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Simplicial maps from the 3-sphere to the 2-sphere

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Abstract. We give minimal simplicial maps $\eta : \mathbb{S}^3_{12} \to \mathbb{S}^2_4$ and $\xi : \mathbb{S}^3_{12} \to \mathbb{S}^2_4$ of Hopf invariant one and two respectively. In general we give, for each $n \geqslant 3$, a simplicial map $\xi : \mathbb{S}^3_{6n} \to \mathbb{S}^2_4$ (not necessarily minimal) of Hopf invariant n. We use the notation $\mathbb{S}^k_v$ to denote a v-vertex triangulation of the k-sphere.

Key words. Hopf invariant, Seifert fibrations, simplicial maps.

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Introduction. Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be any map; we may assume it simplicial relative to some triangulations of $\mathbb{S}^3$ and $\mathbb{S}^2$. Note that the pre-image, $f^{-1}(x)$, of an interior point x of any 2-simplex of $\mathbb{S}^2$ is a simple closed curve in $\mathbb{S}^3$ and by using orientations of $\mathbb{S}^2$ and $\mathbb{S}^3$ a natural orientation can be assigned to it. As we know any simple closed curve bounds in $\mathbb{S}^3$; so by subdividing $\mathbb{S}^3$ suitably, we can choose a 2-chain Σ^2 (say) bounded by $f^{-1}(x)$ and orient all its 2-simplices such that the induced orientation on $f^{-1}(x)$ coincides with its natural orientation. In this way we get a homomorphism $f_* : \mathrm{H}_2(\Sigma^2, f^{-1}(x)) \to \mathrm{H}_2(\mathbb{S}^2, x)$, defined as $f_*([\alpha]) = d \cdot [\beta]$ for some integer d. The number d is independent of the choice of the interior point x, it depends only on the homotopy class of the map f and is called *Hopf invariant* of the map f.

We know that any two maps from the 3-sphere to the 2-sphere are homotopic if and only if they have the same Hopf invariant. In this article we give, for each $n \in \mathbb{N}$, a simplicial map (from the 3-sphere to the 2-sphere) of Hopf invariant n and we observe that for $n = 1, 2$ these are *minimal simplicial maps.*

We say that a simplicial map $\xi : \mathbb{S}^3_{v(n)} \to \mathbb{S}^2_4$ is a minimal simplicial map of Hopf invariant n if there is no simplicial map $\mathbb{S}^3_{v(n)-1} \to \mathbb{S}^2_4$ of Hopf invariant n. It seems to us that it is a hard problem to find such minimal simplicial maps. However if we restrict ourselves to the category of some special maps, e.g. if we consider only those maps $\mathbb{S}^3 \to \mathbb{S}^2$ in which the inverse image of each point of $\mathbb{S}^2$ is homeomorphic to $\mathbb{S}^1$ then we hope to get minimal simplicial maps of any given Hopf invariant. Note that in each homotopy class of maps $\mathbb{S}^3 \to \mathbb{S}^2$, such a special map exists. So we shall give simplicial maps $\mathbb{S}^3_{v(n)} \to \mathbb{S}^2_4$ in this category.

Indeed we are going to triangulate a particular type of the *Seifert fibrations* [3] of the 3-sphere. We know that any Seifert fibration of $\mathbb{S}^3$ gives a quotient map $\mathbb{S}^3 \to \mathbb{S}^2$ (see [3]) under which a circle maps to a point and at most two points of $\mathbb{S}^2$ can be exceptional and all other points are ordinary. But in our constructions at most one point of $\mathbb{S}^2$ will be exceptional. We first recall definitions of ordinary and exceptional points of $\mathbb{S}^2$ for a given quotient map $\mathbb{S}^3 \to \mathbb{S}^2$ having only one exceptional point.

Ordinary points. Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be a map in which $f^{-1}(x)$ is homeomorphic to $\mathbb{S}^1$ for all points x of $\mathbb{S}^2$ and let $\mathbb{D}^2$ denote the unit 2-disk in $\mathbb{R}^2$ with centre at the origin. We say $x \in \mathbb{S}^2$ is an ordinary point or a regular point if there exists a neighbourhood $U_x \subseteq \mathbb{S}^2$ of x, and homeomorphisms

$$\begin{array}{ccc} f^{-1}(U_x) & \xrightarrow{h} & \mathbb{D}^2 \times \mathbb{S}^1 \\ \downarrow f & & \downarrow p \\ U_x & \xrightarrow{\tilde{h}} & \mathbb{D}^2 \end{array}$$

$h : f^{-1}(U_x) \to \mathbb{D}^2 \times \mathbb{S}^1$ and $\tilde{h} : U_x \to \mathbb{D}^2$ with $\tilde{h}(x) = 0$ for which the above diagram commutes.

Exceptional points. Let $\rho : \mathbb{D}^2 \to \mathbb{D}^2$ be a rotation map defined as $\rho(r \cdot e^{i\theta}) = r \cdot e^{i(\theta + 2\pi/n)}$ for some $n \geqslant 2$. Consider the quotient space, Γ_n, obtained from $\mathbb{D}^2 \times I$ by identifying the points $(x, 0)$ with $(\rho(x), 1)$ for all $x \in \mathbb{D}^2$. This identification gives a quotient map $q_n : \mathbb{D}^2 \times I \to \Gamma_n$ under which exactly n fibers, i.e. $x \times I$, $\rho(x) \times I$, $\rho^2(x) \times I, \ldots, \rho^{n-1}(x) \times I$ (here $x \neq 0$), of $\mathbb{D}^2 \times I$, together form a single circle of Γ_n and the fiber $0 \times I$ of $\mathbb{D}^2 \times I$ maps to the middle circle of Γ_n.

Let a map $\eta : \mathbb{D}^2 \to \mathbb{D}^2$ be defined as $\eta(z) = z^n$ for some $n \geqslant 2$. Then there is a unique map $g : \Gamma_n \to \mathbb{D}^2$ which makes the following diagram commutative.

$$\begin{array}{ccc} \mathbb{D}^2 \times I & \xrightarrow{q} & \Gamma_n \\ \downarrow p & & \downarrow g \\ \mathbb{D}^2 & \xrightarrow{\eta} & \mathbb{D}^2 \end{array}$$

We call a point x of $\mathbb{S}^2$ an *exceptional* or a *singular point* of multiplicity n, for a given map $f : \mathbb{S}^3 \to \mathbb{S}^2$, if for each neighbourhood U_x of x there are homeomorphisms $h : f^{-1}(U_x) \to \Gamma_n$ and $\tilde{h} : U_x \to \mathbb{D}^2$ with $\tilde{h}(x) = 0$ which make the following diagram commutative.

$$\begin{array}{ccc} f^{-1}(U_x) & \xrightarrow{h} & \Gamma_n \\ \downarrow f & & \downarrow g \\ U_x & \xrightarrow{\tilde{h}} & \mathbb{D}^2 \end{array}$$

A fiber corresponding to an exceptional point will be called an *exceptional fiber*.

Theorem. *There exist minimal simplicial maps $\eta, \xi : \mathbb{S}^3_{12} \to \mathbb{S}^2_4$ of Hopf invariants one and two respectively, and in general for each $n > 2$ there is a simplicial map $\xi : \mathbb{S}^3_{6n} \to \mathbb{S}^2_4$ of Hopf invariant n.*

Proof. A simplicial map $\eta : \mathbb{S}^3_{12} \to \mathbb{S}^2_4$ has been shown and defined by the vertex labelling in Figures 1 and 2.

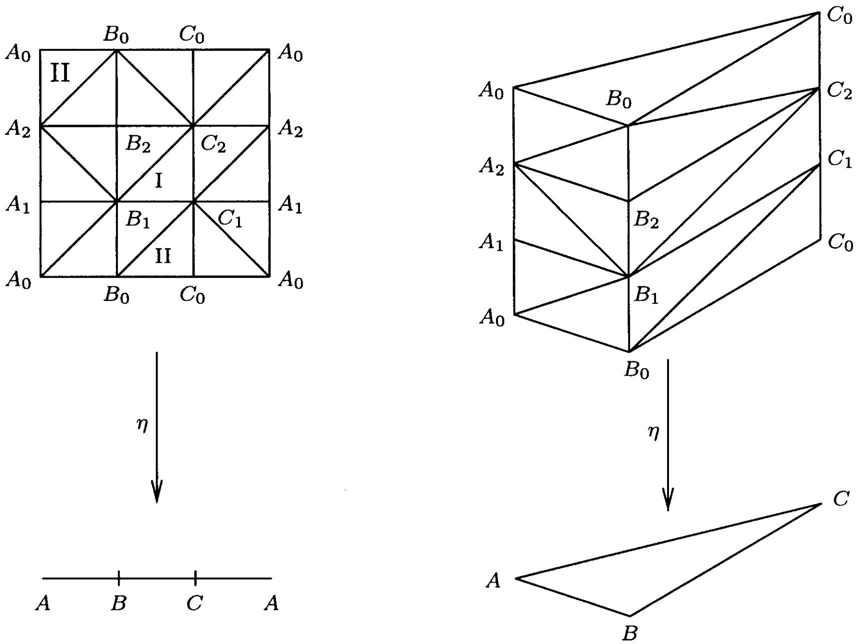

Figure 1. $T^2_9 = \partial M^3_9$ and M^3_9

We have proved in [2] that the map η is the minimal triangulation of the well-known Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$, so the map has Hopf invariant one. Now we shall give a minimal simplicial map $\xi : \mathbb{S}^3_{12} \to \mathbb{S}^2_4$ of Hopf invariant two.

Construction of a minimal simplicial map of Hopf invariant two. As we wish to make simplicial maps in which the pre-image of each point of $\mathbb{S}^2_4$ (it is the minimal triangulation of the 2-sphere) is homeomorphic to $\mathbb{S}^1$, so corresponding to four vertices of $\mathbb{S}^2_4$ there are four simplicial circles in $\mathbb{S}^3$ and each will have at least three vertices. So at least 12 vertices are needed in $\mathbb{S}^3$ to make a simplicial map of any non-zero Hopf invariant. *Moreover it is interesting to know that any simplicial map* $\mathbb{S}^3_v \to \mathbb{S}^2_4$, *with* $v \leqslant 11$, *is a homotopically trivial map* (see Theorem IIa of [1]).

The simplicial map $\xi : \mathbb{S}^3_{12} \to \mathbb{S}^2_4 = \partial[ABCD]$. Our simplicial complex $\mathbb{S}^3_{12}$ is the union of two solid tori. One of them, M^3_9, is the pre-image of a 2-simplex ABC. Its triangulated boundary has been shown in Figure 3 below. This solid torus consists of nine

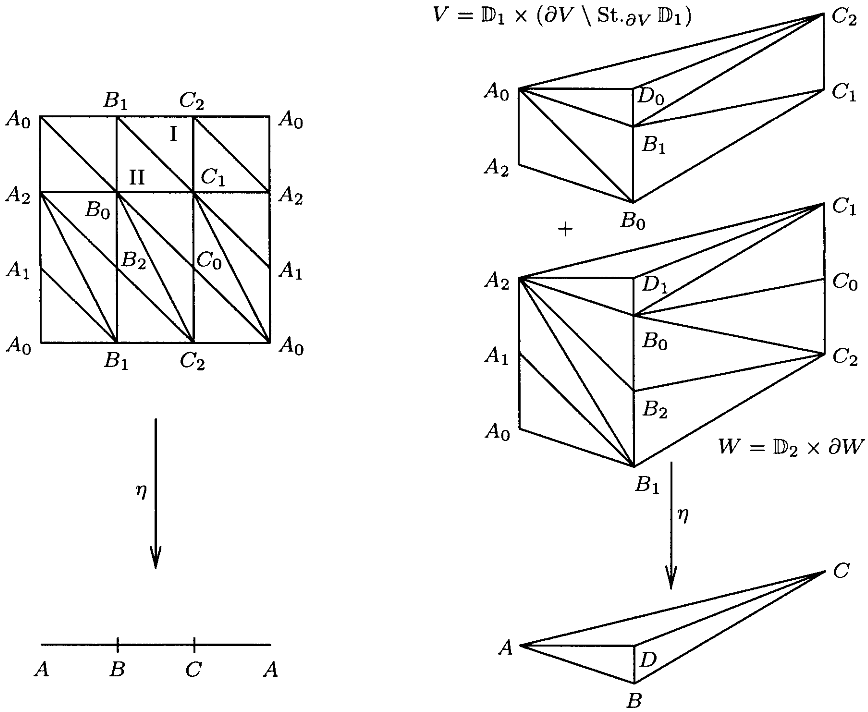

Figure 2. $T_9^2 = \partial N_{12}^3$ and N_{12}^3

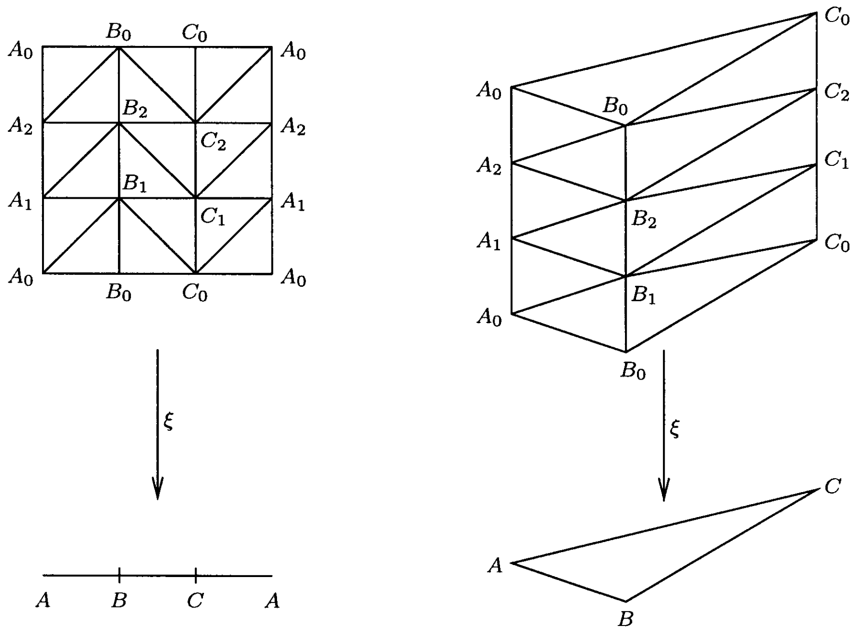

Figure 3. $T_9^2 = \partial M_9^3$ and M_9^3

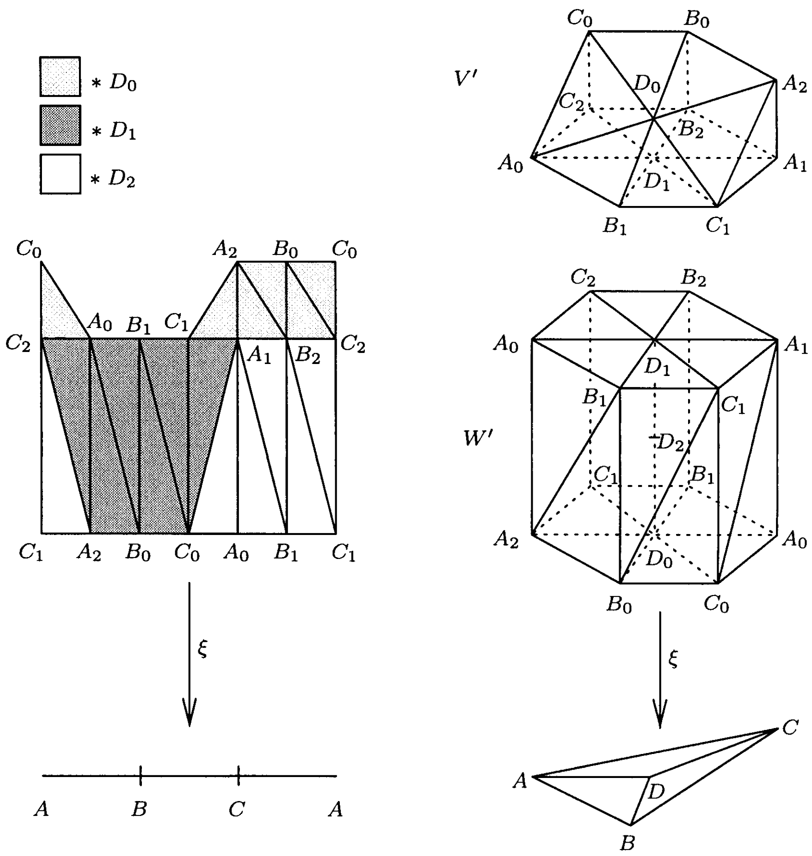

Figure 4. $T^2_9 = \partial N^3_{12}$ and N^3_{12}

3-simplices, three of them are $A_0B_0C_0B_1$, $A_0A_1B_1C_0$, $A_1B_1C_1C_0$ and the remaining six can be obtained from these by using the permutation $\alpha = (A_0A_1A_2)(B_0B_1B_2)\cdot(C_0C_1C_2)(D_0D_1D_2)$.

The second solid torus, N^3_{12}, is the pre-image of $\mathbb{S}^2_4\backslash\text{Int}.\,ABC$. Its triangulated boundary (shown in Figure 4 above) is isomorphic to, and will be identified with, the boundary of M^3_9. It has thirty-six 3-simplices, twelve of them are $A_0C_0C_2D_0$, $A_1A_2C_1D_0$, $A_1A_2B_2D_0$, $A_2B_0B_2D_0$, $B_0B_2C_2D_0$, $B_0C_0C_2D_0$, $A_0B_1D_0D_1$, $B_1C_1D_0D_1$, $A_1C_1D_0D_1$, $A_1B_2D_0D_1$, $B_2C_2D_0D_1$, $A_0C_2D_0D_1$ and the remaining twenty-four can be obtained from these by using the permutation α. Note that the solid torus N^3_{12} is homeomorphic to Γ_2 under a fiber preserving homeomorphism, its middle fiber $\xi^{-1}(D)$ is exceptional of multiplicity 2.

The simplicial map $\xi : \mathbb{S}^3_{12} \to \mathbb{S}^2_4$ given by $X_i \mapsto X$ for all $X \in \{A, B, C, D\}$ is well defined, as under this map simplices of $\mathbb{S}^3_{12}$ get mapped onto the simplices of $\mathbb{S}^2_4$.

Remarks. 1. The solid torus N^3_{12} contains pre-images of CAD, ABD and BCD. We

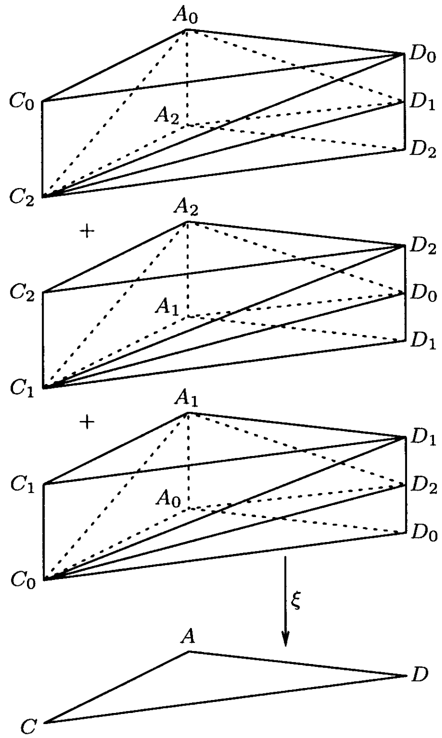

Figure 5

have given simplices of $\xi^{-1}(CAD)$, in Figure 5, explicitly and simplices of $\xi^{-1}(ABD)$ and $\xi^{-1}(BCD)$ can be obtained similarly. The simplicial complex $\xi^{-1}(CAD)$ consists of twelve 3-simplices, four of them are $A_0C_0D_0D_2$, $A_0A_1C_0D_2$, $A_1C_0D_1D_2$, $A_1C_0C_1D_1$ and the remaining eight can be obtained from these by using the permutation α.

2. From here it is very clear that pre-images of AD, BD and CD are Möbius strips bounded by $\xi^{-1}(A)$, $\xi^{-1}(B)$ and $\xi^{-1}(C)$ respectively. In each case the middle circle of the Möbius strip is $\xi^{-1}(D)$. Further note that pre-images of AB, BC and CA are cylinders.

3. Here vertices A, B, C are ordinary vertices while the vertex D is an exceptional vertex of multiplicity 2. So in order to verify the Hopf invariant of the map ξ, we choose a 2-chain, in $\mathbb{S}^3_{12}$, bounded by one of the pre-images of A, B or C and see the restriction of the map ξ to this 2-chain. In particular let us see in Figure 6 the restriction of ξ to a 2-chain bounded by the pre-image of the vertex A of $\mathbb{S}^2_4$.

It is clear from Figure 6 that the restricted map $\xi : (\Sigma^2, \xi^{-1}(x)) \to (\mathbb{S}^2, x)$ has degree ± 2, so the Hopf invariant of the map is ± 2 depending upon the choice of the orientations of $\mathbb{S}^3$ and $\mathbb{S}^2$.

Now we shall give for each $n \geqslant 3$, a simplicial map $\xi : \mathbb{S}^3_{6n} \to \mathbb{S}^2_4$ of Hopf invariant n but their minimality is yet to be verified.

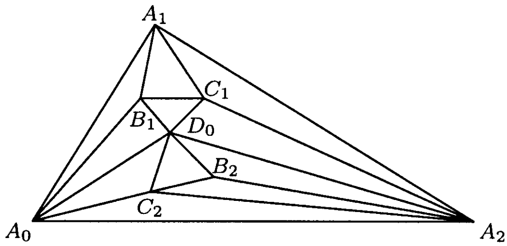

Figure 6

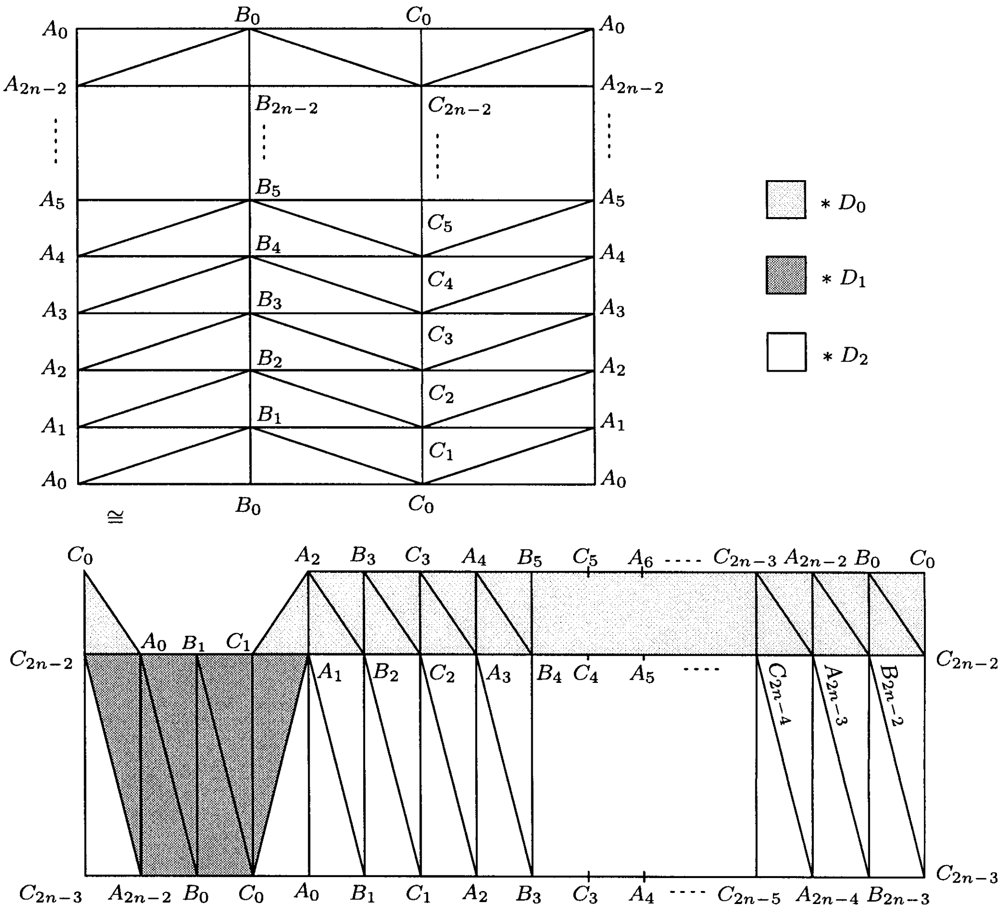

Figure 7

In order to make a simplicial map $\xi : \mathbb{S}^3_{6n} \to \mathbb{S}^2_4$ of Hopf invariant $n \geqslant 3$, we take a sphere $\mathbb{S}^3_{6n}$ which is the union of two solid tori (M and N say) whose common boundary is shown in Figure 7 above.

The solid torus M has been triangulated with $3(2n-1)$ 3-simplices. Three of them are $A_0B_0C_0B_1$, $A_1B_1C_1C_0$, $A_0A_1B_1C_0$ and rest of the 3-simplices can be obtained from these by using the permutation $\alpha = \prod_{X\in\{A,B,C,D\}}(X_0X_1\dots X_{2n-2})$.

The second solid torus has been triangulated with $3(2n-1)(3n-2)$ 3-simplices; $3(3n-2)$ of these are $C_iA_iA_{i+1}D_0$, $A_iA_{i+1}B_{i+1}D_0$, $A_{i+1}B_{i+1}B_{i+2}D_0$, $B_{i+1}B_{i+2}C_{i+1}D_0$, $B_{i+2}C_{i+1}C_{i+2}D_0$, $C_{i+1}C_{i+2}A_{i+2}D_0$, $A_0B_1D_0D_1$, $B_1C_1D_0D_1$, $A_1C_1D_0D_1$, $A_iB_{i+1}D_0D_1$, $B_{i+1}C_{i+1}D_0D_1$, $C_{i+1}A_{i+2}D_0D_1$ for each odd $i \in \mathbb{Z}/(2n-1)$ i.e. i lies in the set $\{1, 3, 5, \dots, 2n-3\}$ and the rest of the 3-simplices can be obtained from these by using the permutation α. It is easy to verify that the simplicial map $\xi : \mathbb{S}^3_{6n} \to \mathbb{S}^2_4$ defined as $X_i \to X$ for all $X \in \{A, B, C, D\}$ has Hopf invariant n.

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