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Osculatory behavior and second dual varieties of Del Pezzo surfaces

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Abstract. Let S be a smooth surface embedded in a projective space, whose general osculating space has the expected dimension. Inside the dual variety of S one can consider the second discriminant locus, which parameterizes the hyperplane sections of S having some singular point of multiplicity $\geqslant 3$. In this paper the various components of the second discriminant loci of Del Pezzo surfaces are investigated from a unifying point of view. This allows us to describe the second dual varieties of such surfaces and to understand their singular loci.

Key words. Surface (complex, projective), Jet bundles, k-jet spannedness, Osculating spaces, Duality

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Introduction

In [5], [6] we studied higher order dual varieties of projective surfaces focusing on the physiology rather than the pathology. While k-regularity is a very nice (but strong) condition for handling k-th dual varieties [5], generic k-regularity looks more acceptable and still is in the range of physiology [6]. However in this setting the most appropriate object to look at in order to have a full understanding of the osculatory behavior of a surface S seems to be the k-th discriminant locus, which parameterizes all hyperplane sections having some singular point of multiplicity $\geqslant k+1$. Of course the k-th dual variety of S is a component of it, but the loci of S where the k-regularity property fails give rise to certain extra components. It is very instructive to describe this discriminant locus in detail for some class of surfaces. In this paper we do that for Del Pezzo surfaces. By the way this allows us to supplement some information on the higher order embedding properties of some line bundles provided by several authors ([3], [4]).

Our motivation for considering Del Pezzo surfaces is the following. Smooth surfaces in $\mathbb{P}^N$ ($N \geqslant 5$) of sectional genus 0 are either rational scrolls or the Veronese surface (for which the second dual variety is not defined). Smooth surfaces of sectional genus 1 are either elliptic scrolls or Del Pezzo surfaces embedded by the anticanonical system. Higher order dual varieties of rational and elliptic scrolls have been extensively studied [11], [8], unlike those of anticanonical Del Pezzo surfaces. So it is quite natural to investigate them from the point of view of their second dual varieties.

Let S be a Del Pezzo surface and let $d = K_S^2$. Then $d \leqslant 9$ and for $d \geqslant 3$ we know that the anticanonical line bundle $L := -K_S$ is very ample. Let $J_k L$ be its k-th jet bundle. In this paper, as a first thing we investigate the rank of the vector bundle map $j_k : S \times H^0(S, L) \to J_k L$ sending every section of L to its k-th jet, at every point $x \in S$. If j_k is generically surjective then the k-th osculating space to S embedded by $|L|$ at a general point has dimension $\binom{k+2}{2} - 1$. In this case we say that (S, L) is generically k-regular. Since $h^0(L) = d + 1 \leqslant 10$, the interest is of course on 2-jets, i.e., $k = 2$, except for $S = \mathbb{P}^2$.

If $S \cong \mathbb{P}^2$ ($\mathbb{P}^1 \times \mathbb{P}^1$), then L is not simply 3-regular (2-regular), but even 3-jet (2-jet) ample. In the remaining cases we compute the rank of $j_{2,x}$ at every point $x \in S$ (Section 2). We do this simply by using the plane model of S, which allows us to describe all hyperplane sections admitting a singular point of multiplicity $\geqslant 3$ at x in a very simple and transparent way (Section 1). Among them we recognize the osculating hyperplane sections. For $d = 6$ we also provide the details for computing directly $\operatorname{rk} j_{2,x}$ in terms of local coordinates through the concrete expression of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^6$ defining S. We insist on this case for two reasons: first generically 2-regular surfaces in $\mathbb{P}^6$ have a single osculating hyperplane at the general point, so that they represent an analog of hypersurfaces in the classical theory of duality; second, the explicit description of the second dual variety $S^\vee$ in this case allows us to understand an interesting example discovered by Togliatti from our point of view. We also have the opportunity to recover and improve some sentences in the literature concerning this example (Section 4). Moreover we prove a sort of self-duality, showing that the second dual variety of the Del Pezzo surface of degree 6 is exactly the surface of Togliatti's example.

In fact L is always generically 2-regular for $d \geqslant 5$ while $S^\vee$ is defined only for $d \geqslant 6$. By using the plane model we compute the degree of $S^\vee$ and of all extra components of the second discriminant locus (Section 2). Moreover, in Section 3 we describe a natural stratification of $S^\vee$ through the singular loci. We also obtain a precise description of the fibres of the morphism $\pi_2 : \overline{\mathbb{P}(\mathscr{K})} \to S^\vee$, where $\mathscr{K}$ is the dual of the kernel of the vector bundle homomorphism defined by j_2 on the dense Zariski open subset of S where it is surjective. In particular it turns out that π_2 is always birational (and in fact finite for $d = 6$). This fact was already known for the Del Pezzo surface of degree 8 isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, which can be seen as a rational geometric conic bundle [5, Proposition 2.6]. Section 5 is devoted to the study of $S^\vee$ in this special case.

We also discuss the degrees $d = 4$ and 3, in which cases (S, L) is nowhere 2-regular (Section 2). When $d \leqslant 2$, L is not very ample, but one can replace it with $-2K_S$ and $-3K_S$ for $d = 2, 1$ respectively. The analysis of the k-jet spannedness properties of these line bundles requires different techniques and will be done elsewhere.

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1 Background material

(1.1) Let X be a smooth complex projective manifold, let $L \in \mathrm{Pic}(X)$ and consider a vector subspace $W \subseteq H^0(X, L)$. Let $x \in X$, let $\mathfrak{m}_x$ be the ideal sheaf of x, and for every integer $k \geqslant 0$ consider the homomorphism

$$j_{k,x} : H^0(X, L) \to \Gamma(L \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1}),$$

sending every section $\sigma \in H^0(X, L)$ to its k-th jet evaluated at x. Note that the range of the homomorphism above is simply $(J_kL)_x$, the fibre of the k-th jet bundle J_kL at the point x. According to [2], we say that: L is *k-jet spanned at x with respect to W* if $j_{k,x|W}$, the homomorphism $j_{k,x}$ restricted to W, is surjective; L is *k-jet spanned on $\mathcal{U}$ with respect to W* if this happens for all $x \in \mathcal{U} \subset X$. We simply say that L is *generically k-jet spanned with respect to W* if $\mathcal{U}$ is a dense Zariski open subset of X and that L is *k-jet spanned with respect to W* if $\mathcal{U} = X$. Moreover we always omit the expression "with respect to W" to mean that $W = H^0(X, L)$. Assume furthermore that $|W|$ is a very ample linear system, i.e., the map $\varphi_W : X \dashrightarrow \mathbb{P}(W)$ defined by W is an embedding; then we shift to the more classical terminology, saying that (X, W) ((X, L), if $W = H^0(X, L)$) is (generically) *k-regular* to mean that L is (generically) k-jet spanned with respect to W. So, in accordance with [5], the expression "(X, L) is k-regular" means that L is very ample and k-jet spanned.

(1.2) Now consider $X \subset \mathbb{P}(W) = \mathbb{P}^N$ embedded by φ_W and identify L with $\mathcal{O}_X(1)$. Let $\mathcal{U} \subseteq X$ be the Zariski dense open subset where $j_{k,x|W}$ attains its maximum, say $s(k) + 1$. The *k-th osculating subspace* to X at a point $x \in \mathcal{U}$ is defined as the $s(k)$-plane $\mathrm{Osc}_x^k(X) := \mathbb{P}(\mathrm{Im}\, j_{k,x|W}) \subset \mathbb{P}(W)$, and a hyperplane $H \in \mathbb{P}^{N\vee}$ is said to be *k-th osculating to X at x* if $H \supseteq \mathrm{Osc}_x^k(X)$. The *$k$-th dual variety* $X_k^\vee$ of (X, W) is defined as the closure in $\mathbb{P}^{N\vee}$ of the locus parameterizing all k-th osculating hyperplanes to X at points of $\mathcal{U}$.

(1.3) Assume that (X, W) is generically k-regular on $\mathcal{U}$. Then $s(k) = \mathrm{rk}(J_kL) - 1 = \binom{n+k}{n} - 1$, where $n = \dim X$. Moreover $j_{k|X \times W}$ gives rise to the exact sequence of vector bundles on $\mathcal{U}$

$$0 \to \mathcal{K}_k^\vee \to W \otimes \mathcal{O}_{\mathcal{U}} \to (J_kL)_{\mathcal{U}} \to 0$$

and we see that $X_k^\vee$ is the image of $\overline{\mathbb{P}(\mathcal{K}_k)} \subset X \times |W|$ via the second projection of $X \times |W|$. We denote by

$$\pi_k : \overline{\mathbb{P}(\mathcal{K}_k)} \to X_k^\vee$$

the morphism induced by this projection.

(1.4) Note that for $x \in \mathcal{U}$, the fact that $H \in |W|$ is a k-th osculating hyperplane to X at x is equivalent to the fact that $H = (\sigma)_0$, where $\sigma \in W$ with $j_{k,x}(\sigma) = 0$. Equivalently, this means that $H \in |W - (k+1)x|$ i.e., the hyperplane section cut out by H on X has a point of multiplicity $\geqslant (k+1)$ at x. Note however that if $x \notin \mathcal{U}$ and $H \in |W - (k+1)x)|$, this does not mean necessarily that $H \in X_k^\vee$. Actually $H \in X_k^\vee$ if and only if H is a limit of k-th osculating hyperplanes to X at points of $\mathcal{U}$. On the other hand we can also consider the *k-th discriminant locus $\mathcal{D}_k(X, W)$ of (X, W)*, which is defined as the image of

$$\mathcal{J} := \{(x, H) \in X \times |W| \,\big|\, H \in |W - (k+1)x|\}$$

via the second projection of $X \times |W|$. It parameterizes all hyperplane sections of X having some singular point of multiplicity $\geqslant k+1$; of course $\mathcal{D}_k(X, W) \supseteq X_k^\vee$ with equality if (X, W) is k-regular. In general $\mathcal{D}_k(X, W)$ contains some extra components coming from the stratification of $X \backslash \mathcal{U}$ given by the rank of $j_{k,x|W}$.

Throughout all the paper we will deal with second dual varieties of generically 2-regular surfaces (S, L). So, to simplify the notation, we will denote by $S^\vee$ the second dual variety of $S \subset \mathbb{P}^N = \mathbb{P}(H^0(S, L))$.

Let S be a Del Pezzo surface and let $d = K_S^2$. Then $d \leqslant 9$ and for $d \geqslant 3$, the anti-canonical line bundle $-K_S$ is very ample and embeds S into $\mathbb{P}^d$. In this Section we always refer to this embedding in discussing the extrinsic geometry of S. So now let $L := -K_S$ and $W = H^0(S, L)$. In view of the classification of Del Pezzo surfaces, for $d = 8$, S is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$, the plane blown-up at a point. In the former case $-K_S = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)$ is 2-jet ample, being the double of a very ample line bundle [3, Corollary 2.1]. For this reason and the fact that all remaining Del Pezzo surfaces admit a common description for all d in terms of linear systems of plane cubics passing through $9 - d$ points in general position, we do not consider the above case here: Section 5 is devoted to it.

(1.5) Let S be a Del Pezzo surface with $d = K_S^2 \geqslant 3$. If $S \neq \mathbb{P}^1 \times \mathbb{P}^1$, then $S = B_{p_1,\dots,p_t}(\mathbb{P}^2)$ is the plane blown-up at $t := 9 - d \leqslant 6$ points in general position [1, p. 45]. Let $\eta : S \to \mathbb{P}^2$ be the blowing-up and let $e_i = \eta^{-1}(p_i)$ for $i = 1, \dots, t$. Then $L = \eta^* \mathcal{O}_{\mathbb{P}^2}(3) - e_1 - \cdots - e_t$.

(1.6) In order to compute the rank of $j_{2,x}$ at all points $x \in S$, here we describe all hyperplane sections of (S, L) having some point of multiplicity $\geqslant 3$ (i.e., all elements of $\mathcal{D}_2(S, L)$). In particular this will give the description of all osculating hyperplane sections (i.e., the elements of $S^\vee$). First of all, since $g(L) = 1$, all elements of $|L|$ having a point of multiplicity $\geqslant 3$ are reducible. Recall (1.5) and set $P := \{p_1, \dots, p_t\}$ for simplicity. Thus we get the following possibilities:

Type I. Let $x \in S$ be a point not lying on $e_1 \cup \cdots \cup e_t$ and let $x' = \eta(x)$. Then a hyperplane section of (S, L) with a triple point at x corresponds to a plane cubic having a triple point at x' and containing P. Of course such a cubic splits into three lines (not necessarily distinct) all passing through x'. Note that type I represents the general osculating hyperplane section, provided that x' is general and the three lines are distinct.

Now let $x \in e_1$, where e_1 is the exceptional curve corresponding to $p_1 \in P$. Let $l \subset \mathbb{P}^2$ be the line through p_1 corresponding to the direction x and let γ be any conic tangent to l at p_1. If $t \geqslant 2$ assume furthermore that the cubic $l + \gamma$ contains P. Let $\tilde{l} = \eta^{-1}(l)$.

Type II. Suppose that γ is irreducible and denote by $\tilde{\gamma} = \eta^{-1}(\gamma)$ its proper transform. Then the hyperplane section given by $\eta^*(l + \gamma) - e_1 - \cdots - e_t$ contains the divisor $e_1 + \tilde{l} + \tilde{\gamma}$, which consists of three distinct curves all meeting at x. Hence it defines a hyperplane section of (S, L) with a triple point at x.

Let things be as above, but now assume that γ is reducible, i.e., $\gamma = l' + l''$, where l', l'' are lines; since γ is "tangent" to l at p_1 we have two possibilities: either one of these lines coincides with l, or $l' \cap l'' \ni p_1$. They lead to the following types. We set $\widetilde{l'} = \eta^{-1}(l')$, $\widetilde{l''} = \eta^{-1}(l'')$.

Type IIIa. $\gamma = l + l''$, l'' being any line in $\mathbb{P}^2$. Then the hyperplane section given by $\eta^*(2l + l'') - e_1 - \cdots - e_t$ contains the divisor $2\tilde{l} + \widetilde{l''} + e_1$, which has a point of multiplicity 3 at x.

Type IIIb. $\gamma = l' + l''$ and both lines l', l'' contain p_1. Then the hyperplane section given by $\eta^*(l + l' + l'') - e_1 - \cdots - e_t$ contains the divisor $\tilde{l} + \widetilde{l'} + \widetilde{l''} + 2e_1$, which has a point of multiplicity 3 at x (and at two other distinct points of e_1 in general).

If both circumstances occur at the same time we call this

Type IV. $\gamma = l + l''$ and l'' contains p_1. In this case the hyperplane section given by $\eta^*(2l + l'') - e_1 - \cdots - e_t$ contains the divisor $2\tilde{l} + \widetilde{l''} + 2e_1$, which has a point of multiplicity 4 at x.

Finally, if also l'' coincides with l, we get

Type V. $\gamma = 2l$. In this case the hyperplane section given by $\eta^*(3l) - e_1 - \cdots - e_t$ contains the divisor $3\tilde{l} + 2e_1$, which has triple points along $\tilde{l}$ and a point of multiplicity 5 at x.

Note that types III (hence IV and V) also occur as limit of hyperplane sections of type I. The general osculating hyperplane section is of type I; since $S^\vee$ is the closure of the set of all osculating sections, some elements of types III, IV and V may occur in $S^\vee$.

2 The second discriminant locus

Let S be as in (1.5) and note that $h^0(L) = d + 1 = 10 - t$. So, though L is very ample, it is nowhere 2-jet spanned if $t \geqslant 5$ because rk $j_{2,x} \leqslant h^0(L) \leqslant 5$ for all $x \in S$. Moreover, apart from the case $t = 0$, in which $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ is 3-regular, the pair (S, L) contains lines, hence it is not 2-regular [5, Proposition 1.3]. However it is generically 2-regular for $t \leqslant 4$, as we will see.

Analyzing the compatibility of the types described in (1.6) with the combinatorics allowed by the cardinality $t = 9 - d$ of P gives rise to the following results. For $t \leqslant 5$ we denote by $\mathscr{E}$ the union of the exceptional curves $e_1, \dots, e_t$ and the proper transforms of the lines $l_{ij} := \langle p_i, p_j \rangle$, $i < j$, joining pairs of distinct points of P. Moreover we denote by V the set of vertices of $\mathscr{E}$, i.e., the set of points at which pairs of curves of $\mathscr{E}$ meet. Of course $\mathscr{E}$ is empty for $d = 9$, while V is empty for $d \geqslant 8$ and consists of $2, 6, 15, 35$ points according to whether $d = 7, 6, 5, 4$ respectively.

(2.1) Theorem. *Let S be a Del Pezzo surface as in* (1.5), *of degree $d = K_S^2 \geqslant 4$, and consider the vector bundle map $j_2 : S \times H^0(S, L) \to J_2L$.*

For $d = 9$ the pair (S, L) is 3*-regular*; *in particular* $\mathrm{rk}\, j_{2,x} = 6$ *for every $x \in S$.*

Let $5 \leqslant d \leqslant 8$. Then

$$\mathrm{rk}\, j_{2,x} = \begin{cases} 6, & \text{for } x \in S \setminus \mathscr{E} \\ 5, & \text{for } x \in \mathscr{E} \setminus V \\ 4, & \text{for } x \in V. \end{cases}$$

Let $d = 4$. Then

$$\mathrm{rk}\, j_{2,x} = \begin{cases} 5, & \text{for } x \in S \setminus V \\ 4, & \text{for } x \in V. \end{cases}$$

Proof. For $d = 9$ we have $(S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$, so that L is 3-jet ample [3, Corollary 2.1], hence 3-regular.

For the second group we describe the argument in case $d = 5$; cases $d = 6, 7, 8$ can be dealt with in a similar way. Recall that $h^0(L) = 6$. If $x \notin \mathscr{E}$, then no line l_{ij} contains $x' = \eta(x)$. On the other hand type I cannot occur, since P consists of 4 points, and therefore there is no hyperplane section of (S, L) with a triple point at x. This shows that $\mathrm{Ker}(j_{2,x}) = 0$, hence $j_{2,x}$ has rank 6. Let $x \in \mathscr{E} \setminus V$. If $x \notin e_1 \cup \cdots \cup e_4$ then $x' = \eta(x)$ lies on a line l_{ij} and type I occurs. Thus there is exactly one hyperplane of (S, L) having a triple point at x. Thus $\mathrm{Ker}(j_{2,x})$ has dimension 1, and so $j_{2,x}$ has rank 5. If x belongs to some e_i, e.g., $x \in e_1$, then call l the line in $\mathbb{P}^2$ through p_1 corresponding to x. Since P is in general position there exists only one conic γ tangent to l at p_1 and passing through p_2, p_3, p_4, which is irreducible. Thus type II occurs and again there exists only one hyperplane section of (S, L) with a triple point at x. Hence $j_{2,x}$ has rank 5 as before. Now, let $x \in V$. If $x \notin e_1 \cup \cdots \cup e_4$ then $x' = \eta(x)$ is a point at which two lines joining distinct pairs of points of P meet. Adding to these two lines any line of the pencil through x' we see that type I occurs exactly for a 1-dimensional family. Thus there is exactly a pencil of hyperplane sections of (S, L) with a triple point at x. Thus $\mathrm{Ker}(j_{2,x})$ has dimension 2, and so $j_{2,x}$ has rank 4. Finally let $x \in V \cap (e_1 \cup \cdots \cup e_4)$, e.g., suppose that $x = e_1 \cap \eta^{-1}(l_{12})$. Then there is a pencil of conics tangent to l_{12} at p_1 and passing through p_3, p_4, whose general element is irreducible. So type II occurs for a 1-dimensional family. Thus there is exactly a pencil of hyperplane sections of (S, L) with a triple point at x and as before we conclude that $j_{2,x}$ has rank 4.

Finally let $d = 4$; in this case $h^0(L) = 5$. Let $x \notin e_1 \cup \cdots \cup e_5$; since P consists of 5 points, we see that type I cannot occur unless $x' = \eta(x)$ is collinear with two distinct pairs of points of P, i.e., $x \in V$. So either $\mathrm{Ker}(j_{2,x})$ is trivial and then $j_{2,x}$ has rank 5, or $x \in V$, in which case there is a unique hyperplane section with a triple point at x, i.e., $\mathrm{Ker}(j_{2,x})$ has dimension 1, hence $j_{2,x}$ has rank 4. On the other hand, if e.g., $x \in e_1$, call l the line in $\mathbb{P}^2$ through p_1 whose direction corresponds to x. Again since

P consists of 5 points, we see that type II and degenerate types cannot occur unless $l = l_{1j}$, for some $j = 2, \dots, 5$, which means that $x \in V$. Moreover in this case since P is in general position there is only one hyperplane section of type II and as before we conclude that $j_{2,x}$ has rank 4.

For $d = 3$ our S is a cubic surface and the situation depends on the fact that S is general in moduli. Recall that if three coplanar lines of a smooth cubic surface of $\mathbb{P}^3$ have a common point, this is called an Eckardt point [13, p. 6]. Since the general cubic surface has no such points [9], we call *special* those cubic surfaces admitting some Eckardt point. Then the result is as follows.

(2.2) Theorem. *Let S, d, and j_2 be as in Theorem* (2.1) *with* $d = 3$. *Then* $\operatorname{rk} j_{2,x} = 4$ *except if S is special and x is an Eckardt point*, *in which case* $\operatorname{rk} j_{2,x} = 3$.

Proof. Let $x \notin e_1 \cup \cdots \cup e_6$; then a hyperplane section of (S, L) with a triple point at x, of type I, can occur only if $x' = \eta(x)$ is collinear with three pairs of points of P. If this is not the case, then $\operatorname{Ker}(j_{2,x}) = 0$ and so $\operatorname{rk} j_{2,x} = h^0(L) = 4$. Assume that, e.g., l_{12}, l_{34}, l_{56} are in a pencil. Then the hyperplane section of (S, L) given by $\eta^{-1}(l_{12}) + \eta^{-1}(l_{34}) + \eta^{-1}(l_{56})$ consists of three lines meeting at x; thus x is an Eckardt point. Moreover $\operatorname{Ker}(j_{2,x})$ has dimension 1, and so $\operatorname{rk} j_{2,x} = 3$. Now let $x \in e_1 \cup \cdots \cup e_6$, e.g., $x \in e_1$ and let l be the line in $\mathbb{P}^2$ through p_1 whose direction corresponds to x. There is only one conic γ passing through p_1 and 4 other points of P; moreover γ is irreducible since P is in general position; if either γ is not tangent to l at p_1 or l omits the remaining point of P, then there are no hyperplane sections of (S, L) with a triple point at x. This means that $\operatorname{Ker}(j_{2,x}) = 0$ and so $\operatorname{rk} j_{2,x} = h^0(L) = 4$. On the other hand, if l is tangent to γ at p_1 and contains the sixth point of P, then the hyperplane section of (S, L) given by

$$\eta^*(\gamma + l) - \sum_{i=1}^{6} e_i = \eta^{-1}(\gamma) + \eta^{-1}(l) + e_1$$

consists of three lines meeting at x; thus x is an Eckardt point. Moreover $\operatorname{Ker}(j_{2,x})$ has dimension 1, and so $\operatorname{rk} j_{2,x} = 3$.

By Theorem (2.1) we know that if S is a Del Pezzo surface with $d \geqslant 5$, then (S, L) is generically 2-regular. If, in addition, $d \geqslant 6$, then the second dual variety $S^\vee$ is defined and the discussion in (1.6) shows that the morphism

$$\pi_2 : \overline{\mathbb{P}(\mathscr{K})} \to S^\vee$$

is birational, where $\mathscr{K}$ stands for the dual of the kernel of the surjective homomorphism of vector bundles

$$H^0(L) \otimes \mathcal{O}_{S \setminus \mathscr{E}} \to (J_2 L)_{S \setminus \mathscr{E}}.$$

Actually for every $x \in S \setminus \mathscr{E}$ the only element in $|L - 3x|$ is the proper transform via η

of three distinct lines meeting at $\eta(x)$, whose union contains P. In particular recalling the discussion made in (1.3), we get

$$\dim S^\vee = 2 + \mathrm{rk}(\mathscr{K}) - 1 = 2 + h^0(L) - 1 - \mathrm{rk}(J_2 L) = d - 4.$$

By using (1.6) again we can also compute the degrees of the second dual variety of S and of the extra components in $\mathscr{D}_2(S, L)$ coming from the loci where the rank of $j_{2,x}$ drops.

For every $p_i, p_j \in P$, $i < j$, let $\widetilde{l_{ij}} = \eta^{-1}(l_{ij})$, and set

$$\Gamma_{ij} := \overline{\bigcup_{x \in \widetilde{l_{ij}} \setminus V} \{\text{hyperplane sections of type I at } x\}};$$

for every $p_i \in P$, set

$$\Delta_i := \overline{\bigcup_{x \in e_i \setminus V} \{\text{hyperplane sections of type II at } x \text{ such that } l \cap P = \{p_i\}\}}.$$

Finally let $\Theta_{ij;i}$ ($\Theta_{ij;j}$ respectively) be the closure of the set consisting of hyperplane sections of type II coming from l_{ij} and a conic γ tangent to l_{ij} at p_i (at p_j respectively).

Note that for $d \geqslant 6$ the second discriminant locus $\mathscr{D}_2(S, L)$ contains, in addition to $S^\vee$, $9 - d$ extra components of type Δ, $\binom{9-d}{2}$ of type Γ and $(9 - d)(8 - d)$ of type Θ. For $d = 5$ there are 3 more components, each one corresponding to a point of $V \setminus (\bigcup_{i=1}^4 e_i)$.

(2.3) Theorem. *Let S be a Del Pezzo surface as in* (1.5), *of degree $d \geqslant 6$, embedded by L. Then $S^\vee$ and the extra components above have dimension $d - 4$. Their degrees are*:

$$\deg S^\vee = 15 - 3\binom{9-d}{2},$$

$$\deg \Delta_i = d - 3, \quad \deg \Theta_{ij;i} = \deg \Theta_{ij;j} = 1, \quad \textit{and} \quad \deg \Gamma_{ij} = 3.$$

Proof. We already observed that $\dim S^\vee = d - 4$. From Theorem (2.1) it is clear that the hyperplane sections with a point of multiplicity $\geqslant 3$ at a fixed general point of either e_i or $\widetilde{l_{ij}}$ constitute a linear space of dimension $d - 5$, hence $\dim \Delta_i = \dim \Gamma_{ij} = d - 4$. Moreover the hyperplane sections with a point of multiplicity $\geqslant 3$ at $e_i \cap \widetilde{l_{ij}}$ constitute a linear space of dimension $d - 4$, so that $\deg \Theta_{ij;i} = 1$. Of course also $\deg \Theta_{ij;j} = 1$. To compute the remaining degrees we intersect our components with a linear space $\Lambda \subset \mathbb{P}^{d\vee}$ of codimension $d - 4$. We choose Λ corresponding to the linear system of plane cubics passing through P and $Q := \{q_1, \ldots, q_{d-4}\}$, where $P \cup Q$ consists of $9 - d + d - 4 = 5$ points in general position. Thus we immediately see that $\deg \Delta_i = d - 3$ enumerating the elements of $\Delta_i \cap \Lambda$: actually one is given by the cubic consisting of the irreducible conic containing $P \cup Q$ and its tangent line at p_i; the others correspond to cubics consisting each one of the irreducible conic through $P \cup (Q \setminus \{q_j\})$ tangent at p_i to the line $\langle p_i, q_j \rangle$ plus the line itself. There is such a cubic for any point of Q, so that they are $(d-4)$. In the same way we see that $\deg \Gamma_{ij} = 3$, since there are exactly three cubics of type I whose triple point lies on l_{ij}. To compute $\deg S^\vee$ we have to count how many cubics of type I and II, not already taken into account, appear in the linear system Λ. Those of type I are $15 - 3\binom{9-d}{2}$ since those having their triple points on lines l_{ij} have already been counted. On the other hand, also all hyperplane sections of type II have been counted.

(2.4) Remarks. i) Recall that $c_2(S) = 3 + (9-d)$ in view of (1.5). So we get (e.g. see [5, (0.1.3)])

$$c_2(J_2L) = 5c_2(S) = 15 + 5(9-d).$$

It deserves to note that for any $d = 6, \dots, 9$ this is the sum of the degrees of the various components of the second discriminant locus. E.g., for $d = 7$ we have

$$c_2(J_2L) = 25 = \deg S^\vee + \deg \Delta_1 + \deg \Delta_2 + \deg \Gamma_{12} + \deg \Theta_{12;1} + \deg \Theta_{12;2}.$$

ii) In case $d = 5$ note that there is no $S^\vee$ and $\mathscr{D}_2(S, L)$ contains 25 1-dimensional components: 4 components of type Δ, 6 of type Γ, 12 of type Θ plus three lines corresponding to the three points of V lying outside $\bigcup_{i=1}^4 e_i$. Arguing as in the proof of Theorem (2.3) one can see that $\deg \Delta_i = 3$ and that all components of type Θ are lines. On the contrary, for $d = 5$ we have that all Γ_{ij}'s are conics. Actually, imposing to cubics to pass through a general point $q \in \mathbb{P}^2$ we have that e.g., the cubic consisting of l_{12}, l_{34} and the line joining the point y, where they meet, with q is not in Γ_{12} (and of course not even in Γ_{34}). On the other hand this cubic represents the section of the new 1-dimensional component corresponding to y with the hyperplane of $\mathbb{P}^{5\vee}$ defined by the condition of passing through q. In view of the above, summing up the degrees of all components we get again $4 \cdot 2 + 6 \cdot 2 + 12 + 3 = 35 = c_2(J_2L)$.

iii) In case $d = 5$ note also that

$$\sum_{i=1}^{4} e_i + \sum_{i<j} \widetilde{l_{ij}} = \eta^*(l_{12} + l_{13} + l_{34}) - \sum_{i=1}^{4} e_i + \eta^*(l_{23} + l_{24} + l_{14}) - \sum_{i=1}^{4} e_i.$$

This shows that the inflectional locus $\mathscr{E}$ of (S, L) is the support of a divisor in $|2L| = |4K_S + 6L|$, in accordance with [14, Proposition (0.3), a)].

3 The second dual variety for $S \not\cong \mathbb{P}^1 \times \mathbb{P}^1$

In this Section we describe the second dual varieties of surfaces as in (1.5) with $d \geqslant 6$. We write S_d instead of S since we have to consider distinct surfaces of this type at the same time. Accordingly we will denote by $(S_d)^\vee$, $\Gamma_{ij}(S_d)$, $\Delta_i(S_d)$, $\Theta_{ij;i}(S_d)$ the second dual variety and the other components of the second discriminant locus $\mathscr{D}(S_d)$ of $(S_d, -K_{S_d})$. For a further discussion of $(S_6)^\vee$ see also Section 5. As we already said the morphism $\pi_2 : \overline{\mathbb{P}(\mathscr{K})} \to (S_d)^\vee$ is always birational and $\dim(S_d)^\vee = d - 4$. By using (1.6) we can also describe the positive dimensional fibres of π_2 and understand the singularities of $(S_d)^\vee$.

(3.1) We start our study from $(S_9)^\vee$. We know that $\mathscr{D}(S_9) = (S_9)^\vee$, $\dim(S_9)^\vee = 5$ and $\deg(S_9)^\vee = 15$. Since $S_9 \cong \mathbb{P}^2$ and $\mathscr{E} = \emptyset$ we have that $\mathbb{P}(\mathscr{K})$ is a $\mathbb{P}^3$-bundle over $\mathbb{P}^2$. Note that $\mathbb{P}(\mathscr{K})$ parameterizes pairs (x, H) consisting of a point $x \in \mathbb{P}^2$ and a triplet of lines of $\mathbb{P}^2$ each passing through x; hence by duality we see that $\mathbb{P}(\mathscr{K}) \cong \mathrm{Al}^3\,\mathbb{P}^2$, the Hilbert scheme of collinear triplets of points in $\mathbb{P}^2$ [7]. This can provide an alternative description of the morphism π_2. Now let $H \in (S_9)^\vee$; then $\pi_2^{-1}(H)$ is a single point, unless $H = 3l$ is given by three coinciding lines, in which case $\pi_2^{-1}(H)$ is a $\mathbb{P}^1$. So π_2 contracts the $\mathbb{P}^1$-bundle isomorphic to the incidence variety of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$ to a subset $T \subset (S_9)^\vee$, which is isomorphic to $\mathbb{P}^{2\vee}$. Now consider the locus $\Sigma \subset (S_9)^\vee$ defined by the elements of the form $H = 2l + l'$, where l, l' are lines. Of course $T \subset \Sigma$.

(3.1.1) Lemma. *Let* $(x, H) \in \mathbb{P}(\mathscr{K})$. *Then*

$$\mathrm{rk}((d\pi_2)(x, H)) = \begin{cases} 5, & \text{if } H \in (S_9)^\vee \backslash \Sigma \\ 4, & \text{if } H \in \Sigma \backslash T \\ 2, & \text{if } H \in T. \end{cases}$$

Proof. Fix affine local coordinates u, v on $S_9 \cong \mathbb{P}^2$ and let

$$u^3, u^2v, uv^2, v^3, u^2, uv, v^2, u, v, 1$$

be the local expression of the basis of $H^0(S_9, L)$ giving homogeneous coordinates on $\mathbb{P}^{9\vee}$. Let (a, b) be the affine coordinates of x. Then H is defined by a cubic polynomial of the form

$$((u-a) + \lambda(v-b))((u-a) + \mu(v-b))((u-a) + \nu(v-b)) = 0,$$

for some $\lambda, \mu, \nu \in \mathbb{C}$. Therefore π_2 is defined around (x, H) by the map

$$(a, b, \lambda, \mu, \nu) \mapsto (1 : \lambda + \mu + \nu : \lambda\mu + \lambda\nu + \mu\nu : \lambda\mu\nu : \cdots).$$

So, simply forgetting the first homogeneous coordinate we get the affine local coordinates of $\pi_2(x, H)$. Making explicit the terms replaced by dots in the expression above, and taking derivatives with respect to a, b, λ, μ, ν we see that $(d\pi_2)(x, H)$ is represented by a matrix of the form

$$\begin{pmatrix} 0 & A \\ B & C \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ \mu + \nu & \lambda + \nu & \lambda + \mu \\ \mu\nu & \lambda\nu & \lambda\mu \end{pmatrix},$$

the rows of C are linear combinations of those of A, and

$$B = \begin{pmatrix} -3 & -\sigma_1 \\ -2\sigma_1 & -2\sigma_2 \\ -\sigma_2 & -3\sigma_3 \\ * & * \\ \cdot & \cdot \\ \cdot & \cdot \\ * & * \end{pmatrix},$$

where σ_i, $i = 1, 2, 3$, stands for the elementary symmetric function of degree i of λ, μ, ν. Thus $(d\pi_2)(x, H)$ has the same rank as the matrix

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix},$$

hence

$$\mathrm{rk}((d\pi_2)(x, H)) = \mathrm{rk}\, A + \mathrm{rk}\, B. \qquad (+)$$

We have $\mathrm{rk}\, A = 3, 2$ or 1 according to whether λ, μ, ν are distinct, not all three distinct, or all coinciding, respectively. Moreover it is immediate to check that the submatrix of B consisting of the first three rows has rank 1 if and only if the polynomial

$$\xi^3 - \sigma_1\xi^2 + \sigma_2\xi - \sigma_3,$$

whose roots are λ, μ, ν, is a cube, i.e., if and only if $\lambda = \mu = \nu$. On the other hand, computing explicitly the terms replaced by stars in B, one can check that $\mathrm{rk}\, B$ is in fact 1 for $\lambda = \mu = \nu$. Then the assertion follows from $(+)$, recalling the equation of H.

Now we can describe $\mathrm{Sing}((S_9)^\vee)$.

(3.1.2) Proposition. *We have* $\mathrm{Sing}((S_9)^\vee) = \Sigma \cong \mathbb{P}^2 \times \mathbb{P}^2$. *Moreover* $\deg \Sigma = 6$ *and* $T \subset \Sigma$ *is a* 2*-plane.*

Proof. Recall that π_2 is injective except on $\pi_2^{-1}(T)$ and $T \subset \Sigma$. Since π_2 ramifies along $\pi_2^{-1}(\Sigma)$ by Lemma (3.1.1), we conclude that $\mathrm{Sing}((S_9)^\vee) = \Sigma$. Of course Σ is parameterized by $\mathbb{P}^{2\vee} \times \mathbb{P}^{2\vee}$ via the map $(l, l') \mapsto 2l + l'$, T corresponding to the diagonal. In particular $\dim \Sigma = 4$. In order to compute the degree we intersect Σ with a linear space $\Lambda \subset \mathbb{P}^{9\vee}$ of codimension 4. Choose Λ corresponding to the linear system of plane cubics passing through 4 general points $q_1, \dots, q_4 \in \mathbb{P}^2$. Then we see that $\deg \Sigma = 6$, the number of the ordered pairs of lines whose union contains the 4 points. In the same way we see that $\deg T = 1$.

(3.2) Now consider S_8. We know that $\dim (S_8)^\vee = 4$ and $\deg (S_8)^\vee = 15$. Moreover, since the hyperplane sections of S_8 are represented by the plane cubics through p_1, we conclude that

$$(S_8)^\vee = (S_9)^\vee \cap \Lambda(p_1)$$

is the section of $(S_9)^\vee$ with the hyperplane of $\mathbb{P}^{9\vee}$ defined by the condition of passing through p_1. Note that $\Lambda(p_1)$ is not in general position with respect to $(S^9)^\vee$. In fact we have

(3.2.1) Proposition. $(S_8)^\vee \cap \Sigma$ *consists of two irreducible components* Σ_i, $i = 1, 2$, *each being the image of* $\mathbb{P}^2 \times \mathbb{P}^1$ *via the Segre embedding. Moreover* $\Sigma_1 \cap \Sigma_2$ *is a quadric surface and* $(S_8)^\vee \cap T$ *is a line.*

Proof. Actually $\Lambda(p_1) \cap \Sigma$ consists of two irreducible components Σ_1 and Σ_2, whose general point corresponds to a plane cubic of type $H = 2l + l'$, such that either $p_1 \in l$ or $p_1 \in l'$, respectively. Let $\mathbb{P}^1_{(1)}$ be the pencil of lines through p_1. Then Σ_1 is parameterized by $\mathbb{P}^1_{(1)} \times \mathbb{P}^{2\vee}$ via the map $(l, l') \mapsto 2l + l'$. Similarly Σ_2 is parameterized by $\mathbb{P}^{2\vee} \times \mathbb{P}^1_{(1)}$ via the map $(l, l') \mapsto 2l + l'$. Both maps restricted to $\mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(1)}$ show that $\Sigma_1 \cap \Sigma_2$ is parameterized by $\mathbb{P}^1 \times \mathbb{P}^1$. Each component Σ_i has degree 3. To see this, intersect Σ_1 with a linear space $\Lambda \subset \mathbb{P}^{8\vee}$ of codimension 3. We choose Λ corresponding to the linear system of plane cubics passing through p_1 and three other points $q_1, q_2, q_3 \in \mathbb{P}^2$, such that the 4 points are in general position. In this linear system there are exactly three elements of the form $2l + l'$ with $l \ni p_1$, which correspond to the three lines $\langle p_1, q_i \rangle$, $i = 1, 2, 3$. Hence $\deg \Sigma_1 = 3$. Similarly we can see that $\deg \Sigma_2 = 3$. It thus follows that Σ_i is a Segre product. Moreover, the intersection $\Sigma_1 \cap \Sigma_2$ has degree 2, as we can see by computing the degree as before. Finally, $\Lambda(p_1) \cap T \cong \mathbb{P}^1_{(1)}$, and its degree is 1 since both $\Lambda(p_1)$ and T are linear.

As to the second discriminant locus, note that $\mathscr{D}(S_8)$ consists of $\mathscr{D}(S_9) \cap \Lambda(p_1)$ plus the new component $\Delta_1(S_8)$, which in fact has degree 5, according to Theorem (2.3).

(3.3) Now consider S_7. We know that $\dim(S_7)^\vee = 3$ and $\deg(S_7)^\vee = 12$. The hyperplane sections of S_7 are represented by the plane cubics passing through p_1 and p_2, However, the section of $(S_8)^\vee$ with the hyperplane $\Lambda(p_2)$, defined by the condition of passing through p_2, contains the general element of $(S_7)^\vee$ and the elements having a triple point on $l_{12} \backslash \{p_1, p_2\}$. So, in accordance with Theorem (2.3), we have that

$$(S_7)^\vee + \Gamma_{12}(S_7) = (S_8)^\vee \cap \Lambda(p_2).$$

Moreover $(S_7)^\vee$ intersects both Σ_1 and Σ_2 along surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, while $(S_7)^\vee \cap T$ consists of a single point representing the section of type V given by $3\widetilde{l_{12}} + 2e_1 + 2e_2$.

As to the second discriminant locus, note that

$$\Delta_1(S_8) \cap \Lambda(p_2) = \Delta_1(S_7) + \Theta_{12;1}(S_7).$$

The appearance of the linear Θ component explains the decreasing by 1 of the degree of Δ_1 passing from S_8 to S_7 (see Theorem (2.3)). On the other hand (compare also Remark(2.4, i)), we see that $\mathscr{D}(S_7)$ consists of the hyperplane section $\mathscr{D}(S_8) \cap \Lambda(p_2)$ plus the new components $\Delta_2(S_7)$ and $\Theta_{12;2}(S_7)$, the sum of whose degrees is in fact 5, according to Theorem (2.3).

(3.4) Finally consider S_6. Here $\dim(S_6)^\vee = 2$ and $\deg(S_6)^\vee = 6$. The hyperplane sections of S_6 are represented by the plane cubics passing through p_1, p_2 and p_3, However, the section of $(S_7)^\vee$ with the hyperplane $\Lambda(p_3)$, defined by the condition of passing through p_3, contains the general element of $(S_6)^\vee$ and the elements having a triple point on $(l_{13} \cup l_{23}) \backslash \{p_1, p_2, p_3\}$. So, in accordance with Theorem (2.3), we have that

$$(S_6)^\vee + \Gamma_{13}(S_6) + \Gamma_{23}(S_6) = (S_7)^\vee \cap \Lambda(p_3).$$

Note that Σ intersects each $\Gamma_{ij}(S_6)$ along two irreducible curves, say λ_{ij} and ε_h, with $h \neq i, j$: λ_{ij} corresponds to the cubics of the form $2l_{ij} + \langle p_h, x \rangle$ as x varies on l_{ij}, while ε_h corresponds to those of the form $l_{ij} + 2\langle p_h, x \rangle$ as x varies on l_{ij}. So λ_{ij} and ε_h also lie on $(S_6)^\vee$. It is immediate to check that all six curves $\lambda_{12}, \lambda_{13}, \lambda_{23}, \varepsilon_1, \varepsilon_2, \varepsilon_3$ are rational and have degree 1. Moreover, $\lambda_{ij} \cap \varepsilon_h = \emptyset$, while λ_{ij} meets ε_i and ε_j at two distinct points for every $(i, j), i < j$. This shows that the six lines obtained by intersecting Σ with $\Gamma_{12}(S_6) \cup \Gamma_{13}(S_6) \cup \Gamma_{23}(S_6)$ draw on $(S_6)^\vee$ a 1-cycle, say $\mathscr{E}^\vee$, dual to the 1-cycle $\mathscr{E}$ on S_6. The six vertices of $\mathscr{E}^\vee$ represent the hyperplane sections of S_6 coming from the cubics which consist of two lines among l_{12}, l_{13}, l_{23}, one of which being counted twice. Note that e.g., the hyperplane section $H = 2\widetilde{l_{ij}} + \widetilde{l_{ih}} + 2e_i + e_j$ has in fact a point of multiplicity 4 at the vertex $v = \widetilde{l_{ij}} \cap e_i$. However, in spite of this, $(S_6)^\vee$ is smooth at H. In fact we have

(3.4.1) Theorem. *For $d = 6$ the morphism π_2 is an isomorphism. In other words, $S_6 \cong (S_6)^\vee$.*

Proof. The second assertion comes from the first one because $\overline{\mathbb{P}(\mathscr{K})}$ is isomorphic to S_6. To see this note that $\mathscr{E}$ has codimension 1 in S_6 and that for every $v \in V$ there is only one well defined limit of sections of type I at x, for $x \to v$. As to the first assertion, note that $(S_6)^\vee \cap T = \emptyset$, hence π_2 is a finite morphism, and in fact one-to-one. So it is enough to show that $d\pi_2$ has rank 2 everywhere. This will be done in Section 4 since it requires an appropriate choice of coordinates, which we will do there.

Finally, a discussion similar to the one made for $d = 7$ can be done for the second discriminant locus.

4 The Del Pezzo surface of degree 6 and Togliatti's example

The case $d = 6$ is interesting in several respects. According to (1.5), S is $\mathbb{P}^2$ blown-up via η at three non-collinear points p_1, p_2, p_3. Choosing homogeneous coordinates in $\mathbb{P}^2$ in such a way that $p_1 = (1:0:0), p_2 = (0:1:0), p_3 = (0:0:1)$, the vector space $H^0(S, L)$ can be generated by the 7 monomials

$$x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_2^2, x_1^2x_2, x_1x_2^2, x_0x_1x_2. \qquad (*)$$

In other words, using affine coordinates $u = \frac{x_1}{x_0}$ and $v = \frac{x_2}{x_0}$, our surface $S \subset \mathbb{P}^6$ is the closure of the image of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^6$ given by

$$(u, v) \mapsto (u : v : u^2 : v^2 : u^2v : uv^2 : uv). \qquad (**)$$

We know from Theorem (2.1) that $j_{2,x}$ has rank 6, 5, or 4 according to whether x is in $S \setminus \mathscr{E}$, $\mathscr{E} \setminus V$ or V, respectively. This can be confirmed by a direct computation using $(*)$. The check is straightforward for $x \notin e_1 \cup e_2 \cup e_3$. For e.g., $x \in e_1$ use local coordinates (u, t) on the blow-up so that $v = tu$; then, after dividing all terms by u, we see that around x the map $(**)$ is defined by

$$(u, t) \mapsto (1 : t : u : ut^2 : u^2t : u^2t^2 : ut).$$

Thus it is immediate to check that the matrix representing $j_{2,x}$ at points $x \in e_1$, where $u = 0$, has rank 5 if $t \neq 0$ and 4 if $t = 0$.

The appropriate choice of coordinates we made before makes easy to complete the proof of Theorem (3.4.1) showing that $d\pi_2$ has rank 2 everywhere. Let $(x, H) \in \overline{\mathbb{P}(\mathscr{K})}$ (which, as we already observed, is isomorphic to S). First suppose that $x \notin e_1 \cup e_2 \cup e_3$ and let (a, b) be the affine coordinates of $x' = \eta(x)$. Since H has to contain the three points p_1, p_2, p_3, it is defined by the following cubic polynomial:

$$(b(u - a) - a(v - b))(v - b)(u - a).$$

Thus $\pi_2 : \overline{\mathbb{P}(\mathscr{K})} \to \mathbb{P}^{6\vee}$ is described around (x, H) by the map

$$(a, b) \mapsto (ab^2 : -a^2b : -b^2 : a^2 : b : -a : 0).$$

Recall that $(a, b) \neq (0, 0)$, since $x' \neq p_1$. So, to get affine coordinates for $\pi_2(x, H)$ we can divide by the last but one homogeneous coordinate if $a \neq 0$, and by the previous one if $a = 0$. Then, taking derivatives with respect to a and b, it is easy to see that $(d\pi_2)(x, H)$ has rank 2. Now let $x \in e_1 \cup e_2 \cup e_3$, e.g., suppose that $x \in e_1$. By using local coordinates (u, t) on S near x, as before, we get $v = tu$, hence $b = ta$, so that π_2 is locally defined by

$$(a, t) \mapsto (a^3t^2 : -a^3t : -a^2t^2 : a^2 : at : -a : 0).$$

So, dividing by $-a$ and forgetting the last but one homogeneous coordinate we see that $\pi_2(x, H)$ has the following affine local coordinates: $(-a^2t^2, a^2t, at^2, -a, -t, 0)$. Thus, taking derivatives with respect to a and t, and then letting $a = 0$, we see that $(d\pi_2)(x, H)$ has rank 2 also for every $x \in e_1$. This concludes the proof.

Now look at $S^\vee$. Recall that the 2-osculating hyperplane section at the general point $x \in S$ is of type I. So, let $\mathbb{P}^1_{(i)}$ be the pencil of lines through p_i, $i = 1, 2, 3$ and inside the product $\mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)} \times \mathbb{P}^1_{(3)}$ consider the surface

$$\mathscr{I} := \{(l_1, l_2, l_3) \in \mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)} \times \mathbb{P}^1_{(3)} \mid l_1 \cap l_2 \cap l_3 \neq \varnothing\}.$$

Let

$$\psi : \mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)} \times \mathbb{P}^1_{(3)} \to \mathbb{P}^6$$

be the composition of the Segre embedding in $\mathbb{P}^7$ and the projection to $\mathbb{P}^6$ corresponding to the subspace of $H^0(\mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)} \times \mathbb{P}^1_{(3)}, \mathcal{O}_{\mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)} \times \mathbb{P}^1_{(3)}}(1,1,1))$ generated by the monomials in $(*)$. Identifying $\mathbb{P}^6$ with the projective space $|L|$, ψ sends the triplet of lines (l_1, l_2, l_3) to the plane cubic $l_1 + l_2 + l_3$ regarded as a member of $|L|$; hence $S^\vee = \psi(\mathcal{I})$. On the other hand, by using the same homogeneous coordinates as before, $\psi(l_1, l_2, l_3)$ corresponds to the plane cubic of equation

$$(a_1x_1 + a_2x_2)(b_0x_0 + b_2x_2)(c_0x_0 + c_1x_1) = 0,$$

where

$$\begin{vmatrix} 0 & a_1 & a_2 \\ b_0 & 0 & b_2 \\ c_0 & c_1 & 0 \end{vmatrix} = 0,$$

i.e., this cubic has no term $x_0x_1x_2$. This means that $S^\vee$ is contained in the hyperplane of $|L| = \mathbb{P}^{6\vee}$ corresponding by duality to the last term appearing in $(*)$.

By dualizing this fact we get that all 2-osculating hyperplanes to S contain the point $c = (0 : \cdots : 0 : 1) \in \mathbb{P}^6$. So we have

(4.1) Theorem. *Let (S, L) be the Del Pezzo surface of degree 6. All 2-osculating hyperplanes to $S \subset \mathbb{P}(H^0(S, L))$ have a common point.*

This interesting property was discovered by Togliatti [15, p. 259]. On the other hand, L is 2-jet spanned on $S \backslash \mathcal{E}$ according to Theorem (2.1) and since we are in $\mathbb{P}^6$ the 2-osculating hyperplanes are 2-osculating spaces. So we have that all the 2-osculating spaces to $S \subset \mathbb{P}(H^0(S, L))$ at points $x \in S \backslash \mathcal{E}$ contain c. However, as we will see, this does not mean that all 2-osculating spaces contain c.

Now let $W \subset H^0(S, L)$ be the subspace generated by the first 6 monomials in $(*)$ and project $S \subset \mathbb{P}^6$ from the point c to $\mathbb{P}(W) = \mathbb{P}^5$. Since c lies outside the secant variety of S, this is an isomorphic projection. So the image $Y := \varphi_W(S)$ is smooth and isomorphic to S. Moreover all 2-osculating spaces to Y have dimension $\leqslant 4$, in view of Theorem (4.1). Translating this into our terminology we have

(4.2) Corollary. *Let (S, L) be the Del Pezzo surface of degree 6 and let W be the subspace of $H^0(S, L)$ defined above. Then (S, W) is nowhere 2-regular.*

We can check all details of the discussion above by using affine coordinates (u, v) as before. Actually, recalling $(**)$, it turns out that Y is the closure of the image of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^5$ given by

$$(u, v) \mapsto (u : v : u^2 : v^2 : u^2v : uv^2).$$

By using the 6 monomials above as a basis for W we immediately see that $j_{2|S\times W}$ is represented at the point x corresponding to (u,v) by the following matrix

$$A = \begin{pmatrix} u & 1 & 0 & 0 & 0 & 0 \\ v & 0 & 1 & 0 & 0 & 0 \\ u^2 & 2u & 0 & 2 & 0 & 0 \\ v^2 & 0 & 2v & 0 & 2 & 0 \\ u^2v & 2uv & u^2 & 2v & 0 & 2u \\ uv^2 & v^2 & 2uv & 0 & 2u & 2v \end{pmatrix}.$$

Looking at the columns of A it is easy to check that for every $\sigma \in W$ we have

$$2\sigma - 2u\frac{\partial\sigma}{\partial u} - 2v\frac{\partial\sigma}{\partial v} + u^2\frac{\partial^2\sigma}{\partial u^2} + v^2\frac{\partial^2\sigma}{\partial v^2} + uv\frac{\partial^2\sigma}{\partial v\partial u} = 0.$$

In particular $j_{2|S\times W}$ has rank $\leqslant 5$ for every $x \in S$, which shows that (S,W) is nowhere 2-regular. Moreover, in Togliatti's terminology, the differential equation above is the "Laplace equation represented by Y" (e.g., see [15, p. 255]). As to the map

$$j_{2|S\times W} : S \times W \to J_2L,$$

the same direct computation mentioned at the beginning of this section, simply forgetting the second jet of the last monomial in $(*)$, gives the following result (see also [10, Example (2.4)]).

(4.3) Proposition. *Let* (S,L) *and* W *be as before. Then*

$$\operatorname{rk} j_{2,x|W} = \begin{cases} 5, & \text{for } x \in S \backslash V \\ 3, & \text{for } x \in V. \end{cases}$$

In terms of $Y \subset \mathbb{P}^5$, this means that $\operatorname{Osc}^2_x(Y)$ is a hyperplane except at the six points of V, where it is a 2-plane. This improves the assertion in [14, p. 248]. On the other hand, in terms of $S \subset \mathbb{P}^6$ this can be rephrased in the following way: all 2-osculating spaces $\operatorname{Osc}^2_x(S)$ contain the point c except those at points $x \in \mathscr{E}\backslash V$. This makes what is asserted in ([12, p. 223]) more precise.

Finally we point out some connections among the three surfaces Y, S and the Del Pezzo surface of degree 5, concerning their 2-osculating spaces. To do that we restore the notation used in Section 3 so that $S = S_6$. As Y is the projection of S_6 from the point $c \notin S_6$, we get that $Y^\vee$ is isomorphic to the section of $(S_6)^\vee$ with the hyperplane $\Lambda(c)$ of $\mathbb{P}^{6\vee}$ consisting of hyperplanes of $\mathbb{P}^6$ passing through c. On the other hand, as we have seen before, $(S_6)^\vee$ lies in the hyperplane $\Lambda(c)$; hence $(S_6)^\vee$ is isomorphic to $Y^\vee$. On the other hand $(S_6)^\vee \cong S_6$ by Theorem (3.4.1), and Y is the isomorphic projection of S_6 from c. This gives the following self-duality result.

(4.4) Corollary. $Y \cong Y^\vee$.

Now consider S_5. According to (1.5) it can be obtained as a special projection of S_6 from a point $b \in S_6$ (which can be identified with the point p_4 in $\mathbb{P}^2$). So, if we consider the hyperplane $\Lambda(b)$ in $\mathbb{P}^{6\vee}$, according to the discussion made in Section 3, we get that the section $(S_6)^\vee \cap \Lambda(b)$ consists of $(S_5)^\vee$ plus other components. But $(S_5)^\vee$ is empty. By arguing as in Section 3 we thus see that

$$(S_6)^\vee \cap \Lambda(b) = \Gamma_{14}(S_5) + \Gamma_{24}(S_5) + \Gamma_{34}(S_5).$$

All three components of this intersection belong to $\mathscr{D}(S_5)$ and are conics, as shown in Remark (2.4, ii). But the intersection above also coincides with $Y^\vee \cap \Lambda(b) \subset \Lambda(c)$, so this shows that the second dual variety of $Y \subset \mathbb{P}^5$ has a special hyperplane section consisting of three conics. One can easily see that these three conics meet at a single point, representing the plane cubic $l_{14} + l_{24} + l_{34}$. Note that this fits into the self-duality of Y. Actually the hyperplane section of Y corresponding to that cubic consists of the fibres through $\eta^{-1}(p_4)$ of the three distinct conic fibrations of Y coming from the pencils of lines through p_i, $i = 1, 2, 3$.

5 The second dual variety for $S \cong \mathbb{P}^1 \times \mathbb{P}^1$

So far we restricted our attention to Del Pezzo surfaces as in (1.5). In this Section we consider the case $(S, L) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2))$. We replace $\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ with the more efficient notation $[c + f]$ where c and f represent the fibres of the two rulings of S.

As a first thing we have the following result. Though it could be deduced from [5, Proposition 2.6 and Theorem 1.4] since (S, L) is a 2-regular rational conic bundle, we prefer to give a direct proof for it, closer to the spirit of the present paper.

(5.1) Proposition. *$S^\vee$ has dimension 4 and degree $c_2(J_2L) = 20$.*

Proof. First of all $\mathscr{D}_2(S, L) = S^\vee$ by (1.4), since (S, L) is 2-regular. Note that $Lc = Lf = 2$. So, imposing a triple point at $x \in S$ to an element $H \in |2c + 2f|$ implies that $H = c_x + f_x + R$, where c_x and f_x denote the section and the fibre through x and $R \in |c + f - x|$. Recall that $[c + f]$ is very ample and embeds S in $\mathbb{P}^3$ as a quadric surface Q. The general point of $S^\vee$ corresponds to such an H, with R irreducible. Hence the morphism π_2 is birational, and recalling (1.3), this says that $\dim S^\vee = \dim \mathbb{P}(\mathscr{K}) = 4$. Note that in fact π_2 is finite, since, even if R is reducible, H contains only a finite number of singular points of multiplicity $\geqslant 3$. To compute the degree intersect $S^\vee$ with a linear space $\Lambda \subset \mathbb{P}^{8\vee}$ of codimension 4. We choose Λ as the linear system $|2c + 2f - q_1 - \cdots - q_4|$ where $q_1, \ldots, q_4 \in S$ are 4 points in general position, i.e., no two of them lie on the same horizontal or vertical fibre of S. Then the elements of Λ having a singular point x of multiplicity $\geqslant 3$ are of the following types. Let $\{i, j, h, k\}$ be a permutation of $\{1, 2, 3, 4\}$. Type (a): $c_{q_i} + f_{q_j} + R$, where R is the unique element of $|c + f|$ passing through q_h, q_k and the point $x := c_{q_i} \cap f_{q_j}$

(recall that $h^0(c+f) = 4$). Type (b): $c_{q_i} + R + f_x$, where R is the unique element of $|c + f - q_j - q_h - q_k|$ and $x := c_{q_i} \cap R$ (recall that $(c+f)c = 1$). Type (c): $f_{q_i} + R + c_x$, where R is the unique element of $|c + f - q_j - q_h - q_k|$ and $x := f_{q_i} \cap R$. Note that any element of type (a) corresponds bijectively to the ordered pair (i, j), $i \neq j$ defining it. So there are exactly $2\binom{4}{2} = 12$ elements of type (a), while those of type (b) are obviously 4, as well as those of type (c). This shows that $\deg S^\vee = 12 + 4 + 4 = 20$.

Now look at $\mathrm{Sing}(S^\vee)$. As observed in the proof of Proposition(5.1), π_2 is finite and birational, hence the singular points of $S^\vee$ correspond to the elements $H \in S^\vee$ having more than a single triple point.

(5.2) Proposition. $\mathrm{Sing}(S^\vee)$ *consists of two irreducible components* Σ_i, $i = 1, 2$, *both isomorphic to* $\mathbb{P}^2 \times \mathbb{P}^1$; *moreover* $\deg \Sigma_i = 3$, *for* $i = 1, 2$.

Proof. As we said, any element $H \in S^\vee$ is of the form $H = c_x + f_x + R$ with $R \in |c + f - x|$, for some $x \in S$, and for the general H, R is irreducible, hence smooth. If R degenerates, it can only be of type $c_y + f_z$ for some points $y, z \in S$ and since it has to pass through x we have that either $y = x$ or $z = x$. This shows that $\mathrm{Sing}(S^\vee)$ consists of two components Σ_i, $i = 1, 2$, the elements $H' = 2c_x + f_x + f_z$ and $H'' = c_x + 2f_x + c_y$ representing the general point of Σ_1 and Σ_2 respectively. Since Σ_1 is parameterized by the elements of the form $2c_0 + f_1 + f_2$, with $c_0 \in |c|$ and $f_1, f_2 \in |f|$, we conclude that Σ_1 is isomorphic to $\mathbb{P}^1$ times the quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ by the symmetric group. Therefore $\Sigma_1 \cong \mathbb{P}^1 \times \mathbb{P}^2$. On the other hand the same conclusion holds for Σ_2, which is isomorphic to Σ_1 under the involution induced by exchanging the rulings of S. Finally, since Σ_i has dimension 3, we compute its degree by intersecting it with a linear subspace Λ of $|L|$ of codimension 3, which can be represented by the linear system $|2c + 2f - q_1 - q_2 - q_3|$ where $q_1, q_2, q_3 \in S$ and no two of them lie on the same horizontal or vertical fibre. Then we immediately see that $\deg \Sigma_i = 3$, $i = 1, 2$.

Note that the general point of Σ_i is a double point for $S^\vee$, corresponding to a section with two distinct triple points. On the other hand $\Sigma_1 \cap \Sigma_2$ is parameterized by sections of the form $2c_x + 2f_x$. They have a 4-tuple point at x so that they correspond to the 3-osculating hyperplane to S at x (note that for any $x \in S$ there is just one such section). Thus $\Sigma_1 \cap \Sigma_2$ coincides with the third discriminant locus $\mathscr{D}_3(S, L)$. It is immediate to note that it is a smooth quadric surface: the dual of Q.

Proposition (5.2) also says that Σ_1 and Σ_2 are the images of $\mathbb{P}^1 \times \mathbb{P}^2$ and $\mathbb{P}^2 \times \mathbb{P}^1$ via the Segre embedding. Hence their linear spans $\langle \Sigma_1 \rangle, \langle \Sigma_2 \rangle$ in $\mathbb{P}^{8\vee}$ are two 5-planes. Since they meet along the linear span of the dual of Q, which is a $\mathbb{P}^3$, we conclude that the linear span of $\mathrm{Sing}(S^\vee)$ is a hyperplane. By duality this means that all hyperplanes of $\mathbb{P}^8$ corresponding to points of $\mathrm{Sing}(S^\vee)$ pass through a point p. Note that the same property holds for the other Del Pezzo surface of degree 8 discussed in (3.2).

Finally let

$$\mathscr{I} := \{(x, R) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{3\vee} \mid x \in R\}.$$

Recalling that the incidence variety of $\mathbb{P}^3 \times \mathbb{P}^{3\vee}$ is the projective bundle $\mathbb{P}(T_{\mathbb{P}^3})$, we can intepret $\mathscr{I}$ as the $\mathbb{P}^2$-bundle $\mathbb{P}(T_{\mathbb{P}^3}|_Q)$, where $Q \subset \mathbb{P}^3$ is the smooth quadric surface. Using the same notation as above consider the map sending $(x, R) \in \mathscr{I}$ to the point representing the osculating section to S given by $c_x + f_x + R$. This defines an obvious morphism $\mathscr{I} \to S^\vee$, which provides an alternative description of π_2.

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