Adv. Geom. 2 (2002), 107–122

Advances in Geometry

Hyperplanes of dual polar spaces of rank 3 with no subquadrangular quad

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(Communicated by A. Pasini)

Abstract. Let Δ be a thick dual polar space of rank 3, and let H be a hyperplane of Δ. Calling the elements of Δ *points*, *lines* and *quads*, we call a quad $\alpha \not\subset H$ *singular* if $H \cap \alpha = P^{\perp} \cap \alpha$ for some point P, *subquadrangular* if $H \cap \alpha$ is a subquadrangle, and *ovoidal* if $H \cap \alpha$ is an ovoid. A point $P \in H$ of a quad α is said to be *deep with respect to* α if $P^{\perp} \cap \alpha \subset H$, and it is called *deep* if $P^{\perp} \subset H$.

We investigate hyperplanes H of Δ such that no quad is subquadrangular. We generalize a result of Shult proving that, if all quads are singular, then the polar space $\Pi = \Delta^*$ is an orthogonal polar space $\mathcal{Q}_6(K)$ for some (not necessarily finite) field K, and the hyperplane H is a split Cayley hexagon $H(K)$.

If both singular and ovoidal quads exist, then one of the following holds:

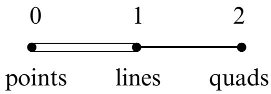
1. $H = \bigcup_{P \in O} P^{\perp}$ where O is an ovoid of a quad ω.
2. There exists one deep point $P \in H$ such that all quads containing P are singular and the remaining quads are ovoidal.
3. The set $\mathcal{P}$ of deep points with respect to the singular quads is a locally singular hyperplane of a dual polar space Δ_0. Δ_0 is the dual of an orthogonal polar space $\Pi_0 \cong \mathcal{Q}_6(K)$ for some field K and Π_0 is a subspace of the dual Π of Δ where the lines of Π_0 are lines of Π. The set $\mathcal{P}$ together with the lines of Δ contained in H form a split Cayley hexagon $H(K)$. The hyperplane H contains all points of Δ on lines of $H(K)$.

1 Introduction and results

A subspace of a linear space is a subset of the point set of the linear space that contains all points of a line l if l contains at least two points of the subspace. A *geometric hyperplane* or shortly, a *hyperplane*, of a point-line geometry is a proper subspace H such that each line meets H.

Let Π be a thick polar space of rank 3 (not necessarily finite) and let Δ be its dual. So, Δ has the diagram

* Supported by a scholarship of the *Graduiertenförderung des Landes Hessen*.



and as indicated, we call the elements of Δ *points*, *lines* and *quads*. Note that finite thick polar spaces are classical by Tits [12]. Considering a classical thick polar space, the residue $\mathrm{Res}_{\Pi}(P)$ in Π of a point $P \in \Pi$ is a classical generalized quadrangle. The residue $\mathrm{Res}_{\Delta}(\alpha)$ in Δ of a quad α of Δ is isomorphic to the generalized quadrangle dual of $\mathrm{Res}_{\Pi}(P)$.

If H is a hyperplane of Δ (the latter being regarded as point-line geometry) and α a quad of Δ, then either α is contained in H or H meets the generalized quadrangle α in a hyperplane. If $\alpha \not\subset H$, the hyperplane $\alpha \cap H$ of α is the perp of a point or a full subquadrangle or an ovoid (cf. Payne and Thas [7, 2.3.1]). Accordingly, we call the quad α *singular*, *subquadrangular*, or *ovoidal*, respectively. If α is a singular quad, we call the point $P \in \alpha$ with $\alpha \cap H = P^{\perp} \cap \alpha$ the *deep point with respect to* α, where $\perp$ denotes collinearity in Δ and where $P^{\perp}$ is the set of points of Δ collinear with P, including P. If α is a quad contained in H and P a point of α, then $P^{\perp} \cap \alpha \subset H$. Accordingly, we also say that every point of a quad $\alpha \subset H$ is a *deep point with respect to* α. Furthermore, we call a point P *deep* if $P^{\perp} \subset H$. So, if a point P is deep, it is deep with respect to all quads containing P.

Pasini and Shpectorov [6] call a hyperplane H of a dual polar space *locally ovoidal* (respectively *locally singular* and *locally subquadrangular*), if each quad α not contained in H meets the hyperplane in an ovoid (respectively in the perp of a point or in a subquadrangle). A hyperplane is called *uniform* if it is locally ovoidal or locally singular, or locally subquadrangular. Their main result is the non-existence of finite locally subquadrangular hyperplanes apart from two small examples, one in $\mathcal{Q}_6(2)$ and one in $\mathcal{H}_5(4)$.

The existence of locally ovoidal hyperplanes is an outstanding problem. Pasini and Shpectorov [6] prove that, given a locally ovoidal hyperplane H of a dual polar space Δ of rank 3, its complement $\Delta \backslash H$ cannot be flag-transitive.

For locally singular hyperplanes, there are two possibilities. The points at non-maximal distance from a given point P form a locally singular hyperplane called the *singular hyperplane* with *deep point P*. If all quads are singular and there is no deep point, then the hyperplane is a generalized hexagon by Shult [10]. More precisely, there is a bijection from the set of quads of Δ onto the set of deep points of the (singular) quads. In the finite case, Shult [10] shows that the polar space $\Pi = \Delta^*$ is an orthogonal space $\mathcal{Q}_6(q)$ and that the generalized hexagon H is a split Cayley hexagon $H(K)$ for a finite field $K = \mathrm{GF}(q)$ (for definition of $H(K)$, see Section 4.1). By means of a theorem due to Ronan [8], we prove the following Theorem 1 in Section 4.2 without assuming finiteness; it has also been proved by H. Van Maldeghem in 1999 (unpublished).

Theorem 1. *Let H be a hyperplane of a dual polar space* Δ *of rank* 3 *such that every quad is singular and H does not admit any deep point. Then the polar space* $\Pi \cong \Delta^*$

is an orthogonal polar space $\mathcal{Q}_6(K)$ and the hyperplane H is a split Cayley hexagon $H(K)$.

Our aim is to determine the non-uniform hyperplanes H of Δ such that no quad is subquadrangular, i.e. quads are either singular or ovoidal or contained in H. We prove the following:

Theorem 2. *Let H be a hyperplane of a thick dual polar space Δ of rank 3 such that no subquadrangular, but both singular and ovoidal quads exist. Then one of the following holds*:

1. $H = \bigcup_{P \in O} P^{\perp}$ *where O is an ovoid of a quad ω.*
2. *There exists one deep point $P \in H$ such that all quads containing P are singular and the remaining quads are ovoidal.*
3. *The set $\mathcal{P}$ of deep points with respect to the singular quads is a locally singular hyperplane of a dual polar space Δ_0. Δ_0 is the dual of an orthogonal polar space $\Pi_0 \cong \mathcal{Q}_6(K)$ for some field K and Π_0 is a subspace of the dual Π of Δ where the lines of Π_0 are lines of Π. The set $\mathcal{P}$ together with the lines of Δ contained in H form a split Cayley hexagon $H(K)$. The hyperplane H contains all points of Δ on lines of $H(K)$.*

Remarks. 1. The class of hyperplanes of the form $\bigcup_{P \in O} P^{\perp}$ where O is an ovoid of a quad has not yet been determined since ovoids of generalized quadrangles are not yet classified.

2. If we do not require the existence of ovoidal quads, Case 3 includes the possibility $\Delta_0 = \Delta$. Then we are back to Theorem 1.

Case 2 of Theorem 2 is non-empty. In Section 2, we describe examples for Cases 1 and 2 of Theorem 2.

No example as in Case 3 is known to the author in which Π_0 is not a hyperplane of Π. Thus it is an open problem whether the existence of a hyperplane as in Case 3 forces Π_0 to be a hyperplane of Π. This is of great interest because of

Corollary 1. *Assume the hypotheses of Theorem 2 and suppose that the hyperplane H is as in Case 3 of Theorem 2. If Π_0 is a hyperplane of Π, then H contains exactly the points of Δ on lines of $H(K)$ and we have $\Pi \cong \mathcal{Q}_7^-(K)$.*

In particular, if Δ is finite and the hyperplane H is as in Case 3 of Theorem 2, then Π is isomorphic to $\mathcal{Q}_7^-(q)$.

Remarks. 1. Clearly, the situation described in Corollary 1 occurs whenever $\Pi = \mathcal{Q}_7^-(K)$: just take $\Pi_0 = \mathcal{Q}_6(K) \subset \Pi$, $H(K)$ inside Π_0 and define H as in Corollary 1.

2. In the finite case, if $\Pi = \mathcal{Q}_6(q)$ with q odd, neither ovoidal nor subquadrangular quads exist, whence hyperplanes of its dual are locally singular.

If $\Pi = \mathcal{Q}_7^-(q)$ or $H_6(q^2)$, then Δ admits no subquadrangular quad. The example of Case 2 of Theorem 2 exposed in Section 2 is infinite. No finite example is known to the author. If Case 2 of Theorem 2 could be shown to be impossible in the finite case, then by means of Theorem 2 and the known results on locally uniform hyperplanes, the hyperplanes of the duals of $\mathscr{Q}_7^-(q)$ and $H_6(q^2)$ would be classified.

After a few preliminaries in Section 3, we introduce the split Cayley hexagon in Section 4.1 and prove Theorem 1 in Section 4.2.

The main part of the proof of Theorem 2 characterizes the hyperplane H when H has no deep point, which leads to Case 3 of Theorem 2. One step of this proof is a result on the extension of hyperplanes of dual polar spaces stated below in Theorem 3 and proved in Section 5.1. The proof of Theorem 2 will be finished in Section 5.2.

In the following, Π_0 is a non-degenerate polar subspace of rank 3 of a non-degenerate polar space Π of rank 3, such that the lines of Π_0 are lines of Π. We denote their duals by Δ_0 and Δ, respectively. Furthermore, σ (respectively σ_0) is the shadow operator of Δ (respectively Δ_0), i.e. given a line l of Δ_0, $\sigma(l)$ (respectively $\sigma_0(l)$) is the set of points of Δ (respectively Δ_0) on l. If X is a point subset of Δ_0, we denote by $\sigma(X)$ the set of points of Δ that are on some line m with $\sigma_0(m) \subseteq X$. Then the following holds.

Theorem 3. *Let H_0 be a locally singular hyperplane of the dual Δ_0 of the polar space Π_0 of rank 3. Then Π_0 is a hyperplane of Π (and its lines are lines of Π) if and only if $\sigma(H_0)$ is a hyperplane of the dual Δ of Π. In that case and if $\sigma(H_0)$ has no deep point, then all quads of Δ not in Δ_0 are ovoidal with respect to $\sigma(H_0)$.*

Acknowledgment. The author wishes to thank the referee for some remarkable improvements.

2 Examples

The points collinear with those of a hyperplane of a quad. The following proposition gives an easy method to construct hyperplanes of dual polar spaces of finite rank.

Proposition 2. *If ω is a quad of a dual polar space Δ of finite rank $n \geqslant 3$ and H_ω is a hyperplane of ω, then the set H of points at distance at most $n-2$ from some point of H_ω is a hyperplane of Δ.*

We omit the straightforward proof. As mentioned in the introduction, the hyperplane H_ω is one of the three kinds of hyperplanes of generalized quadrangles, namely an ovoid, a subquadrangle or the perp of a point.

We focus again on a rank three dual polar space Δ. If α is a quad meeting ω in a line l, then $P^\perp \cap \alpha \subseteq \alpha \cap H$ for all $P \in l \cap H_\omega$. So, if l meets H_ω in a single point X, α is singular with deep point X. If $l \subset H_\omega$, then α is contained in H.

Finally, if a quad α misses ω, then $\alpha \cap H$ is the image of H_ω in the projection of ω onto α (for definition of the projection, see Section 3). In particular, if H_ω is an ovoid, then α is ovoidal.

An example of Case 2 of Theorem 2. We state a lemma first.

Lemma 3. *Given a dual polar space Δ of rank 3 and a hyperplane H of Δ, suppose that H admits a deep point P. Then the following are equivalent*:

(i) *H is as in Case 2 of Theorem 2, namely P is the unique deep point of H and no two points of $H \backslash P^{\perp}$ are collinear*;

(ii) *$Q^{\perp} \cap H = \{Q\}$ for every point $Q \in H$ at distance 3 from P*;

(iii) *$Q^{\perp} \cap H = \{Q\}$ for some point $Q \in H$ at distance 3 from P.*

Proof. Clearly, (i) implies (ii) and (ii) implies (iii). We shall prove that (iii) implies (i). Assume (iii) and suppose that (i) is false, to get a contradiction. So, there exist distinct collinear points X, Y in $H \backslash P^{\perp}$. As $X, Y \in H$, the line $l = XY$ is contained in H. Let P' be the point of l at minimal distance from P. If $d(P, P') = 1$, then l and P belong to a quad α. Otherwise, $d(P, P') = 2$ and there exist a quad α containing both P and P'. In any case, we get a quad α containing P and a point $Z \in H$ at distance 2 from P (take $Z = X$ or P' according to whether $d(P, P') = 1$ or 2). As $P^{\perp} \subset H$, $\alpha \cap H \supseteq (P^{\perp} \cap \alpha) \cup \{Z\} \supset P^{\perp} \cap \alpha$. Therefore, $\alpha \subseteq H$. Hence the point $Q' \in \alpha$ collinear with Q belongs to H, which forces $QQ' \subseteq H$, contrary to (iii). □

We are now ready to describe an example for Case 2 of Theorem 2. Given a field K, the dual $S_5(K)^*$ of the symplectic polar space $S_5(K)$ is embeddable into a projective space $\mathscr{P} \cong \mathrm{PG}(13, K)$ by means of the mapping $\varepsilon : \mathrm{Gr}_3(V) \to \bigwedge^3 V$ from the Grassmannian of planes of $\mathrm{PG}(V)$ (where $V = V(6, K)$) into the 3-fold wedge product $\bigwedge^3 V$ (cf. Cooperstein [4, Proposition 5.1]; note that no hypotheses on the underlying field is needed in Proposition 5.1 of [4]).

Explicitly, suppose the polar space $S_5(K)$ is represented by the symplectic form

$$f(\boldsymbol{x}, \boldsymbol{y}) = x_0 y_3 - x_3 y_0 + x_1 y_4 - x_4 y_1 + x_2 y_5 - x_5 y_2$$

for $\boldsymbol{x} = \sum_{i=0}^{5} x_i \boldsymbol{e}_i$ and $\boldsymbol{y} = \sum_{i=0}^{5} y_i \boldsymbol{e}_i$ with $\boldsymbol{e}_0, \dots, \boldsymbol{e}_5$ a basis of V. The vectors $\boldsymbol{e}_{ijk} := \boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k$ for $0 \leqslant i < j < k \leqslant 5$ form a basis of $\bigwedge^3 V$. The coordinates of a vector

$$\sum_{0 \leqslant i < j < k \leqslant 5} x_{ijk} \boldsymbol{e}_{ijk}$$

of $\bigwedge^3 V$ representing a totally isotropic plane $\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle$ of $\mathrm{PG}(5, K)$ satisfy the six equations

$$\begin{aligned} 0 &= x_{013} - x_{125} = x_{014} + x_{025} = x_{023} + x_{124} \\ &= x_{034} - x_{245} = x_{035} + x_{145} = x_{134} + x_{235}. \end{aligned} \tag{1}$$

where, for $1 \leqslant i < j < k \leqslant 5$, x_{ijk} is the determinant of the matrix obtained from the 3×6 matrix formed by the three vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ by picking the i-th, j-th and k-th column. Equations (1) define a 13-dimensional subspace $\mathscr{P} < \mathrm{PG}(\bigwedge^3 V)$. The image of $S_5(q)^*$ via ε spans $\mathscr{P}$ (Cooperstein [4, Proposition 5.1]; see also [5, Teorema 4.15]).

Also, if $\overline{H}$ is a hyperplane of PG($\bigwedge^3 V$) not containing $\mathscr{P}$, then the point set $\{X \in S_5(K)^* : \varepsilon(X) \in \overline{H}\}$ is a hyperplane of $S_5(q)^*$. So, in view of Lemma 3, we only need to find a hyperplane $\overline{H}$ of PG($\bigwedge^3 V$) satisfying the following:

$$\varepsilon(P^\perp) \subset \overline{H}, \tag{2.1}$$

$$\varepsilon(Q^\perp) \cap \overline{H} = \{\varepsilon(Q)\} \text{ for some point } Q \text{ at distance 3 from } P. \tag{2.2}$$

(Note that (2.2) implies $\overline{H} \not\supseteq \mathscr{P}$). We choose $P = \langle \boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2 \rangle$ and $Q = \langle \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5 \rangle$. So, the points of $P^\perp \backslash \{P\}$ correspond to the following (singular) planes of $S_5(K)$ where $r, s, t \in K$:

$$\langle \boldsymbol{e}_0 + s\boldsymbol{e}_1, \boldsymbol{e}_1 + t\boldsymbol{e}_2, st\boldsymbol{e}_3 - t\boldsymbol{e}_4 + \boldsymbol{e}_5 + r\boldsymbol{e}_0 \rangle \tag{3.1}$$

$$\langle \boldsymbol{e}_0 + s\boldsymbol{e}_1, \boldsymbol{e}_2, -s\boldsymbol{e}_3 + \boldsymbol{e}_4 + r\boldsymbol{e}_0 \rangle \tag{3.2}$$

$$\langle \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_0 + r\boldsymbol{e}_3 \rangle \tag{3.3}$$

Their coordinates in PG$(13, K)$ are as follows:

for (3.1): $x_{012} = rst$, $x_{013} = x_{125} = st$, $x_{014} = -x_{025} = -t$, $x_{015} = 1$, $x_{023} = -x_{124} = st^2$, $x_{024} = -t^2$, $x_{123} = s^2t^2$ and $x_{ijk} = 0$ if $i \geqslant 2$

for (3.2): $x_{012} = rs$, $x_{023} = -x_{124} = -s$, $x_{024} = 1$, $x_{123} = -s^2$, and $x_{ijk} = 0$ for all remaining choices of $\{i, j, k\}$

for (3.3): $x_{012} = 1, x_{123} = r$ and $x_{ijk} = 0$ for $\{i, j, k\} \neq \{0, 1, 2\}, \{1, 2, 3\}$

So, considering that the above scalars r, s and t vary arbitrarily in K, if

$$\sum_{1 \leqslant i < j < k \leqslant 5} a_{ijk} x_{ijk} = 0$$

is the equation of $\overline{H}$, condition (2.1) forces

$$\begin{aligned} a_{012} = a_{015} = a_{024} = a_{123} = 0 \\ a_{025} - a_{014} = a_{124} - a_{023} = a_{125} + a_{013} = 0 \end{aligned} \tag{4.1}$$

In its turn, the condition $\varepsilon(Q) \in \overline{H}$ forces

$$a_{345} = 0. \tag{4.2}$$

However, according to (2.2), we also want that $\varepsilon(Q^\perp \backslash \{Q\}) \cap \overline{H} = \emptyset$. The points of $Q^\perp \backslash \{Q\}$ correspond to the following (singular) planes of $S_5(K)$ where $r, s, t \in K$

$$\langle \boldsymbol{e}_4 + s\boldsymbol{e}_5, \boldsymbol{e}_3 + t\boldsymbol{e}_4, st\boldsymbol{e}_0 - s\boldsymbol{e}_1 + \boldsymbol{e}_2 + r\boldsymbol{e}_3 \rangle \tag{5.1}$$

$$\langle \boldsymbol{e}_5, \boldsymbol{e}_3 + t\boldsymbol{e}_4, -t\boldsymbol{e}_0 + \boldsymbol{e}_1 + r\boldsymbol{e}_3 \rangle \tag{5.2}$$

$$\langle \boldsymbol{e}_5, \boldsymbol{e}_4, \boldsymbol{e}_0 + r\boldsymbol{e}_3 \rangle \tag{5.3}$$

and their coordinates in $PG(13, K)$ are as follows:

for (5.1): $x_{034} = x_{245} = -st$, $x_{035} = -x_{145} = -s^2t$, $x_{045} = -s^2t^2$, $x_{134} = -x_{235} = s$, $x_{135} = s^2$, $x_{234} = -1$, $x_{345} = -rst$ and $x_{ijk} = 0$ if $j \leqslant 2$;

for (5.2): $x_{035} = -x_{145} = t$, $x_{045} = t^2$, $x_{135} = -1$, $x_{345} = rst$ and $x_{ijk} = 0$ for all remaining choices of $\{i, j, k\}$;

for (5.3): $x_{045} = -1$, $x_{345} = r$ and $x_{ijk} = 0$ for $\{i, j, k\} \neq \{0, 4, 5\}, \{3, 4, 5\}$;

In view of (4.1) and (4.2), none of the above points belongs to $\overline{H}$ if and only if all of the following hold for any $s, t \in K$:

$$(a_{034} + a_{245})st + (a_{035} - a_{145})s^2t + a_{045}s^2t^2 - (a_{134} - a_{235})s - a_{135}s^2 + a_{234} \neq 0 \tag{6.1}$$

$$a_{045}t^2 + (a_{035} - a_{145})t - a_{135} \neq 0 \tag{6.2}$$

$$a_{345} = 0 \quad \text{and} \quad a_{045} \neq 0 \tag{6.3}$$

Needless to say, the existence of solutions for the inequalities (6.1)–(6.3) depends on the field we consider. In order to describe some of them, we may assume some additional conditions on the scalars a_{ijk} involved in (6.1)–(6.3). Note first that, as $a_{234} \neq 0$ by (6.1), we may always assume $a_{234} = 1$. Suppose furthermore that $a_{145} - a_{035} = a_{034} + a_{245} = a_{235} - a_{134} = 0$ (which is allowed by (6.1)–(6.3)) and put $a = a_{045}$ (which is non-zero by (6.3)) and $b = a_{135}/a$ (which is non-zero by (6.2)). Then (6.1) and (6.2) read as follows:

$$s^2(t^2 - b) \neq -\frac{1}{a} \quad \text{for any } s, t \in K \tag{7.1}$$

$$t^2 \neq b \quad \text{for any } t \in K \tag{7.2}$$

Condition (7.2) just says that b is a non-square. So, suppose that K is the field of real numbers and choose $b = -1$. Then (7.2) is satisfied and (7.1) says that $-1/a$ is not the product of two positive numbers, which is true whenever $a > 0$.

Note. The trick we have exploited above when K is the field of real numbers does not work if K is a finite field. Indeed, when $K = GF(q)$, for (7.2) to hold we only need to choose a non-square for b (which can always be done, provided that q is odd).

However, even so, the equation $s^2(t^2 - b) = -\frac{1}{a}$ always has a solution in $GF(q)$, contrary to (7.1).

Problem. Is there any example for Case 2 of Theorem 2 in the finite case?

3 Notations and preliminaries

For basic properties of dual polar spaces, we refer to Cameron [2]. If α is a quad of a dual polar space Δ and P is a point not in α, there exists exactly one point in α nearest to P. If the rank of Δ is 3, there exists exactly one point in α collinear with P.

Definition 4. Let α be a quad of the dual polar space Δ of rank 3. We call the map sending every point $P \in \Delta$ to the point of α nearest to P the *projection onto* α and denote it by π_α.

It is well known that π_α is a surjective morphism from Δ, regarded as a point-line geometry, to α. It is easy to see that, if α' is a quad disjoint from α, π_α induces an isomorphism from α' onto α.

Given a hyperplane H of a dual polar space Δ of rank 3, we introduce the following notations. We call a line (quad) a $-$-*line* (respectively $-$-*quad*) if it is contained in H and a $+$-*line* (respectively $+$-*quad*) if it is not contained in H.

4 On locally singular hyperplanes

4.1 The split Cayley hexagon. We introduce the split Cayley hexagon $H(K)$ for a field K following Van Maldeghem [13, 2.4] for notation. Most of the results are due to Tits [11].

Let $\mathscr{P}^{(0)}$ denote the points of the quadric $\mathscr{Q}_7^+(K)$ for a field K, let $\mathscr{L}$ denote the set of lines of $\mathscr{Q}_7^+(K)$, and let $\mathscr{P}^{(1)}$ and $\mathscr{P}^{(2)}$ denote the two classes of 3-spaces of $\mathscr{Q}_7^+(K)$. The D_4-geometry $\Omega(K)$ associated to $\mathscr{Q}_7^+(K)$ consists of the elements of $\mathscr{L}$ and the elements of $\mathscr{P}^{(i)}$ for $i = 0, 1, 2$ where incidence is symmetrized inclusion from $\mathscr{Q}_7^+(K)$ and two elements of $\mathscr{P}^{(1)}$ and $\mathscr{P}^{(2)}$ are incident if the corresponding 3-spaces of $\mathscr{Q}_7^+(K)$ intersect in a plane.

For the following, the most important property of $\Omega(K)$ is that the geometry arising from $\Omega(K)$ by any permutation of $\{\mathscr{P}^{(0)}, \mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ is isomorphic to $\Omega(K)$. Hence for each $i \in \{0, 1, 2\}$, the set of elements of $\mathscr{P}^{(i)}$ may be identified with the points of $\mathscr{Q}_7^+(K)$.

A *triality* of $\Omega(K)$ is an incidence preserving map $\theta : \mathscr{L} \to \mathscr{L}$, $\mathscr{P}^{(i)} \to \mathscr{P}^{(i+1) \bmod 3}$ for $i = 0, 1, 2$ such that $\theta^3 = 1$. An element x of $\mathscr{P}^{(i)}$ is called *absolute* if x and $\theta(x)$ are incident. A line is *absolute* if it is fixed by θ.

Now, identifying the elements of $\mathscr{P}^{(i)}$ with the points of $\mathscr{Q}_7^+(K)$ for each $i = 0, 1, 2$, there is an algebraic description of $\Omega(K)$ available by a trilinear form $\mathscr{T} : V \times V \times V \to K$ where V is the 8-dimensional vector space over K in which $\mathscr{Q}_7^+(K)$ is defined by a quadratic form. Two elements $x \in \mathscr{P}^{(0)}$ and $y \in \mathscr{P}^{(1)}$ are incident in $\Omega(K)$ if and only if $\mathscr{T}(x, y, z) = 0$ for all $z \in V$. The other incidences follow similarly.

Given a vector space representation of $\mathscr{Q}_7^+(K)$ as above, denote by τ the triality sending (x, y, z) onto (z, x, y) induced on $\Omega(K)$ by the identity on V. The *split Cayley hexagon* $H(K)$ consists of the absolute points and lines of τ. The absolute points of τ are the singular points of a non-degenerate hyperplane $\mathrm{PG}(6, K)$ of the embedding projective space $\mathrm{PG}(7, K)$ of $\mathscr{Q}_7^+(K)$. For the direct embedding of $H(K)$ in the parabolic quadric $\mathscr{Q}_6(K)$ due to Tits [11], see Van Maldeghem [13, 2.4.13].

4.2 Proof of Theorem 1. Let L denote the set of lines contained in H. By Shult [10, Theorem 1], $\mathscr{H} = (H, L)$ is a generalized hexagon, and there exists a bijection δ from H onto the set of quads of Δ that maps every point $X \in H$ onto the quad $\delta(X)$ of which X is the deep point.

We follow Van Maldeghem [13, 1.9] for the notation: For a point P of $\mathscr{H}$, the *perp-geometry in P* has point set the points of $P^{\perp} \cap H$. There are two different kinds of lines in the perp-geometry in P, firstly the lines of $\mathscr{H}$ through P, and secondly the point sets $\Gamma_2(P) \cap \Gamma_4(X)$ for every $X \in H$ at distance 3 from P, where $\Gamma_{2n}(X)$ denotes the set of points of $\mathscr{H}$ at distance n from X in the collinearity graph of $\mathscr{H}$. A point P is called *projective* if the perp-geometry in P is a projective plane. A point P is called *distance-2-regular* if two lines of the perp-geometry in P intersect in at most one point.

By Ronan [8], the generalized hexagon $\mathscr{H}$ is a split Cayley hexagon $H(K)$ if all its points are distance-2-regular and if there is at least one projective point in H (we use Ronan's result as it is stated in Van Maldeghem [13, Theorem 6.3.1]). We prove that every point of $\mathscr{H}$ is projective. Then every point is distance-2-regular and the conditions of [13, 6.3.1] are satisfied.

For a point $P \in H$, let $\Gamma_2(P) \cap \Gamma_4(X)$ be a line of the perp-geometry in P where X is a point of H at distance 3 from P. The line $\Gamma_2(P) \cap \Gamma_4(X)$ contains the points of H collinear with P and with $\pi_{\delta(P)}(X)$ in Δ. Hence it is the trace $\{P, \pi_{\delta(P)}(X)\}^{\perp}$ of P and $\pi_{\delta(P)}(X)$ in the generalized quadrangle $\mathrm{Res}(\delta(P))$. For every point $T \in \Gamma_2(P) \cap \Gamma_4(X)$, the quad $\delta(T)$ contains the $+$-line $h(P, X) := P\pi_{\delta(X)}(P)$. Indeed, if $T' \in P^{\perp} \cap \delta(P)$ and $h(P, X) \not\subset \delta(T')$, then the quad $\delta(T')$ is disjoint from $\delta(X)$, whence T' is at distance 3 from X and $T' \notin \Gamma_2(P) \cap \Gamma_4(X)$. Hence the line $\Gamma_2(P) \cap \Gamma_4(X)$ of the perp-geometry in P is uniquely determined by the line $h(P, X)$ of Δ, and the points of $\Gamma_2(P) \cap \Gamma_4(X)$ are the points $\delta^{-1}(\tau)$ for the quads τ of Δ on the line $h(P, X)$.

Hence the points of the perp-geometry in P correspond to the lines of the residue $\mathrm{Res}_{\Delta}(P)$ of P in Δ (these are the quads having points of $P^{\perp} \cap H$ as deep points), and the lines of the perp-geometry in P correspond to the points of $\mathrm{Res}_{\Delta}(P)$ (these are the lines of Δ through P). Since $\mathrm{Res}_{\Delta}(P)$ is a projective plane, the perp-geometry in P is a projective plane isomorphic to the dual of $\mathrm{Res}_{\Delta}(P)$.

Hence the generalized hexagon $\mathscr{H}$ is a split Cayley hexagon $H(K)$ embedded in an orthogonal polar space $\tilde{\Pi} \cong \mathscr{Q}_6(K)$.

It remains to show that $\Pi = \Delta^*$ is isomorphic to the orthogonal polar space $\tilde{\Pi} \cong \mathscr{Q}_6(K)$. The points of Π are the (singular) quads of Δ. These correspond to the points of $\mathscr{H}$ via the bijection δ, whence the quads of Δ correspond to the points of the orthogonal polar space $\tilde{\Pi} = \mathscr{Q}_6(K)$ which is the ambient space of $H(K)$.

The lines of $\tilde{\Pi}$ through a point $P \notin H$ are the lines of the perp-geometry in P. By the above, the perp-geometry in P is a projective plane, in fact a singular plane of $\tilde{\Pi} \cong \mathscr{Q}_6(K)$. On the other hand, we have seen that the lines of the perp-geometry in P correspond to the lines of Δ through P. So, the lines of Δ are the lines of the polar space $\tilde{\Pi} \cong \mathscr{Q}_6(K)$ and we have proved that the dual Π of Δ is isomorphic to $\tilde{\Pi} = \mathscr{Q}_6(K)$. Hence we have proved Theorem 1. □

5 Non-uniform hyperplanes with no subquadrangular quad

5.1 Proof of Theorem 3. We remind the reader of the notations in Theorem 3: H_0 is a locally singular hyperplane of the dual Δ_0 of a polar space Π_0 of rank 3 that is a hyperplane of a polar space Π of rank 3. The shadow operator of Δ (respectively Δ_0) is σ (respectively σ_0). Remark that a line l of Δ belongs to Δ_0 if and only if $\sigma_0(l) \neq \emptyset$. If X is a point subset of Δ_0, $\sigma(X)$ denotes the set of points of Δ that are on some line m with $\sigma_0(m) \subseteq X$.

Suppose Π_0 is a hyperplane of Π. We first show $\sigma(H_0)$ is a subspace of Δ. Let X, Y be two points of $\sigma(H_0)$ on a line l. We have to show $l \subset H_0$. Since Π_0 is a hyperplane, there is a point $\beta \in \Pi_0$ on l, the latter being regarded as a line of Π. As $X, Y \in \sigma(H_0)$, there are lines $h, m \subset H_0$ through X and Y, respectively. If h does not belong to the quad β of Δ_0, then the point $X = h \cap \beta$ belongs to Δ_0, thus $X \in H_0$. If m does neither belong to β, it follows similarly $Y \in H_0$. Then $l \subset H_0$ and we are done. So, assume $h \not\subset \beta$ and $m \subset \beta$. Then, as X and m belong to Δ_0, the line through X concurrent with m, namely the line l, belongs to Δ_0, and we are done. Finally, assume $h, m \subset \beta$. If $\beta \subset H_0$, then $l \subset \beta \subset H_0$ is contained in H_0. If β is singular, h and m are concurrent in the deep point of β. Thus either h, l, m form a triangle, which is impossible, or $l = h = m \subset H_0$. Hence $\sigma(H_0)$ is a subspace of Δ.

It remains to show $\sigma(l) \cap \sigma(H_0) \neq \emptyset$ for every line l of Δ. Since Π_0 is a hyperplane of Π, every line l of Π contains a point α of Π_0. Regarded in Δ, α is a singular quad with deep point P, and l contains a point P_0 collinear with P. Then the line $l_0 := PP_0$ (possibly $l = l_0$) belongs to H_0 since it is contained in the quad α of Δ_0.

Conversely, suppose $\sigma(H_0)$ is a hyperplane of Δ. Then every line l of Δ contains a point P of $\sigma(H_0)$. The point P lies on some line l_0 of Δ_0. The quad $\langle l, l_0 \rangle$ belongs to Δ_0 since it contains the line l_0 of Δ_0. In Π, the quad $\langle l, l_0 \rangle$ is a point of Π_0 on the line l. So, every line of Π meets Π_0, and Π_0 is a hyperplane of Π.

In the remainder, suppose Π_0 is a hyperplane of Π and $\sigma(H_0)$ has no deep point. Given a quad α of Δ not in Δ_0, we have to show that $\alpha \cap \sigma(H_0)$ is an ovoid. Assume to the contrary that α contains two points X, Y of $\sigma(H_0)$ on a line l. Since Π_0 is a hyperplane, there is a point $\beta \in \Pi_0$ on l regarded as line of Π. Since $X, Y \in \sigma(H_0)$, there are lines $h, m \subset H_0$ through X and Y, respectively. If h does not belong to the quad β of Δ_0, then the intersection point X of the line h of Δ_0 with β belongs to Δ_0. Regarding X as plane of Π_0, all of its points belong to Π_0. This forces α to belong to Δ_0 in contradiction to the hypothesis. Thus $h \subset \beta$ and similarly $m \subset \beta$. Since $h, m \subset H_0 \cap \beta$ and β is singular, h and m are concurrent in a point P of Δ_0. Thus either h, l, m form a triangle, which is impossible, or $h = l = m$. However, the second case is impossible, too, since it forces α to belong to Δ_0.

Hence Theorem 3 is proved.

5.2 Proof of Theorem 2. Let H be a hyperplane such that $+$-quads are either singular or ovoidal and assume that there exist both singular and ovoidal quads. In Lemmata 5–9, we investigate some properties of the hyperplane H.

Lemma 5. *Given two quads α, α' with common deep point $P \in H$, P is deep with respect to H.*

Proof. Let β be a quad containing P and not containing the line $\alpha \cap \alpha'$. Then $\beta \cap \alpha \neq \beta \cap \alpha'$ and $\beta \cap \alpha$ and $\beta \cap \alpha'$ are $-$-lines concurrent in P. Hence P is deep with respect to β. It follows that all lines through P are $-$-lines. So, P is deep itself. □

Lemma 6. *Given a quad* $\alpha \subset H$ *and an ovoidal quad* ω, *setting* $O := \omega \cap H$, *the set* $\pi_\alpha(O)$ *is an ovoid and all its points are deep.*

Proof. The quads α and ω are disjoint since an ovoidal quad has no $-$-line. Let $P \in O$ and denote the point $\pi_\alpha(P)$ by P'. The line $l := PP'$ is contained in H. Let β be a quad on l. Then β contains the two $-$-lines l and $\beta \cap \alpha$ through the point P'. Hence P' is deep with respect to β. By Lemma 5, P' is deep. □

Lemma 7. *If there exists an ovoidal quad, any two* $-$*-quads are disjoint.*

Proof. Assume α, α' are $-$-quads on a $-$-line $l = \alpha \cap \alpha'$. By Lemma 5, every point $P \in l$ is deep. If ω is an ovoidal quad, it is disjoint from α and α', hence from l. So, $\pi_\omega(l)$ is a line in ω. Since all points of l are deep, all points of $\pi_\omega(l)$ are contained in H. This is impossible since ω is ovoidal. □

Lemma 8. *If there exists an ovoidal quad, no two deep points of H are collinear.*

Proof. Assume P, P' are collinear deep points. The line $l := PP'$ is contained in H. Given a quad α on l, it is contained in H since P, P' are two deep points with respect to $\alpha \cap H$. Hence all quads on l are $-$-quads in contradiction to Lemma 7. □

Lemma 9. *If* α *is a* $-$*-quad and* ω_1, ω_2 *are ovoidal quads with ovoids* $O_i := \omega_i \cap H$, $i = 1, 2$, *then* $\pi_\alpha(O_1) = \pi_\alpha(O_2)$.

Proof. By Lemma 6, $\pi_\alpha(O_1)$ and $\pi_\alpha(O_2)$ consist of deep points. Since by Lemma 8 no two deep points are collinear, $\pi_\alpha(O_1) \cup \pi_\alpha(O_2)$ is an ovoid of Q, hence $\pi_\alpha(O_1) = \pi_\alpha(O_2)$. □

Propositions 10 and 11 establish Cases 1 and 2 of Theorem 2.

Proposition 10. *The hyperplane H contains at most one quad. If H contains a quad* α, *then* $H = \bigcup_{P \in O} P^\perp$ *where O is an ovoid of* α.

Proof. Assume H contains a quad α. By Lemma 7 and since no ovoidal quad meets a $-$-quad, all quads meeting α are singular.

Since there exists an ovoidal quad ω, by Lemma 6, the points of the ovoid $O := \pi_\alpha(\omega \cap H)$ of the quad α are deep. Let $P \in O$. Given a point $X \in P^\perp \backslash \alpha$, the line $l := PX$ is contained in H and the quads on l are singular with deep point P. Hence l is the only $-$-line through X, and the quads through X not containing l are ovoidal.

Since a quad α' disjoint from α has exactly one point Y in $P^\perp$ (indeed the point $\pi_{\alpha'}(P)$) and since α' does not contain the line YP, according to the previous paragraph, α' is ovoidal. Hence all quads disjoint from α are ovoidal.

By Lemma 9, there is one and only one ovoid O of deep points in α. So, the hyperplane consists of all points collinear with points of O. □

Proposition 11. *If H does not contain any quad, H has at most one deep point. If it has a deep point P, the quads through P are singular and the remaining quads are ovoidal.*

Proof. Let P be a deep point of H. Let $R \in P^{\perp} \backslash \{P\}$ and denote the $-$-line PR by l. Since no quad is contained in H, all quads on l are singular with deep point P. Hence the line l is the only line through R contained in H. So, a quad through R not containing l is ovoidal and meets $P^{\perp}$ in the unique point R.

Since every quad has a point of $P^{\perp}$, it follows that the quads through P are singular with deep point P and the quads not containing P are ovoidal. □

In the remainder, we investigate the hyperplane H when it has no deep point. We prove the assertions of Case 3 of the theorem in several lemmata and propositions.

Let $\mathcal{Q}$ be the set of all singular quads of Δ, let $\mathcal{P}$ be the set of points of H deep with respect to some singular quad, and let $\mathcal{L}$ be the set of lines of Δ that meet the point set $\mathcal{P}$. Denote by Π_0 the substructure $(\mathcal{Q}, \mathcal{L})$ of the polar space Π dual of Δ.

Lemma 12. *Every quad of Δ containing a line of $\mathcal{L}$ is singular.*

Proof. Let $l \in \mathcal{L}$. If $l \subset H$, the assertion is obvious. Assume $l \cap H$ is a point X. By definition of $\mathcal{L}$, X is deep for some singular quad α. Hence every quad containing l meets α in a $-$-line and is singular. □

Lemma 13. *Π_0 is a proper subspace of Π. The lines of Π_0 are lines of Π, and two points $\alpha, \beta \in \Pi_0$ are collinear whenever they are collinear in Π.*

Proof. Let α, β be two quads of $\mathcal{Q}$ that have a line l in common. We claim $l \in \mathcal{L}$. Indeed, if $l \subset H$, all quads on l are singular and have their deep points on l, whence $l \in \mathcal{L}$. If l is a $+$-line that meets H in a point P, both quads α and β are singular and have distinct $-$-lines h, h', respectively, through P. Hence the quad $\langle h, h' \rangle$ is singular with deep point P. So, $P \in \mathcal{P}$ and $l \in \mathcal{L}$. By Lemma 12, all quads on l belong to $\mathcal{Q}$. Hence $\mathcal{Q}$ which is the point set of Π_0, is a subspace of Π, and the lines of Π_0 are precisely the lines of Π contained in $\mathcal{Q}$.

It is a proper subspace since the points of Π corresponding to ovoidal quads of Δ do not belong to Π_0. □

Lemma 14. *Given a point P of Δ, either none, one or all lines through P belong to $\mathcal{L}$.*

Proof. If $P \in \mathcal{P}$, there is nothing to prove. Accordingly, assume in the following that P is not deep for any quad of $\mathcal{Q}$, i.e. $P \notin \mathcal{P}$.

We first show that there is no $-$-line of $\mathcal{L}$ through P. Assume to the contrary that l is a $-$-line through P. If $l' \in \mathcal{L}$ is a second line through P, l' is a $+$-line with $l' \cap H = P$. It follows $P \in \mathcal{P}$ since otherwise l' would not belong to $\mathcal{L}$—a contradiction.

So, the two lines l, l' of $\mathcal{L}$ through P are $+$-lines. Since by Lemma 12, the quad $\langle l, l' \rangle$ is singular, it follows $P \notin H$ since otherwise there would be a $-$-line through P in contradiction to the previous paragraph. Hence given a line h through P not in $\langle l, l' \rangle$, $h \cap H$ is a point $Y \neq P$. Since every quad on l (respectively l') is singular, the quad $\langle h, l \rangle$ (respectively $\langle h, l' \rangle$) is singular. Since $Y \in \langle h, l \rangle \cap H$ (respectively $\in \langle h, l' \rangle \cap H$), there is a $-$-line through Y in $\langle h, l \rangle$ (respectively $\langle h, l' \rangle$). Hence there are two $-$-lines through Y. So, $Y \in \mathscr{P}$, thus implying $h \in \mathscr{L}$.

Given a line m through X in $\langle l, l' \rangle$, the quad $\langle h, m \rangle$ is singular by Lemma 12 since $h \in \mathscr{L}$. Hence there are the two singular quads $\langle l, l' \rangle$ and $\langle h, m \rangle$ on the $+$-line m. It follows that there are two $-$-lines on the point $m \cap H$. Hence $m \cap H \in \mathscr{P}$, which forces $m \in \mathscr{L}$. □

Proposition 15. Π_0 *is a polar space of rank* 3.

Proof. By Buekenhout and Shult [1], Π_0 is a polar space of rank 3 if it satisfies the one-or-all axiom and if its maximal subspaces are planes. We start verifying the one-or-all axiom for polar spaces.

Let $\alpha \in \mathscr{Q}$ be a singular quad with deep point Z, and let $l \in \mathscr{L}$ be a line of $\mathscr{L}$ not contained in α. All quads on l are singular by Lemma 12. If a quad α' on l meets α in a line $h \in \mathscr{L}$, α' is a point of Π_0 on l collinear with α.

We show that the quads on l meeting α intersect α in lines of $\mathscr{L}$. This proves the one-or-all-axiom since on the one hand, all quads on l meet α if l meets α, and on the other hand, there is exactly one quad on l meeting α if l is disjoint from α.

Firstly, assume $l \cap \alpha = \emptyset$. Denote the line $\pi_\alpha(l)$ by h. By Lemma 12, the unique quad $\langle h, l \rangle$ containing l and meeting α is singular, and it remains to show $h \in \mathscr{L}$.

If h goes through Z, h belongs to $\mathscr{L}$ since $Z \in \mathscr{P}$. If h does not go through Z, denote the point $h \cap H = h \cap Z^\perp$ by X. Since $\langle h, l \rangle$ is singular by Lemma 12, there is a $-$-line g through X in $\langle h, l \rangle$ such that the point $X \in h$ is deep for the quad $\langle g, XZ \rangle$. Hence h belongs to $\mathscr{L}$.

Secondly, assume l intersects α in a point X. The quads on l are the points of Π_0 on the line l collinear with the point α by the lines that are the lines in the quad α through the point X. Since all quads on l are singular, it remains to show that all lines through X in α belong to $\mathscr{L}$.

If l is a $-$-line, X is collinear with Z and is deep with respect to the quad $\langle l, XZ \rangle$. Thus $X \in \mathscr{P}$ and all lines through X belong to $\mathscr{L}$ and we are done. If l is a $+$-line, the point $l \cap H$ is deep for a singular quad β since $l \in \mathscr{L}$. We investigate the cases $l \cap H = X$ $(= P \cap l)$ and $l \cap H \neq X$ separately.

Assume $l \cap H \neq X$. Then the quad β is disjoint from α since it does not contain the line l. The point X is not collinear with the deep point Z with respect to α. Let γ be a quad on l. So, γ is singular with deep point D on the line $m := \gamma \cap \beta$ where $D \neq m \cap l$. Furthermore, γ meets α in a line h that is not contained in H since $D \notin h$. It follows $h \cap H = h \cap D^\perp$. The point $Y := h \cap H$ is collinear with Z since Z is the deep point of α. So, $Y \in \mathscr{P}$ since the lines DY and YZ are distinct $-$-lines through Y. Thus, Y is the deep point of the singular quad $\langle D, Z \rangle$. Hence $h \in \mathscr{L}$.

Finally, assume l is a $+$-line with $l \cap H = X$. The line $XZ \subset \alpha$ is contained in H since Z is the deep point of α and XZ belongs to $\mathscr{L}$. So, l, XZ are two lines of $\mathscr{L}$ through X, and by Lemma 14, all lines through X belong to $\mathscr{L}$. Therefore, every quad on l meets α in a line $h \in \mathscr{L}$.

Hence Π_0 satisfies the one-or-all axiom. It remains to show that the maximal subspaces of Π_0 are planes. On the one hand, since Π_0 is a subspace of the polar space Π, the rank of Π_0 is at most 3. On the other hand, by Lemma 14, the maximal subspaces of Π_0 are maximal subspaces of Π. Consequently, Π_0 has rank 3. □

Lemma 16. *A point P of Δ belongs to the dual Δ_0 of Π_0 if and only if all lines of Δ through P belong to $\mathcal{L}$.*

Proof. By Lemma 13, the polar space Π_0 is a subspace of the polar space $\Pi = \Delta^*$. Since the set of all quads through a point of Δ is the point set of a maximal subspace of Π, the quads through a point of Δ_0 are the points of a maximal subspace of Π_0. If X is a point of Δ_0, the quads of Δ_0 through X regarded as points of Π_0 are points of a plane of Π_0. By Proposition 15, all points and lines of that plane belong to Π_0. Hence all lines of Δ through X belong to L. Conversely, if all lines of Δ through X belong to $\mathcal{L}$, then all quads on X are singular by Lemma 12. Thus regarding X as a plane of Π, all its points belong Π_0, whence it is a plane of Π_0, namely a point of Δ_0. □

Proposition 17. *The point set $\mathcal{P}$ is a hyperplane of Δ_0. It is the hyperplane $\Delta_0 \cap H$ induced by H in Δ_0.*

Proof. Since $\mathcal{P}$ consists of the points deep with respect to some singular quad of Δ, all lines through a point $X \in \mathcal{P}$ belong to $\mathcal{L}$, and X is a point of Δ_0 by Lemma 16. Hence $\mathcal{P}$ is a subset of the point set of Δ_0.

To show that $\mathcal{P}$ is a subspace of Δ_0, let $X, X' \in \mathcal{P}$ be distinct points on a line l. Since $\mathcal{P} \subset H$ and H is a subspace of Δ, it follows $l \subset H$. It remains to show that if l is a line of Δ contained in H, every point of Δ_0 on l belongs to $\mathcal{P}$. Let $V \in l$ belong to Δ_0. By Lemma 16, all lines through V belong to $\mathcal{L}$. If α is a quad of Δ_0 through V not containing l, then α is singular by Lemma 12. So, since $V \in H$, there exists a $-$-line m $(\neq l)$ of Δ_0 through V in α. Hence V is deep with respect to the quad $\langle l, m \rangle$ of Δ_0, and it follows $V \in \mathcal{P}$.

Since every quad of Δ_0 meets $\mathcal{P}$ in the perp of a point by construction of Δ_0, $\mathcal{P}$ is a proper subspace of Δ_0. $\mathcal{P}$ is a hyperplane of Δ_0 since the lines of Δ_0 are the lines in $\mathcal{L}$ that meet $\mathcal{P}$ by definition.

Furthermore, we have seen in the previous paragraph that all points of Δ_0 on a line contained in H belong to $\mathcal{P}$. Given a point $P \in \Delta_0 \cap H$ and a quad α of Δ_0 on P, α is singular and $P \in \alpha \cap H$ belongs to some line $l \cup H$. Thus $\mathcal{P} = \Delta_0 \cap H$. □

We denote the set of lines of Δ contained in H by L. An immediate consequence is $L \subseteq \mathcal{L}$ and $\mathcal{L} \nsubseteq L$. Let $\mathcal{H}$ denote the incidence structure $(\mathcal{P}, L)$.

Proposition 18. *$\mathcal{H} = (\mathcal{P}, L)$ is a split Cayley hexagon $H(K)$. The polar space $\Pi_0 = (\mathcal{Q}, \mathcal{L})$ is isomorphic to the orthogonal polar space $\mathcal{Q}_6(K)$ in which $\mathcal{H}$ is embedded.*

Proof. Since the points of $\Pi_0 = \Delta_0^*$ are the singular quads of Δ, each quad of Δ_0 is singular with respect to the hyperplane $\mathcal{P} = \Delta_0 \cap H$ (cf. Proposition 17). By Theorem

1, $\mathscr{H} = (\mathscr{P}, L)$ is a split Cayley hexagon $H(K)$ embedded in $\tilde{\Pi} \cong \mathscr{Q}_6(K)$ and the polar space Π_0 is isomorphic to $\tilde{\Pi}$. □

Hence we have proved the assertions of Theorem 2. The following corollary and proposition establish Corollary 1.

Corollary 19. *Suppose* Π_0 *is a hyperplane of* Π. *Then the hyperplane* H *consists of the points on lines of* L.

Proof. Denote by H' the set of points of Δ on lines of L. Supposing Π_0 is a hyperplane of Π, we apply Theorem 3. Then $H' = \sigma(\mathscr{P})$ is a hyperplane of Δ. We show $H = H'$. The inclusion $H' \subseteq H$ follows from the definition of H'.

Assume there exists a point $P \in H \backslash H'$. Since H' is a hyperplane of Δ, each line through P meets H' in a point distinct from P since $P \notin H'$. So, each line through P is contained in H implying $P \in H'$—a contradiction. □

Remark that Corollary 19 is a special case of a result of Shult [9, Lemma 6.1] according to which every hyperplane of a dual polar space is a maximal subspace.

Corollary 19 together with the following proposition prove the second assertion of Corollary 1.

Proposition 20. *If* Δ *is finite, the polar space* $\Pi = \Delta^*$ *is an orthogonal polar space* $\mathscr{Q}_7^-(q)$. *In particular,* $\Pi_0 \cong \mathscr{Q}_6(q)$ *is a hyperplane of* Π.

Proof. By Theorem 2, the dual Π_0 of Δ_0 is a polar space isomorphic to $\mathscr{Q}_6(K)$ for some finite field $K = \mathrm{GF}(q)$.

If α is a point of Π_0, its residue $\mathrm{Res}_{\Pi_0}(\alpha)$ in Π_0 is a generalized quadrangle of order (q, q) which is a proper subquadrangle of the generalized quadrangle $\mathrm{Res}_{\Pi}(\alpha)$ of order (t, q). Hence $t > q$.

Regarding α as a quad of Δ, the generalized quadrangle $\mathrm{Res}_{\Delta}(\alpha)$ admits an ovoid. Thus by Payne and Thas [7, 3.4.1] and since $t > q$, $\mathrm{Res}_{\Delta}(\alpha)$ is isomorphic to $\mathscr{H}_3(q)$. Hence $\Pi \cong \mathscr{Q}_7^-(q)$. □

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Received 15 January, 2001; revised 12 February, 2001

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