of eigenfunctions is typically the most insightful way to analyze A. In fact, eigenfunctions play an important role in many branches of physics, such as the Schrodinger equation in quantum mechanics. One example is that the solution of the vibrating drum problem is, at any point in time, an eigenfunction of the Laplace operator on a disk. Hence, by incorporating Riemann–Liouville Fractional Derivatives, our endeavor is to find the existence of eigenfunctions of (1.1) and (1.2).

It is well known that in mechanics the boundary value problem (1.1) and (1.2) where $\alpha = 4$ describes the deflection of an elastic beam rigidly fixed at both ends. The integer order boundary value problem (1.3) and (1.4) has been studied extensively, see [6]–[8]. In [7, 8], Yao studied

$$u''''(t) = \lambda f(t, u(t)), \quad 0 < t < 1, \tag{1.3}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0, \tag{1.4}$$

and using a Krasnosel'skii fixed point theorem, derived a λ-interval such that, for any λ-lying in this interval, the beam equation has existence and multiplicity on positive solution.

On the one hand, the boundary value problem in [5] is the particular case of problem (1.1) and (1.2) as the case of $\lambda = 1$. On the other hand, as Xu et al. discussed boundary value problem in the reference [5], we also give some existence results by the fixed point theorem on a cone for boundary value problem (1.1) and (1.2) in this paper. Moreover, the purpose of this paper is to derive a λ-interval such that, for any λ lying in this interval, the problem (1.1) and (1.2) has existence and multiplicity on positive solutions.

There have been a few papers which deal with nonlinear fractional differential equation the boundary value problem. Analogy with boundary value problems for differential equations of integer order, we firstly give the corresponding Green function named by fractional Green's function and some properties of the Green function. Consequently, the problem (1.1) and (1.2) is reduced to a equivalent Fredholm integral equation. Finally, by the properties of the Green function and Guo–Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for nonexistence and existence of at least one or two positive solutions for the boundary value problem are established in this paper. As an application, some examples are presented to illustrate the main results.

2. Preliminaries

For the convenience of the reader, we give some background materials from fractional calculus theory to facilitate analysis of problem (1.1) and (1.2). These materials can be found in the recent literature, see [5, 9, 10].

Definition 2.1. ([9]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2. ([9]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

From the definition of the Riemann–Liouville derivative, we can obtain the following statement.

Lemma 2.3. *([5]) Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, as unique solutions, where n is the smallest integer greater than or equal to α.

Lemma 2.4. *([5]) Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \text{ for some } c_i \in \mathbb{R},\ i = 1, 2, \cdots, n,$$

where n is the smallest integer greater than or equal to α.

In the following, we present the Green function of fractional differential equation boundary value problem.

Lemma 2.5. *([5]) Let $h \in C[0,1]$ and $3 < \alpha \le 4$. The unique solution of problem*

$$D_{0+}^{\alpha} u(t) = h(t), \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0, \tag{2.2}$$

is

$$u(t) = \int_0^1 G(t,s) h(s) ds,$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1} + (1-s)^{\alpha-2} t^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-2} (1-s)^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Here $G(t,s)$ is called the Green function of boundary value problem (2.1) and (2.2).

The following properties of the Green function play important roles in this paper.

Lemma 2.6. *([5]) The function $G(t,s)$ defined by (2.3) satisfies the following conditions:*

(1) $G(t,s) = G(1-s, 1-t)$, *for* $t,\ s \in (0,1)$;
(2) $(\alpha-2) t^{\alpha-2} (1-t)^2 s^2 (1-s)^{\alpha-2} \le \Gamma(\alpha) G(t,s) \le M_0 s^2 (1-s)^{\alpha-2}$, *for* $t,\ s \in (0,1)$;
(3) $G(t,s) > 0$, *for* $t,\ s \in (0,1)$;

(4) $(\alpha-2)s^2(1-s)^{\alpha-2}t^{\alpha-2}(1-t)^2 \le \Gamma(\alpha)G(t,s) \le M_0 t^{\alpha-2}(1-t)^2$, *for* $t,\ s\in(0,1)$,

here $M_0=\max\{\alpha-1,\ (\alpha-2)^2\}$.

The following lemma is fundamental in the proofs of our main results.

Lemma 2.7. *([10]) Let X be a Banach space, and let $P\subset X$ be a cone in X. Assume $\Omega_1,\ \Omega_2$ are open subsets of X with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $S:P\to P$ be a completely continuous operator such that, either*

(A1) $\|Sw\|\le\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\ge\|w\|$, $w\in P\cap\partial\Omega_2$, *or*
(A2) $\|Sw\|\ge\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\le\|w\|$, $w\in P\cap\partial\Omega_2$.

Then S has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

For convenience, we set $q(t)=t^{\alpha-2}(1-t)^2,\ k(s)=s^2(1-s)^{\alpha-2}$, then

$$(\alpha-2)q(t)k(s)\le\Gamma(\alpha)G(t,s)\le M_0k(s).$$

3. Main results

In this section, we establish the existence of positive solutions for boundary value problem (1.1) and (1.2).

Let Banach space $E=C[0,1]$ be endowed with the norm $\|u\|=\sup_{0\le t\le 1}|u(t)|$. Define the cone $P\subset E$ by

$$P=\left\{u\in E: u(t)\ge\frac{\alpha-2}{M_0}q(t)\|u\|,\ t\in[0,1]\right\}.$$

Suppose that u is a positive solution of boundary value problem (1.1) and (1.2). Then

$$u(t)=\lambda\int_0^1 G(t,s)f(u(s))ds,\quad t\in[0,1].$$

We define an operator $A_\lambda:P\to E$ as follows

$$(A_\lambda u)(t)=\lambda\int_0^1 G(t,s)f(u(s))ds,\quad t\in[0,1].$$

By Lemma 2.6, we have

$$\|A_\lambda u\|\le\frac{\lambda M_0}{\Gamma(\alpha)}\int_0^1 k(s)f(u(s))ds,$$

$$\begin{aligned}(A_\lambda u)(t)&\ge\frac{\lambda(\alpha-2)}{\Gamma(\alpha)}\int_0^1 q(t)k(s)f(u(s))ds\\ &\ge\frac{\alpha-2}{M_0}q(t)\|A_\lambda u\|.\end{aligned}$$

Thus, $A_\lambda(P)\subset P$.

Then we have the following lemma.

Lemma 3.1. $A_\lambda:P\to P$ *is completely continuous.*

Proof. The operator $A_\lambda : P \to P$ is continuous in view of continuity of $G(t,s)$ and $f(u)$. By means of Arzela–Ascoli theorem, $A_\lambda : P \to P$ is completely continuous. □

For convenience, we denote

$$F_0 = \lim_{u\to 0^+} \sup \frac{f(u)}{u}, \quad F_\infty = \lim_{u\to +\infty} \sup \frac{f(u)}{u},$$

$$f_0 = \lim_{u\to 0^+} \inf \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to +\infty} \inf \frac{f(u)}{u},$$

$$C_1 = \frac{M_0}{\Gamma(\alpha)} \int_0^1 k(s)ds, \quad C_2 = \frac{(\alpha-2)^2}{\Gamma(\alpha)M_0} \int_0^1 q(s)k(s)ds,$$

$$C_3 = \frac{\alpha-2}{\Gamma(\alpha)} \int_0^1 k(s)ds,$$

Theorem 3.2. *If there exists $l \in (0,1)$ such that $q(l)f_\infty C_2 > F_0 C_1$ holds, then for each*

$$\lambda \in \left((q(l)f_\infty C_2)^{-1},\ (F_0C_1)^{-1}\right), \tag{3.1}$$

the boundary value problem (1.1) and (1.2) has at least one positive solution. Here we impose $(q(l)f_\infty C_2)^{-1} = 0$ if $f_\infty = +\infty$ and $(F_0C_1)^{-1} = +\infty$ if $F_0 = 0$.

Proof. Let λ satisfy (3.1) and $\varepsilon > 0$ be such that

$$(q(l)(f_\infty - \varepsilon)C_2)^{-1} \le \lambda \le ((F_0+\varepsilon)C_1)^{-1}. \tag{3.2}$$

By the definition of F_0, we see that there exists $r_1 > 0$ such that

$$f(u) \le (F_0+\varepsilon)u, \quad \text{for } 0 < u \le r_1. \tag{3.3}$$

So if $u \in P$ with $\|u\| = r_1$, then by (3.2) and (3.3), we have

$$\begin{aligned}
\|A_\lambda u\| &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 M_0 k(s) f(u(s))ds \\
&\le \frac{\lambda M_0}{\Gamma(\alpha)} \int_0^1 k(s)(F_0+\varepsilon)r_1 ds \\
&= \lambda(F_0+\varepsilon)r_1 C_1 \\
&\le r_1 = \|u\|.
\end{aligned}$$

Hence, if we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then

$$\|A_\lambda u\| \le \|u\|, \quad \text{for } u \in P \cap \partial\Omega_1. \tag{3.4}$$

Let $r_3 > 0$ be such that

$$f(u) \ge (f_\infty - \varepsilon)u, \quad \text{for } u \ge r_3. \tag{3.5}$$

If $u \in P$ with $\|u\| = r_2 = \max\{2r_1,\ r_3\}$, then by (3.2) and (3.5), we have

$$\begin{aligned}\|A_\lambda u\| &\geq A_\lambda u(l)\\ &= \lambda \int_0^1 G(l,s)f(u(s))ds\\ &\geq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (\alpha-2)q(l)k(s)f(u(s))ds\\ &\geq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (\alpha-2)q(l)k(s)(f_\infty-\varepsilon)u(s)ds\\ &\geq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 \frac{(\alpha-2)^2q(l)}{M_0}q(s)k(s)(f_\infty-\varepsilon)\|u\|ds\\ &= \lambda q(l)C_2(f_\infty-\varepsilon)\|u\| \geq \|u\|.\end{aligned}$$

Thus, if we set $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then

$$\|A_\lambda u\| \geq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.6}$$

Now, from (3.4), (3.6) and Lemma 2.7, we guarantee that A_λ has a fix point $u \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$, and clearly u is a positive solution of (1.1) and (1.2). □

Theorem 3.3. *If there exists $l \in (0,1)$ such that $q(l)C_2f_0 > F_\infty C_1$ holds, then for each*

$$\lambda \in \left((q(l)f_0C_2)^{-1},\ (F_\infty C_1)^{-1}\right), \tag{3.7}$$

the boundary value problem (1.1) and (1.2) has at least one positive solution. Here we impose $(q(l)f_0C_2)^{-1} = 0$ if $f_0 = +\infty$ and $(F_\infty C_1)^{-1} = +\infty$ if $F_\infty = 0$.

Proof. Let λ satisfy (3.7) and $\varepsilon > 0$ be such that

$$(q(l)(f_0-\varepsilon)C_2)^{-1} \leq \lambda \leq ((F_\infty+\varepsilon)C_1)^{-1}. \tag{3.8}$$

From the definition of f_0, we see that there exists $r_1 > 0$ such that

$$f(u) \geq (f_0-\varepsilon)u, \quad \text{for } 0 < u \leq r_1.$$

Further, if $u \in P$ with $\|u\| = r_1$, then similar to the second part of Theorem 3.2, we can obtain that $\|A_\lambda u\| \geq \|u\|$. Thus, if we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then

$$\|A_\lambda u\| \geq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.9}$$

Next, we may choose $R_1 > 0$ such that

$$f(u) \leq (F_\infty+\varepsilon)u, \quad \text{for } u \geq R_1. \tag{3.10}$$

We consider two cases:

Case 1: Suppose f is bounded. Then there exists some $M > 0$, such that

$$f(u) \leq M, \quad \text{for } u \in (0,+\infty).$$

We define $r_3 = \max\{2r_1,\ \lambda M C_1\}$, and $u \in P$ with $\|u\| = r_3$, then

$$\begin{aligned}\|A_\lambda u\| &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 M_0 k(s) f(u(s)) ds \\ &\le \frac{\lambda M}{\Gamma(\alpha)} \int_0^1 M_0 k(s) ds \\ &\le \lambda M C_1 \\ &\le r_3 \le \|u\|.\end{aligned}$$

Hence,

$$\|A_\lambda u\| \le \|u\|, \quad \text{for } u \in P_{r_3} = \{u \in P : \|u\| \le r_3\}. \tag{3.11}$$

Case 2: Suppose f is unbounded. Then there exists some $r_4 > \max\{2r_1,\ R_1\}$, such that

$$f(u) \le f(r_4), \quad \text{for } 0 < u \le r_4.$$

Let $u \in P$ with $\|u\| = r_4$. Then by (3.8) and (3.10), we have

$$\begin{aligned}\|A_\lambda u\| &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 M_0 k(s) f(u(s)) ds \\ &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 M_0 k(s) (F_\infty + \varepsilon) \|u\| ds \\ &\le \lambda C_1 (F_\infty + \varepsilon) \|u\| \\ &\le \|u\|.\end{aligned}$$

Thus, (3.11) is also true.

In both Cases 1 and 2, if we set $\Omega_2 = \{u \in E : \|u\| < r_2 = \max\{r_3,\ r_4\}\}$, then

$$\|A_\lambda u\| \le \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.12}$$

Now that we obtain (3.9) and (3.12), it follows from Lemma 2.7 that A_λ has a fix point $u \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ with $r_1 \le \|u\| \le r_2$. It is clear u is a positive solution of (1.1) and (1.2). □

Theorem 3.4. *Suppose there exist $l \in (0, 1)$, $r_2 > r_1 > 0$ such that $q(l) > \frac{M_0 r_1}{(\alpha-2) r_2}$, and f satisfy*

$$\min_{\frac{\alpha-2}{M_0} q(l) r_1 \le u \le r_1} f(u) \ge \frac{r_1}{\lambda q(l) C_3}, \quad \max_{0 \le u \le r_2} f(u) \le \frac{r_2}{\lambda C_1}.$$

Then the boundary value problem (1.1) and (1.2) has a positive solution $u \in P$ with $r_1 \le \|u\| \le r_2$.

Proof. Choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then for $u \in P \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|A_\lambda u\| &\geq A_\lambda u(l) \\
&= \lambda \int_0^1 G(l,s) f(u(s)) ds \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-2) q(l) k(s) f(u(s)) ds \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-2) q(l) k(s) \min_{\frac{\alpha-2}{M_0} q(l) r_1 \leq u \leq r_1} f(u) ds \\
&\geq \lambda q(l) C_3 \frac{r_1}{\lambda q(l) C_3} \\
&= r_1 = \|u\|.
\end{aligned}$$

On the other hand, choose $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then for $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
\|A_\lambda u\| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1) k(s) f(u(s)) ds \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1) k(s) \max_{0 \leq u \leq r_2} f(u(s)) ds \\
&\leq \lambda C_1 \frac{r_2}{\lambda C_1} \\
&= r_2 = \|u\|.
\end{aligned}$$

Thus, by Lemma 2.7, the boundary value problem (1.1) and (1.2) has a positive solution $u \in P$ with $r_1 \leq \|u\| \leq r_2$. □

For the reminder of the paper, we will need the following condition.

(H) $(\min_{u \in [\frac{\alpha-2}{M_0} q(l) r,\ r]} f(u))/r > 0$, where $l \in (0,1)$.

Denote

$$\lambda_1 = \sup_{r>0} \frac{r}{C_1 \max\limits_{0 \leq u \leq r} f(u)}, \tag{3.13}$$

$$\lambda_2 = \inf_{r>0} \frac{r}{C_3 \min\limits_{\frac{\alpha-2}{M_0} q(l) r \leq u \leq r} f(u)}. \tag{3.14}$$

In view of the continuity of $f(u)$ and (H), we have $0 < \lambda_1 \leq +\infty$ and $0 \leq \lambda_2 < +\infty$.

Theorem 3.5. *Assume (H) holds. If $f_0 = +\infty$ and $f_\infty = +\infty$, then the boundary value problem (1.1) and (1.2) has at least two positive solutions for each $\lambda \in (0, \lambda_1)$.*

Proof. Define

$$a(r) = \frac{r}{C_1 \max\limits_{0 \leq u \leq r} f(u)}.$$

By the continuity of $f(u)$, $f_0 = +\infty$ and $f_\infty = +\infty$, we have that $a(r) : (0, +\infty) \to (0, +\infty)$ is continuous and

$$\lim_{r\to 0} a(r) = \lim_{r\to+\infty} a(r) = 0.$$

By (3.13), there exists $r_0 \in (0, +\infty)$, such that

$$a(r_0) = \sup_{r>0} a(r) = \lambda_1,$$

then for $\lambda \in (0, \lambda_1)$, there exist constants $c_1,\ c_2\ (0 < c_1 < r_0 < c_2 < +\infty)$ with

$$a(c_1) = a(c_2) = \lambda.$$

Thus,

$$f(u) \le \frac{c_1}{\lambda C_1}, \quad \text{for } u \in [0, c_1], \tag{3.15}$$

$$f(u) \le \frac{c_2}{\lambda C_1}, \quad \text{for } u \in [0, c_2]. \tag{3.16}$$

On the other hand, applying the conditions $f_0 = +\infty$ and $f_\infty = +\infty$, there exist constants $d_1,\ d_2\ (0 < d_1 < c_1 < r_0 < c_2 < d_2 < +\infty)$ with

$$\frac{f(u)}{u} \ge \frac{1}{q^2(l)\lambda C_3}, \quad \text{for } u \in (0, d_1) \cup \left(\frac{\alpha-2}{M_0} q(l) d_2, +\infty\right),$$

then

$$\min_{\frac{\alpha-2}{M_0} q(l) d_1 \le u \le d_1} f(u) \ge \frac{d_1(\alpha-2)}{\lambda C_3 q(l) M_0}, \tag{3.17}$$

$$\min_{\frac{\alpha-2}{M_0} q(l) d_2 \le u \le d_2} f(u) \ge \frac{d_2(\alpha-2)}{\lambda C_3 q(l) M_0}. \tag{3.18}$$

By (3.15) and (3.17), (3.16) and (3.18), combining with Theorem 3.4 and Lemma 2.7, we can complete the proof. $\square$

Corollary 3.6. *Assume (H) holds. If $f_0 = +\infty$ or $f_\infty = +\infty$, then the boundary value problem (1.1) and (1.2) has at least one positive solution.*

Theorem 3.7. *Assume (H) holds. If $f_0 = 0$ and $f_\infty = 0$, then for each $\lambda \in (\lambda_2, +\infty)$, the boundary value problem (1.1) and (1.2) has at least two positive solutions.*

Proof. Define

$$b(r) = \frac{r}{C_3 \min_{\frac{\alpha-2}{M_0} q(l) r \le u \le r} f(u)}.$$

By the continuity of $f(u)$, $f_0 = 0$ and $f_\infty = 0$, we easily see that $b(r) : (0, +\infty) \to (0, +\infty)$ is continuous and

$$\lim_{r\to 0} b(r) = \lim_{r\to+\infty} b(r) = +\infty.$$

By (3.14), there exists $r_0 \in (0, +\infty)$, such that

$$b(r_0) = \inf_{r>0} b(r) = \lambda_2.$$

For $\lambda \in (\lambda_2, +\infty)$, there exist constants $d_1,\ d_2\ (0 < d_1 < r_0 < d_2 < +\infty)$ with

$$b(d_1) = b(d_2) = \lambda.$$

Therefore,

$$f(u) \geq \frac{d_1}{\lambda C_3}, \quad \text{for } u \in \left[\frac{\alpha-2}{M_0} q(l) d_1, d_1\right],$$

$$f(u) \geq \frac{d_2}{\lambda C_3}, \quad \text{for } u \in \left[\frac{\alpha-2}{M_0} q(l) d_2, d_2\right].$$

On the other hand, using $f_0 = 0$, we know that there exists a constant c_1 $(0 < c_1 < d_1)$ with

$$\frac{f(u)}{u} \leq \frac{1}{\lambda C_1}, \quad \text{for } u \in (0, c_1),$$

$$\max_{0\leq u\leq c_1} f(u) \leq \frac{c_1}{\lambda C_1}. \tag{3.19}$$

In view of $f_\infty = 0$, there exists a constant $c_2 \in (d_2, +\infty)$ such that

$$\frac{f(u)}{u} \leq \frac{1}{\lambda C_1}, \quad \text{for } u \in (c_2, +\infty).$$

Let

$$M = \max_{0\leq u\leq c_2} f(u) \quad \text{and} \quad c_2 \geq \lambda C_1 M.$$

It is easily seen that

$$\max_{0\leq u\leq c_2} f(u) \leq \frac{c_2}{\lambda C_1}. \tag{3.20}$$

By (3.19) and (3.20), combining with Theorem 3.4 and Lemma 2.7, the proof is completed. □

Corollary 3.8. *Assume (H) holds. If $f_0 = 0$ or $f_\infty = 0$, then for each $\lambda \in (\lambda_2, +\infty)$, the boundary value problem (1.1) and (1.2) has at least one positive solution.*

By the above theorems, we can obtain the following results.

Corollary 3.9. *Assume (H) holds. If $f_0 = +\infty$, $f_\infty = d$, or $f_\infty = +\infty$, $f_0 = d$, then for any $\lambda \in (0, (dC_1)^{-1})$, the boundary value problem (1.1) and (1.2) has at least one positive solution.*

Corollary 3.10. *Assume (H) holds. If $f_0 = 0$, $f_\infty = d$, or if $f_\infty = 0$, $f_0 = d$, then for any $\lambda \in \left((\frac{\alpha-2}{M_0} q(l) dC_2)^{-1}, +\infty\right)$, the boundary value problem (1.1) and (1.2) has at least one positive solution.*

Remark 3.11. For the integer derivative case $\alpha = 4$, Theorems 3.2–3.7 also hold, we can find the corresponding existence results in [7].

4. Nonexistence of Positive Solution

In this section, we give some sufficient conditions for the nonexistence of positive solution to the problem (1.1) and (1.2).

Theorem 4.1. *Assume (H) holds. If $F_0 < +\infty$ and $F_\infty < +\infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, the boundary value problem (1.1) and (1.2) has no positive solution.*

Proof. Since $F_0 < +\infty$ and $F_\infty < +\infty$, there exist positive numbers $m_1,\ m_2,\ r_1$ and r_2, such that $r_1 < r_2$ and

$$f(u) \le m_1 u, \quad \text{for } u \in [0, r_1],$$

$$f(u) \le m_2 u, \quad \text{for } u \in [r_2, +\infty).$$

Let $m = \max\{m_1,\ m_2,\ \max\limits_{r_1 \le u \le r_2}\{\frac{f(u)}{u}\}\}$. Then we have

$$f(u) \le mu, \quad \text{for } u \in [0, +\infty).$$

Assume $v(t)$ is a positive solution of (1.1) and (1.2). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 := (mC_1)^{-1}$. Since $A_\lambda v(t) = v(t)$ for $t \in [0,1]$,

$$\|v\| = \|A_\lambda v\| \le \frac{\lambda M_0}{\Gamma(\alpha)} \int_0^1 k(s) f(v(s)) ds \le \frac{m\lambda M_0}{\Gamma(\alpha)} \|v\| \int_0^1 k(s) ds < \|v\|,$$

which is a contradiction. Therefore, (1.1) and (1.2) has no positive solution. $\square$

Theorem 4.2. *Assume (H) holds. If $f_0 > 0$ and $f_\infty > 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, the boundary value problem (1.1) and (1.2) has no positive solution.*

Proof. By $f_0 > 0$ and $f_\infty > 0$, we know that there exist positive numbers $n_1,\ n_2,\ r_1$ and r_2, such that $r_1 < r_2$ and

$$f(u) \ge n_1 u, \quad \text{for } u \in [0, r_1],$$

$$f(u) \ge n_2 u, \quad \text{for } u \in [r_2, +\infty).$$

Let $n = \min\{n_1,\ n_2,\ \min\limits_{r_1 \le u \le r_2}\{\frac{f(u)}{u}\}\} > 0$. Then we get

$$f(u) \ge nu, \quad \text{for } u \in [0, +\infty).$$

Assume $v(t)$ is a positive solution of (1.1) and (1.2). We will show that this leads to a contradiction for $\lambda > \lambda_0 := \left(\frac{\alpha-2}{M_0} q(l) n C_2\right)^{-1}$. Since $A_\lambda v(t) = v(t)$ for $t \in [0,1]$,

$$\|v\| = \|A_\lambda v\| \ge \frac{\lambda(\alpha - 2)}{\Gamma(\alpha)} \int_0^1 q(l) k(s) f(v(s)) ds > \|v\|,$$

which is a contradiction. Thus, (1.1) and (1.2) has no positive solution. $\square$

5. Examples

In this section, we will present some examples to illustrate the main results.

Example 5.1. Consider the boundary value problem

$$D_{0+}^{\frac{7}{2}}u(t)=\lambda u^a,\quad 0<t<1,\ a>1, \tag{5.1}$$

$$u(0)=u(1)=u'(0)=0. \tag{5.2}$$

Since $\alpha=7/2$, we have $M_0=\max\{\alpha-1,\ (\alpha-2)^2\}=5/2$,

$$C_1=\frac{M_0}{\Gamma(\alpha)}\int_0^1 k(s)ds=\frac{\frac{5}{2}}{\Gamma(\frac{7}{2})}\int_0^1 s^2(1-s)^{\frac{3}{2}}ds=\frac{64}{945\sqrt{\pi}}=0.03821,$$

$$C_2=\frac{(\alpha-2)^2}{\Gamma(\alpha)M_0}\int_0^1 k(s)ds=\frac{\frac{9}{4}}{\frac{5}{2}\Gamma(\frac{7}{2})}\int_0^1 s^2(1-s)^{\frac{3}{2}}ds=\frac{9}{25}C_1=0.01375.$$

Let $f(u)=u^a$, $a>1$. Then we have $F_0=0$, $f_\infty=+\infty$. Choose $l=1/2$. Then $q(1/2)=\sqrt{2}/16=0.08839$. So $q(l)C_2f_\infty>C_1F_0$ holds. Thus, by Theorem 3.2, the boundary value problem (5.1) and (5.2) has a positive solution for each $\lambda\in(0,+\infty)$.

Example 5.2. Discuss the boundary value problem

$$D_{0+}^{\frac{7}{2}}u(t)=\lambda u^b,\quad 0<t<1,\ 0<b<1, \tag{5.3}$$

$$u(0)=u(1)=u'(0)=0. \tag{5.4}$$

Since $\alpha=7/2$, we have $M_0=5/2$, $C_1=0.03821$, $C_2=0.01375$. Let $f(u)=u^b$, $0<b<1$. Then we have $F_\infty=0$, $f_0=+\infty$. Choose $l=1/2$. Then $q(1/2)=\sqrt{2}/16=0.08839$. So $q(l)C_2f_0>C_1F_\infty$ holds. Thus, by Theorem 3.3, the boundary value problem (5.3) and (5.4) has a positive solution for each $\lambda\in(0,+\infty)$.

Example 5.3. Consider the boundary value problem

$$D_{0+}^{\frac{7}{2}}u(t)=\lambda\frac{(70u^2+u)(2+\sin u)}{u+1},\quad 0<t<1, \tag{5.5}$$

$$u(0)=u(1)=u'(0)=u'(1)=0. \tag{5.6}$$

Since $\alpha=7/2$, we have $M_0=5/2$, $C_1=0.03821$, $C_2=0.01375$. Let $f(u)=(70u^2+u)(2+\sin u)/(u+1)$. Then we have $F_0=f_0=2$, $F_\infty=210$, $f_\infty=70$ and $2u<f(u)<210u$.

(i) Choose $l=1/2$. Then $q(1/2)=\sqrt{2}/16=0.08839$. So $q(l)C_2f_\infty>F_0C_1$ holds. Thus, by Theorem 3.2, the boundary value problem (5.5) and (5.6) has a positive solution for each $\lambda\in(11.7546,13.0855)$.

(ii) By Theorem 4.1, the boundary value problem (5.5) and (5.6) has no positive solution for all $\lambda\in(0,0.1246)$.

(iii) Choose $l=1/2$. Then $q(1/2)=\sqrt{2}/16=0.08839$. By Theorem 4.2, the boundary value problem (5.5) and (5.6) has no positive solution for all $\lambda\in(685.6794,+\infty)$.

Example 5.4. Consider the boundary value problem

$$D^{\frac{7}{2}}_{0^+}u(t)=\lambda\frac{(u^2+u)(2+\sin u)}{50u+1},\quad 0<t<1, \tag{5.7}$$

$$u(0)=u(1)=u'(0)=u'(1)=0. \tag{5.8}$$

Since $\alpha=7/2$, we have $M_0=5/2$, $C_1=0.03821$, $C_2=0.01375$. Let $f(u)=(u^2+u)(2+\sin u)/(50u+1)$. Then we have $F_0=f_0=2$, $F_\infty=3/50$, $f_\infty=1/50$ and $u/50<f(u)<2u$.

(i) Choose $l=1/2$. Then $q(1/2)=\sqrt{2}/16=0.08839$. So $q(l)C_2f_0>F_\infty C_1$ holds. Thus, by Theorem 3.3, the boundary value problem (5.7) and (5.8) has a positive solution for each $\lambda\in(411.4076,436.1859)$.

(ii) By Theorem 4.1, the boundary value problem (5.7) and (5.8) has no positive solution for all $\lambda\in(0,13.0855)$.

(iii) Choose $l=1/2$. Then $q(1/2)=\sqrt{2}/16=0.08839$. By Theorem 4.2, the boundary value problem (5.7) and (5.8) has no positive solution for all $\lambda\in(69477.4534,+\infty)$.

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References

1. K.B. Oldham and J. Spanier, *The fractional calculus*, Academic Press, New York, 1974.
2. J.T. Machado, V. Kiryakova and F. Mainardi, *Recent history of fractional calculus*, Commun. Nonlinear Sci. Numer. Simul. **16** (2011), 1140–1153.
3. D. Delbosco and L. Rodino, *Existence and uniqueness for a nonlinear fractional differential equation*, J. Math. Anal. Appl. **204** (1996), 609–625.
4. S. Zhang, *Positive solutions for boundary-value problems of nonlinear fractional differential equations*, Electron. J. Differential Equations **36** (2006), 1–12.
5. X. Xu, D. Jiang and C. Yuan, *Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation*, Nonlinear Anal. **71** (2009), 4676–4688.
6. W. Li and M. Zhang, *Non-degeneracy and uniqueness of periodic solutions for some superlinear beam equations*, Appl. Math. Lett. **22** (2009), 314–319.
7. Q. Yao, *Positive solutions for eigenvalue problems of fourth-order elastic beam equations*, Appl. Math. Lett. **17** (2004), 237–243.
8. Q. Yao, *Existence and multiplicity of positive solutions to a singular elastic beam equation rigidly fixed at both ends*, Nonlinear Anal. **69** (2008), 2683–2694.
9. A.A. Kilbas, H.H. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B. V., Amsterdam, 2006.
10. M.A. Krasnoselskii, *Positive solution of operator equation*, Noordhoff Groningen, 1964.

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