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A classification of finite homogeneous semilinear spaces

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Abstract. A semilinear space S is *homogeneous* if, whenever the semilinear structures induced on two finite subsets S_1 and S_2 of S are isomorphic, there is at least one automorphism of S mapping S_1 onto S_2. We give a complete classification of all finite homogeneous semilinear spaces. Our theorem extends a result of Ronse on graphs and a result of Devillers and Doyen on linear spaces.

Key words. Semilinear space, polar space, copolar space, partial geometry, automorphism group, homogeneity.

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1 Introduction

A *semilinear space* (or partial linear space) S is a non-empty set of elements called *points*, provided with a collection of subsets called *lines* such that any pair of points is contained in at most one line and every line contains at least two points. Semilinear spaces are a common generalization of graphs (when all lines have exactly two points) and of linear spaces (when any pair of points is contained in exactly one line). A semilinear space which is neither a graph nor a linear space will be called *proper*.

If S' is a non-empty subset of S, the *semilinear structure induced on* S' is the semilinear space whose points are those of S' and whose lines are the intersections of S' with all the lines of S having at least two points in S'.

Given a positive integer d, a semilinear space S is said to be *d-homogeneous* if, whenever the semilinear structures induced on two subsets S_1 and S_2 of S of cardinality at most d are isomorphic, there is at least one automorphism of S mapping S_1 onto S_2; if every isomorphism from S_1 to S_2 can be extended to an automorphism of S, we shall say that S is *d-ultrahomogeneous*. S is called *homogeneous* (respectively *ultrahomogeneous*) if S is d-homogeneous (respectively d-ultrahomogeneous) for every positive integer d.

Gardiner [13], Sheehan [25] and Gol'fand–Klin [15] proved independently (1976)

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that a finite ultrahomogeneous undirected graph is either a disjoint union tK_n of t isomorphic complete graphs K_n or a regular complete multipartite graph $K_{t;n}$ or the 3×3 lattice graph $L_{3,3}$ on 9 vertices or the graph C_5 of the pentagon. Ronse [23] proved in 1978 that the list of finite homogeneous undirected graphs is exactly the same. The homogeneous and ultrahomogeneous linear spaces have also been classified by Devillers and Doyen [12] without any finiteness assumption. We have recently classified the finite ultrahomogeneous semilinear spaces [11]. Our goal now is to give a complete classification of finite homogeneous semilinear spaces.

By $U_{2,3}(n)$ we denote the semilinear space whose points are the 2-subsets of a nonempty set X of cardinality n and whose lines are the 3-subsets of X, the incidence being the natural inclusion of subsets.

The *triangular space* $T(n)$ is the semilinear space whose points are the 2-subsets of a set X of cardinality n and whose lines are the 1-subsets of X, the incidence being the reversed inclusion.

The *collinearity graph* of a semilinear space S is the graph whose vertices are the points of S and in which two vertices are adjacent if and only if the corresponding points are collinear (i.e. contained in some line). S is said to be *connected* if its collinearity graph is connected. The *connected components* of S are the connected components of its collinearity graph.

Our main result is the following classification of all finite connected 4-homogeneous semilinear spaces.

Theorem 1.1. (a) *Any finite connected* 6*-homogeneous semilinear space is homogeneous and is one of the following*:

- (i) *a graph* C_5, $L_{3,3}$, K_n *or* $K_{t;n}$ $(t, n \geqslant 2)$;
- (ii) *a single point or a single line*;
- (iii) *the projective planes* PG(2,2), PG(2,3) *or* PG(2,4) *or the affine plane* AG(2,3);
- (iv) *the* 3×3 *grid*, *i.e. the unique generalized quadrangle of order* (2,1) (*on* 9 *points*);
- (v) *the punctured* AG(2,3) (*obtained from* AG(2,3) *by removing a point and all lines through that point*), *or* AG(2,3) *with one parallel class of lines removed*;
- (vi) *the duals of* AG(2,3) *and* AG(2,4);
- (vii) $T(n)$ *for any integer* $n \geqslant 4$;
- (viii) $U_{2,3}(n)$ *for any integer* $n \geqslant 5$.

All these semilinear spaces are also ultrahomogeneous, *except* PG(2,4), AG(2,3), *the two examples under* (v) *obtained from* AG(2,3), *and the dual of* AG(2,4).

(b) *The only finite connected* 5*- but not* 6*-homogeneous semilinear spaces are the projective planes* PG(2,5) *and* PG(2,8).

(c) *The only finite connected* 4*- but not* 5*-homogeneous semilinear spaces are the projective plane* PG(2,32), *the unique generalized quadrangle of order* (2,4) (*on* 27 *points*), *the Schläfli graph on* 27 *vertices and its complement*.

Note that the Schläfli graph is precisely the collinearity graph of the generalized quadrangle of order $(2, 4)$.

We can extend our classification to non-connected semilinear spaces, due to the following proposition.

Proposition 1.2. (a) *If $d \geqslant 2$ and if S is a d-homogeneous semilinear space which is not connected, then the connected components of S are isomorphic d-homogeneous linear spaces.*

(b) *If S is a* 6*-homogeneous semilinear space which is not connected, then S is homogeneous and the connected components of S are isomorphic homogeneous linear spaces.*

This proposition can be proved as follows: for (a), the arguments in the proof of Theorem 2.0.1 of [11] show that the connected components of S are pairwise isomorphic linear spaces. These connected components are d-homogeneous, because they are blocks (sets of imprimitivity) for the automorphism group of S. For (b) we use (a) and recall from [12] that any 6-homogeneous linear space is homogeneous, hence the connected components of S are isomorphic homogeneous linear spaces.

It remains to prove Theorem 1.1. If S is a 6-homogeneous linear space (finite or infinite), then S is one of the following (see [12]): a single point, a single line, a complete graph, $PG(2,2)$, $PG(2,3)$, $PG(2,4)$ or $AG(2,3)$. This yields (ii) and (iii) in Theorem 1.1. If S is a finite linear space which is 5- but not 6-homogeneous, then S is $PG(2,5)$ or $PG(2,8)$, and if S is 4- but not 5-homogeneous, then S is $PG(2,32)$ (see [10]).

The case where S is a graph (i.e. where all lines have size 2) is treated in Section 2.

It remains then to classify the finite connected 4-homogeneous proper semilinear spaces S. In order to do this, we study the antiflags (p, L), where p is a point of S and L is a line not containing p. The *collinearity index* of an antiflag (p, L) is the number of points of L which are collinear with p; the *non-collinearity index* is the number of points of L which are not collinear with p.

In Section 3 we classify all finite connected 4-homogeneous proper semilinear spaces S where the semilinear structures induced on the antiflags of S are all isomorphic; this leads to the Cases (iv), (vi), (vii) and the second space of (v) in Theorem 1.1. Note that our proof of 1.1 relies at two points (in Section 2 and in the proof of Proposition 3.1) on the classification of finite simple groups. In Sections 4–7, we classify the remaining finite connected 4-homogeneous proper semilinear spaces S (those with different semilinear structures induced on the antiflags). This will complete the proof of Theorem 1.1.

As usual, the *degree* of a point p is the number of lines through p, and the *neighbourhood* of p is the set of all points which are collinear with p and distinct from p.

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2 Graphs

First we consider the semilinear spaces all of whose lines have size exactly 2, i.e. the graphs. A graph S is called *m-regular* if, for any set M of at most m vertices, the number of vertices of S adjacent to every vertex in M depends only on the isomorphism type of the subgraph induced on M. Obviously, any m-homogeneous graph is m-regular. By the note added in [4], if S is a finite connected 5-homogeneous graph, then S is isomorphic to C_5, $L_{3,3}$, K_n or $K_{t;n}$ $(t, n \geqslant 2)$.

Buczak [2] proved that any finite connected 4-regular graph is in this list or is an extremal Smith graph or its complement. An *extremal Smith graph* $B_3(r)$ (see [6]) is a strongly regular graph with parameters

$$v = (2r^2 + 2r - 1)(2r + 1)^2, \quad k = 2r^3(2r + 3),$$
$$\lambda = r(2r - 1)(r^2 + r - 1), \quad \mu = r^3(2r + 3),$$

where r is a nonnegative integer; the value of the parameter $l = 2(r + 1)^3(2r - 1)$ follows easily. Such a graph $B_3(r)$ has the property that its subconstituents and the subconstituents of its subconstituents are also strongly regular (and their parameters are known, see [2] page 33). Note that $k = 2\mu$ both in $B_3(r)$ and in its complement.

$B_3(1)$ is the Schläfli graph, which is 4-homogeneous (and even 4-ultrahomogeneous); it is the unique strongly regular graph with parameters $(27, 10, 1, 5)$. $B_3(2)$ is the McLaughlin graph, the unique strongly regular graph with parameters $(275, 112, 30, 56)$) (see [14]); a computer check shows that its automorphism group is not transitive on the set of cocliques of size 4, hence $B_3(2)$ is not 4-homogeneous. For $r > 2$, the existence of a graph $B_3(r)$ is an unsolved problem.

Suppose that there exists a 4-homogeneous graph S which is a $B_3(r)$ or the complement of a $B_3(r)$, with $r \geqslant 3$. Let X be the graph consisting of 4 vertices and 2 edges sharing a common vertex, and let $\overline{X}$ be the graph complement of X. The graph X contains a non-edge whose vertices have degree 0 and 2 respectively. Let a and b be two non-adjacent vertices of S. Using the parameters of the subconstituents of S, as well as their subconstituents, it is easy to show that a and b are contained in a subgraph isomorphic to X in such a way that a has degree 0 and b has degree 2 in X. Since S is 4-homogeneous, it follows that $\mathrm{Aut}(S)$ is transitive on the ordered pairs of non-adjacent vertices. A similar argument using $\overline{X}$ shows that $\mathrm{Aut}(S)$ is transitive on the ordered pairs of adjacent vertices. Therefore, S must be a rank 3 graph, and so its automorphism group G must be a finite rank 3 permutation group. It is easily seen that a connected 2-homogeneous graph with an imprimitive rank 3 group must be a complete multipartite regular graph, which is not a $B_3(r)$. Hence G is a finite primitive rank 3 group. These groups have been classified (as a corollary of the classification of finite simple groups). They can be found for example in Buekenhout–Van Maldeghem [3], together with (in most cases) the parameters of the associated rank 3 graphs (the missing parameters are given in Hubaut [18]). They fall into three cases: the grid case, the affine case and the almost simple case.

In the grid case, the number v of vertices must be a square, and so $2r^2 + 2r - 1 = u^2$ for some integer u. But $2r^2 + 2r - 1 = 2r(r + 1) - 1 \equiv 3 \bmod 4$, which is never a square.

In the affine case, the number of vertices $v = (2r^2 + 2r - 1)(2r + 1)^2$ must be a prime power p^e. Clearly, $d = \gcd(2r^2 + 2r - 1, 2r + 1)$ is equal to 1 or 3. If $d = 1$, then $(2r^2 + 2r - 1)(2r + 1)^2$ cannot be a prime power. Hence $d = 3$, and so $p = 3$. This means that both $2r^2 + 2r - 1$ and $2r + 1$ are powers of 3, and so $2r + 1$ must be equal to 3, otherwise d would be at least 9. Therefore $r = 1$, contradicting the fact that $r \geqslant 3$.

Finally, consider the almost simple case. Using the fact that v must be odd and that $k = 2\mu$, the rank 3 representations of classical, exceptional and sporadic groups are easily ruled out: it turns out that the only possible graph $B_3(r)$ in this case is $B_3(2)$, the McLaughlin graph. Using the fact that v is odd, that $\mu = 4$ is impossible and that $v = 35$ is also impossible, the rank 3 representations of the alternating groups are also ruled out without any difficulty. Using again the fact that v is odd, that $k = 2\mu$, that k and l cannot be powers of 2 for $r \geqslant 3$, together with a few simple divisibility arguments, the rank 3 representations of the infinite families of Chevalley groups are also ruled out: the only surviving parameters are those of $B_3(1)$, the Schläfli graph.

In conclusion, there is no $B_3(r)$ which is a rank 3 graph for $r \geqslant 3$, and so the only finite 4- but not 5-homogeneous finite graphs are the Schläfli graph and its complement.

3 Partial geometries

Let S be a finite connected 4-homogeneous proper semilinear space where the semilinear structures induced on the antiflags of S are all isomorphic. Then S is a *partial geometry* with parameters s, t, α, β, i.e. the following conditions are satisfied:

(i) each line is incident with $s + 1$ points ($s \geqslant 1$),

(ii) each point is incident with $t + 1$ lines ($t \geqslant 1$),

(iii) each antiflag has collinearity index α ($\alpha \geqslant 1$) and non-collinearity index β ($\beta \geqslant 1$), where $\alpha + \beta = s + 1$.

Since we exclude graphs, we have $s \geqslant 2$. Note that the case $\alpha = 1$ (i.e. generalized quadrangles) was already dealt with in [11] (section on polar spaces), where we proved that the only 4-homogeneous proper polar spaces are the 3×3 grid, which is the unique generalized quadrangle of order $(2, 1)$ and which is ultrahomogeneous, and the unique generalized quadrangle of order $(2, 4)$, which is 4-ultrahomogeneous but not 5-homogeneous. This yields Case (iv) and part of (c) in Theorem 1.1. The remaining cases $\alpha, \beta \geqslant 2$ and $\beta = 1$ are covered by the following results 3.2, 3.3, 3.4.

Proposition 3.1. *If S is a finite 4-homogeneous partial geometry with $\alpha \geqslant 2$ and $\beta \geqslant 2$, then $\alpha = 2$ and $t = 1$.*

Proof. We claim that the automorphism group of S is transitive on the ordered pairs of collinear points and on the ordered pairs of non-collinear points.

Let x_1 and x_2 be two collinear points of S, and let L be a line of S containing x_1 but not x_2. Since $\beta \geqslant 2$, L contains two points y and z non-collinear with x_2. Because of the 4-homogeneity of S, the automorphism group of S is transitive on the 4-subsets

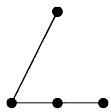

Figure 1

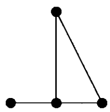

Figure 2

inducing the same semilinear structure as $\{x_1, x_2, y, z\}$, i.e. the semilinear space of Figure 1. Since this semilinear space contains a unique point x_1 of degree 2 and also a unique point x_2 whose neighbourhood has size 1, the automorphism group of S is transitive on the ordered pairs (x_1, x_2) of collinear points.

Let y_1 and y_2 be two non-collinear points of S, and let L be a line of S through y_1. Since $\alpha \geqslant 2$, L contains two points u and v collinear with y_2. Because of the 4-homogeneity of S, the automorphism group of S is transitive on the 4-subsets inducing the same semilinear structure as $\{y_1, y_2, u, v\}$, i.e. the semilinear space of Figure 2. Since this semilinear space contains a unique point y_1 of degree 1 and also a unique point y_2 which is not on the unique line of size 3, the automorphism group of S is transitive on the ordered pairs (y_1, y_2) of non-collinear points.

The dual S^* of S, whose points are the lines of S and whose lines are the points of S, with the same incidence as in S, is a linear space, because each point of S is incident with at least two lines, and any two lines of S meet according to [11] Proposition 2.3.1. We have just shown that the automorphism group of S^* is transitive on the ordered pairs of intersecting lines and on the ordered pairs of disjoint lines.

So far, we have not yet used the finiteness of S. Delandtsheer [8] proved that a finite linear space with the transitivity properties obtained above for S^* is isomorphic to one of the following: (i) a single line; (ii) a Desarguesian affine plane $\mathrm{AG}(2, q)$; (iii) a Desarguesian projective space $\mathrm{PG}(d, q)$ with $d \geqslant 2$; (iv) a linear space all of whose lines have size 2. Note that the proof given in [8] relies on the classification of finite simple groups.

Since the dual of a single line is not a semilinear space, Case (i) can be ruled out. Since S contains a 3-subset inducing a semilinear space consisting of 3 points and one line of size 2 (because $\beta \geqslant 2$), S^* contains two intersecting lines both of which are disjoint from a third one, and so S^* cannot be an affine plane. Since S contains a pair of non-collinear points, S^* contains a pair of disjoint lines, and so S^* cannot be a projective plane. The automorphism group of S is transitive on the subsets consisting of 3 collinear points (because S is 4-homogeneous, and so in particular 3-homogeneous), therefore the automorphism group of S^* is transitive on the sets consisting of 3 intersecting lines, which is obviously not the case if $S^* = \mathrm{PG}(d, q)$ with $d \geqslant 3$.

We conclude that S^* is a linear space with lines of size 2. Hence $t = 1$, and, since S contains no pair of disjoint lines, it follows that $\alpha = 2$. □

The following theorem yields Case (vii) in Theorem 1.1.

Theorem 3.2. *The finite* 4*-homogeneous partial geometries* S *with* $\alpha \geqslant 2$ *and* $\beta \geqslant 2$ *are exactly the triangular spaces. Every triangular space* (*finite or infinite*) *is ultrahomogeneous.*

Proof. By 3.1 we have $\alpha = 2$ and $t = 1$, hence S contains no pair of disjoint lines. The assertions follow from [11] Lemma 2.3.2 and Proposition 2.3.4. □

A *transversal design* $TD(m,n)$ is a semilinear space with point set $X \times Y$ (where X and Y are sets of cardinality m and n respectively) such that (i) each line of $TD(m,n)$ meets every set $\{x\} \times Y$ with $x \in X$, and (ii) two points (x_1, y_1) and (x_2, y_2) of $TD(m,n)$ are joined by a line if and only if $x_1 \neq x_2$. We will call equivalence classes of the $TD(m,n)$ the sets $\{x\} \times Y$ where $x \in X$.

Lemma 3.3. *Any* 1*-homogeneous partial geometry* S *with* $\beta = 1$ *is a transversal design* $TD(m,n)$.

Proof. The relation "is non-collinear with" (defined on the point set of S) is an equivalence relation. Indeed this relation is obviously reflexive and symmetric; it is also transitive, otherwise there would exist three points a, b, c of S such that a is non-collinear with b, b is non-collinear with c and a is collinear with c, which would force the antiflag (b, ac) to have non-collinearity index $\beta \geqslant 2$, a contradiction.

Let X be the set of equivalence classes of this relation. By the transitivity of the automorphism group on points, all the equivalence classes have the same cardinality. Let Y be any set having this cardinality. We may identify the point set of S with $X \times Y$. It remains to check that S, identified with $X \times Y$, satisfies properties (i) and (ii) of a transversal design.

Let L be a line of S and let $x \in X$. Since any two points of x are not collinear, L meets x in at most one point. Suppose that L does not meet x; then any point p in x would be collinear with all the points of L, contradicting the fact that S has non-collinearity index $\beta = 1$. Therefore L meets x in exactly one point, and so S satisfies (i). Two points of S are collinear if and only if they lie in different equivalence classes, and so S satisfies (ii). We conclude that S is a transversal design $TD(m,n)$. □

Note that, in order for a $TD(m,n)$ to be proper, we have to require $n \geqslant 2$ (otherwise $t < 1$) and $m \geqslant 3$ (otherwise $s < 2$). The following theorem yields Case (vi) and parts of Cases (v) and (vii) in 1.1.

Theorem 3.4. *The only* 5*-homogeneous proper transversal designs* $TD(m,n)$ *are the following*:

$TD(3,2)$, *which is isomorphic to* $T(4)$ *and to* $U_{2,3}(4)$ *and to the dual of* AG$(2,2)$,
$TD(3,3)$, *which is* AG$(2,3)$ *with one parallel class of lines removed*,
$TD(4,3)$, *which is the dual of* AG$(2,3)$, *and*
$TD(5,4)$, *which is the dual of* AG$(2,4)$.

All these transversal designs are homogeneous and uniquely determined by their parameters. There is no finite 4*-homogeneous but not* 5*-homogeneous transversal design.*

Proof. Suppose first that $\boldsymbol{m = 3}$. Since $TD(3,2)$ is isomorphic to the triangular space $T(4)$, it is ultrahomogeneous. On the other hand, $TD(3,3)$ is isomorphic to AG$(2,3)$ from which one class of parallel lines has been removed and it is homogeneous.

Now let S be a 4-homogeneous $TD(3, n)$ with $n \geqslant 4$, and let $X \times Y$ be the point set of S, where $X = \{x_0, x_1, x_2\}$. S contains two distinct lines L and L' intersecting in (x_0, y_0). Then by condition (i) in the definition of a transversal design, L has two points $a = (x_1, y_1)$, $b = (x_2, y_2)$, and L' has two points $c = (x_1, y_1')$, $d = (x_2, y_2')$; the third point of ad (resp. bc) is (x_0, w_1) (resp. (x_0, w_2)), where $w_1 \neq y_0 \neq w_2$.

Since $n \geqslant 4$, there is an element $u \in Y \backslash \{y_0, w_1, w_2\}$. Let $e = (x_0, u)$ and let f be the third point of the line be. The point f is distinct from (and non-collinear with) a and c. The semilinear structures induced on $\{a, b, d, f\}$ and $\{a, b, c, d\}$ are isomorphic. By the 4-homogeneity of S, we deduce that the lines ab and df must intersect in S. Hence df must contain the third point of ab, namely (x_0, y_0). This is a contradiction because there are two lines through (x_0, y_0) and d, namely cd and df. This proves that there is no 4-homogeneous transversal design $TD(3, n)$ with $n \geqslant 4$.

Suppose now that $\boldsymbol{m \geqslant 4}$. We claim that if S is 5-homogeneous or 4-homogeneous and finite, then S is a dual affine plane.

Let $X \times Y$ be the point set of a transversal design $S = TD(m, n)$ with $m \geqslant 4$. Since $n \geqslant 2$, S contains two distinct lines L and L' intersecting in (x_0, y_0). Let $x_1, x_2, x_3 \in X \backslash \{x_0\}$. Then by condition (i) L has three points $a = (x_1, y_1)$, $b = (x_2, y_2)$ and $c = (x_3, y_3)$, and L' has two points $d = (x_1, y_1')$ and $e = (x_2, y_2')$.

Suppose by way of contradiction that S contains two disjoint lines M and M'. By condition (i), M contains three points $a' = (x_1, z_1)$, $b' = (x_2, z_2)$ and $c' = (x_3, z_3)$, and M' contains two points $d' = (x_1, z_1')$ and $e' = (x_2, z_2')$. The semilinear structures induced on $\{a, b, c, d, e\}$ and $\{a', b', c', d', e'\}$ are isomorphic. If there is an automorphism α of S mapping the first set onto the second one, then α maps necessarily $\{a, b, c\}$ onto $\{a', b', c'\}$, and so the pair $\{d, e\}$ is mapped onto $\{d', e'\}$, which implies that α maps the lines L and L' onto the lines M and M', a contradiction. Hence α does not exist, contradicting the 5-homogeneity of S. It follows that there is no pair of disjoint lines in S.

On the other hand, suppose that $m \geqslant 4$ is finite and that S is 4-homogeneous. Consider the points a', d' and e' on the disjoint lines M and M' as above, and let N be the line $a'e'$. The point d' is collinear with all the points of M, except a'. Among the $m - 1$ lines through d' meeting M, at most $m - 2$ meet N (because none of these lines meets N in a' or e'). Hence there is a line N' through d' meeting M in f' and disjoint from N. Assume that f' is not collinear with e'. Then the semilinear structures induced on the sets $\{a, b, d, e\}$ and $\{a', d', e', f'\}$ are isomorphic, but there is no automorphism mapping the first set onto the second one, otherwise this automorphism would map a pair of intersecting lines onto a pair of disjoint lines. This contradicts the 4-homogeneity of S, and so f' is collinear with e'. Therefore the semilinear structures induced on the sets $\{a, c, d, e\}$ and $\{a', d', e', f'\}$ are isomorphic, but there is no automorphism mapping the first set onto the second one. This contradicts again the 4-homogeneity of S, and so we have proved that there is no pair of disjoint lines in S.

In both cases, we have proved that S is a dual affine plane, hence $m = n + 1$.

Suppose that S is a 5-homogeneous $TD(n + 1, n)$ with $n \geqslant 5$ (n may be infinite) and let $X \times Y$ be the point set of S (with $|X| \geqslant 6$ and $|Y| \geqslant 5$). Let $a = (x_1, y_1)$, $b = (x_1, y_2)$, $c = (x_2, y_1)$, $d = (x_2, y_2)$, where x_1, x_2 are two distinct elements of X

and y_1, y_2 are two distinct elements of Y. The points a and b are non-collinear, as well as c and d. The lines ac and bd meet in $e \in \{x_3\} \times Y$ (where $x_3 \in X$ and $x_3 \neq x_1, x_2$) and the lines ad and bc meet in $f \in \{x_4\} \times Y$ (where $x_4 \in X$ is different from x_1 and x_2 but might be equal to x_3). If $x_3 = x_4$, let g be a point distinct from e and f in $\{x_3\} \times Y$, and if $x_3 \neq x_4$, let g be a point of $\{x_3\} \times Y$ distinct from e and not on the lines ad and bc (such a point exists since $|Y| \geqslant 5$). In both cases, let $A = \{a, b, c, d, g\}$ and let $x_5 \in X \backslash \{x_1, x_2, x_3, x_4\}$. The four lines ac, ad, bc, bd intersect $\{x_5\} \times Y$ in four distinct points. Since $|Y| \geqslant 5$, there is a point h in $\{x_5\} \times Y$ distinct from these four points. Let $B = \{a, b, c, d, h\}$. The semilinear structures induced on A and B are isomorphic, and any automorphism of S mapping A onto B must map g onto h and leave invariant the set consisting of the four lines ac, ad, bc, bd. This contradicts the 5-homogeneity of S, because g is non-collinear with one of the points (namely e) lying on two of the four lines ac, ad, bc, bd, but this is not the case for h.

Therefore, a 5-homogeneous transversal design $TD(m, n)$ with $m \geqslant 4$ must be a $TD(4, 3)$ or a $TD(5, 4)$. Each of them is unique up to isomorphism: indeed, they are both obtained by deleting one point (and all the lines through this point) from a projective plane whose lines have size 4 or 5; moreover, $\mathrm{PG}(2, 3)$ and $\mathrm{PG}(2, 4)$ are unique up to isomorphism and have an automorphism group acting transitively on points. It is easily checked by computer that $TD(4, 3)$ is ultrahomogeneous and that $TD(5, 4)$ is homogeneous (but not ultrahomogeneous, as shown in [11]).

Suppose now that S is a finite 4-homogeneous transversal design. We already know that S is a dual affine plane $TD(n+1, n)$. If we add to S one new point ∞, and $n+1$ new lines consisting of the union of the point ∞ with each equivalence class of S, we obtain a non-trivial linear space with no pair of disjoint lines, i.e. a projective plane P. Hence S is a punctured projective plane of order n. If $n = 2, 3, 4$, we know that $TD(n+1, n)$ is homogeneous; so we assume $n \geqslant 5$ and aim for a contradiction.

$\mathrm{Aut}(S)$ is a collineation group of the dual projective plane P^* of P which acts transitively on the lines of P^* which are distinct from ∞ (by the 1-homogeneity of S). By a theorem of Wagner [28] (see also [9], p. 214), P^* is a translation plane with translation line ∞. Furthermore $\mathrm{Aut}(S)$ is also doubly transitive on the equivalence classes of the transversal design S (use the 3-homogeneity and consider a 3-subset consisting of one point in the first equivalence class and two points in the second). Therefore $\mathrm{Aut}(S)$ is doubly transitive on the points of the line ∞ of P^*. By results of Czerwinski [7], Schulz [24] and Kallaher ([19], Theorem (16) page 181), a finite translation plane with this property is either Desarguesian or a Lüneburg plane. In addition, the structure induced on any set of 4 points in one equivalence class of S is always the same (namely 4 points with no line), hence the stabilizer in $\mathrm{Aut}(S)$ of an equivalence class (i.e. the stabilizer of a point on the line ∞ of P^*) is 4-homogeneous on that equivalence class. We show that this leads to a contradiction in both cases.

If P^* is Desarguesian, then $n = p^e$ is a prime power, and the stabilizer of an equivalence class of $TD(n+1, n)$ is isomorphic to the group $A\Gamma L(1, n)$ of order $n(n-1)e$. This group can be 4-homogeneous on an equivalence class only if $\binom{n}{4}$ divides $n(n-1)e$, i.e. if $(p^e - 2)(p^e - 3)$ divides $24e$, and it is easy to see that this holds only for $n \leqslant 5$ (note that the inequality $(p^e - 2)(p^e - 3) \leqslant 24e$ already implies $n = p^e \leqslant 9$). Therefore, we are left with $TD(6, 5)$. A computer check shows that the automorphism group of $TD(6,5)$ is not transitive on the set of 4-subsets whose induced semilinear structure consists of 4 points and 5 lines of size 2. Indeed, this semilinear structure contains two pairs of disjoint lines. Since $TD(6,5)$ is a dual affine plane, the lines inducing each of these two pairs intersect in $TD(6,5)$. These two points of intersection may or may not belong to the same equivalence class of $TD(6,5)$. Therefore $TD(6,5)$ is not 4-homogeneous either (note that we do not have such a problem with $TD(5,4)$ because in $\mathrm{PG}(2,4)$ the three diagonal points of a quadrangle are always collinear, and so the two "diagonal" points of the induced semilinear structure described above cannot belong to the same equivalence class).

The *Lüneburg planes* $\mathscr{L}(q)$ [21, 22] are translation planes of order q^2 (where $q = 2^{2m+1} \geqslant 8$) associated with the Suzuki groups $\mathrm{Sz}(q)$. The automorphism group of $\mathscr{L}(q)$ is $T.G_0$, where T is the subgroup of translations (of order q^4) and G_0, the stabilizer of an affine point, is isomorphic to $\mathrm{Aut}(\mathrm{Sz}(q))$. Hence, by Suzuki [26], the order of the automorphism group of $\mathscr{L}(q)$ is

$$(q^2)^2(q^2+1)q^2(q-1)(2m+1) = q^6(q^2+1)(q-1)(2m+1).$$

Suppose that P^* is a Lüneburg plane $\mathscr{L}(q)$. The stabilizer of a point of P^* on ∞ has order $q^6(q-1)(2m+1)$. By 4-homogeneity, $\binom{q^2}{4}$ divides $q^6(q-1)(2m+1)$, i.e. $(q+1)(q^2-2)(q^2-3)$ divides $24q^4(2m+1)$. Since $q^2-3 = 4^{2m+1}-3$ is coprime to 24, we obtain that $4^{2m+1}-3$ divides $2m+1 \geqslant 3$, which is absurd.

This contradiction proves that there is no finite 4- but not 5-homogeneous transversal design. □

4 Types of antiflags

In this section, we prepare the treatment of semilinear spaces with different semilinear structures induced on the antiflags.

Proposition 4.1. (a) *If S is a connected 4-homogeneous proper semilinear space, then all the antiflags of S with collinearity index at least 3 are isomorphic.*

(b) *If S is a connected 3-homogeneous proper semilinear space, then all the antiflags of S with non-collinearity index at least 2 are isomorphic.*

Proof. (a) Suppose that S contains two non-isomorphic antiflags (p,L) and (p',L') with collinearity index at least 3. Let a,b,c be three points of L collinear with p, and a',b',c' three points of L' collinear with p'. The semilinear structures induced on $A = \{a,b,c,p\}$ and $A' = \{a',b',c',p'\}$ are isomorphic, and any automorphism of S mapping A onto A' must clearly map p onto p' and the three points a, b, c onto a', b', c', and so the line L onto L', contradicting the 4-homogeneity of S and our assumptions on (p,L) and (p',L'). This proves that all the antiflags of S with collinearity index at least 3 are isomorphic.

(b) follows with the same argument as (a) with $A = \{a,b,p\}$ and $A' = \{a',b',p'\}$, where a,b (resp. a',b') are two points of L (resp. L') non-collinear with p (resp. p'). □

If S is a connected 4-homogeneous proper semilinear space with at least two non-isomorphic antiflags, it follows from Proposition 4.1 that

(i) if the lines of S have size 3, then S may have antiflags whose collinearity indices are all in the set $\{0, 2, 3\}$ or in $\{1, 2, 3\}$,

(ii) if the lines of S have size at least 4, then S has at most two non-isomorphic types of antiflags: one with collinearity index 0, 1 or 2 and one with non-collinearity index 0 or 1 (altogether, this yields 6 different cases).

Proposition 4.2. *Let S be a proper connected semilinear space having exactly two isomorphism types of antiflags. The following situations cannot occur*:

(i) *S has an antiflag with collinearity index* 0 *and an antiflag with non-collinearity index* 0;

(ii) *S is* 3-*homogeneous and has an antiflag with collinearity index* 1 *and an antiflag with non-collinearity index* 1;

(iii) *S is* 4-*homogeneous and has an antiflag with collinearity index* 1 *and an antiflag with non-collinearity index* 0;

(iv) *S is* 4-*homogeneous*, *with all lines of size at least* 4, *and has an antiflag with collinearity index* 2 *and an antiflag with non-collinearity index* 1.

Proof. (i) Suppose that S has all its antiflags with collinearity index 0 or with non-collinearity index 0. Let (p, L) be an antiflag with collinearity index 0. Since S is connected, there is a minimal path $p_0, p_1, p_2, \dots, p_d = p$ where $p_0 \in L$ and p_i is collinear with p_{i+1} for every $i = 0, 1, \dots, d-1$. The point p_0 is non-collinear with p_2 (because the path is minimal) and collinear with p_1, and so the antiflag (p_0, p_1p_2) has collinearity index at least 1 and non-collinearity index at least 1, contradicting our assumptions on S.

(ii) Suppose that S contains an antiflag (p, L) with collinearity index 1 and an antiflag (p', L') with non-collinearity index 1. There exists a point a of L (resp. a' of L') non-collinear with p (resp. p') and a point o of L (resp. o' of L') collinear with p (resp. p'). The semilinear structures induced on $A = \{o, a, p\}$ and $A' = \{o', a', p'\}$ are isomorphic. Since the antiflags (p, L) and (p', L') are not isomorphic, any automorphism of S mapping A onto A' cannot map p onto p', and so must map a onto p'. Hence the antiflag (a, op) has non-collinearity index 1. Let b be a point of the line op distinct from o and p. Since b is collinear with o and a, the antiflag (b, L) has non-collinearity index 1, and so b is non-collinear with a unique point of $L = oa$, say c. Since c is non-collinear with b and p, the antiflag (c, op) has collinearity index 1. The semilinear structures induced on the sets $C = \{o, c, p\}$ and A' are isomorphic. By the 3-homogeneity of S, one of the antiflags (p, L) or (c, op) must be mapped onto (p', L') by some automorphism of S. This is impossible because both (p, L) and (c, op) have collinearity index 1, while (p', L') has non-collinearity index 1.

(iii) Suppose that S has all its antiflags with collinearity index 1 or with non-collinearity index 0. Then S is a polar space. We have proved in [11] that all the antiflags of a 4-homogeneous proper polar space have collinearity index 1, contradicting the fact that S contains an antiflag with non-collinearity index 0.

(iv) Note first that this case obviously makes no sense if the lines of S have size 3. Suppose that S contains an antiflag (p, L) with collinearity index 2 and an antiflag (p', L') with non-collinearity index 1. Since the lines of S have size at least 4, these two antiflags are non-isomorphic. There exist two points a, b of L (resp. a', b' of L') collinear with p (resp. p') and a point c (resp. c') non-collinear with p (resp. p'). The semilinear structures induced on $A = \{a, b, c, p\}$ and $A' = \{a', b', c', p'\}$ are isomorphic, and any automorphism of S mapping A onto A' must clearly map p onto p' and L onto L'. By the 4-homogeneity of S, this is a contradiction since (p, L) and (p', L') are non-isomorphic antiflags. □

Corollary 4.3. *If S is a 4-homogeneous proper connected semilinear space having exactly two isomorphism types of antiflags, then either all the antiflags of S have collinearity index* 2 *or non-collinearity index* 0, *or all the antiflags of S have collinearity index* 0 *or non-collinearity index* 1.

The two cases arising in this corollary will be examined in Sections 5 and 6, and the remaining case, where S has three isomorphism types of antiflags (and hence all lines of S have size 3), will be considered in Section 7.

5 Copolar spaces

A semilinear space whose antiflags have either collinearity index 0 or non-collinearity index 1 is called a *copolar space* [16]. In this section we classify all proper finite connected copolar spaces which are 4-homogeneous.

In addition to the copolar spaces $U_{2,3}(n)$ defined in the introduction, we will also need the copolar spaces $NQ^{\pm}(2n+1, 2)$ and $\overline{W(2n+1, \mathbb{K})}$, as well as the Moore spaces $\overline{M(k)}$. The points of $NQ^{\pm}(2n+1, 2)$ are those of a finite odd-dimensional projective space over $GF(2)$ which are not on a fixed non-degenerate quadric Q, and the lines of $NQ^{\pm}(2n+1, 2)$ are the lines of $\mathrm{PG}(2n+1, 2)$ disjoint from Q (with $+$ or $-$ according as Q is hyperbolic or elliptic). The points of $\overline{W(2n+1, \mathbb{K})}$ are those of a finite odd-dimensional projective space over a field $\mathbb{K}$, and the lines of $\overline{W(2n+1, \mathbb{K})}$ are the hyperbolic lines for some fixed non-degenerate symplectic polarity (i.e. the lines which are not totally isotropic). If $\mathbb{K}$ is of order q, we will write $\overline{W(2n+1, q)}$ instead of $\overline{W(2n+1, \mathbb{K})}$. A *Moore graph* is a graph of diameter 2, containing no circuit of length 3 or 4 and having no vertex adjacent to all the others. Hoffman and Singleton [17] proved that a finite Moore graph is regular of valency $k = 2, 3, 7$ or 57. In the first three cases, it is well known that such a graph exists and is unique up to isomorphism. The points of the *Moore space* $\overline{M(k)}$ corresponding to a given Moore graph $M(k)$ are the vertices of the graph, and the lines are the neighbourhoods of the vertices (hence the lines have size k). The given Moore graph is the non-collinearity graph of the corresponding Moore space ([16] p. 424).

Proposition 5.1. *For* $n = 5, 6, 8$, $U_{2,3}(n)$ *is isomorphic to* $NQ^-(3,2)$, $\overline{W(3,2)}$ *and* $NQ^+(5,2)$, *respectively.*

Proof. These isomorphisms can be deduced from isomorphisms between copolar graphs, given in [16]. However we will give here direct and self-contained proofs of these isomorphisms.

It is well known that the symplectic space $W(3,2)$, also known as the generalized quadrangle $W(2)$, can be constructed as follows: the points are the unordered pairs of elements of the set $X = \{1,2,3,4,5,6\}$ and the (totally isotropic) lines are the partitions of X into three pairs. So, a hyperbolic line consists of three pairs which mutually intersect. Moreover, these three pairs are disjoint from three other pairs which also form a hyperbolic line (correspondence under the symplectic polarity). Hence the union of the three pairs forming a hyperbolic line is a 3-subset of X. This shows that $\overline{W(3,2)} \cong U_{2,3}(6)$.

$U_{2,3}(5)$ is the substructure of $U_{2,3}(6)$ obtained from the latter by deleting 6 and all the pairs of X containing 6; it is well known that this translates to $W(2)$ as deleting an ovoid, which is an elliptic quadric Q in $PG(3,2)$. Hence $U_{2,3}(5)$ contains all the points of $PG(3,2)$ except those of Q, and the lines of $U_{2,3}(5)$ are exactly the lines of $U_{2,3}(6)$ which are disjoint from Q. Moreover, all lines of $PG(3,2)$ not meeting Q are obviously hyperbolic lines of $\overline{W(3,2)}$ (because any totally isotropic line is a partition of $\{1,2,3,4,5,6\}$ into three pairs, and so has necessarily a pair containing 6). The isomorphism $NQ^-(3,2) \cong U_{2,3}(5)$ follows.

Finally, consider $NQ^+(5,2)$. The quadric $Q^+(5,2)$ encodes (by the Klein correspondence) the projective space $PG(3,2)$ in such a way that the points of $Q^+(5,2)$ correspond to the lines of $PG(3,2)$. Any point p of $NQ^+(5,2)$ can be identified with the intersection of its polar hyperplane $\pi(p)$ (with respect to the polarity π related to $Q^+(5,2)$) and the Klein quadric $Q^+(5,2)$. This intersection is a quadric $Q(4,2)$, and it is mapped by the Klein correspondence to a symplectic space $W(2)$ in $PG(3,2)$ (see for example [27], p. 64). This geometry is obtained from a symplectic polarity ρ_p of $PG(3,2)$. Symplectic polarities are outer automorphisms of order 2 of $PGL(4,2)$. Using the exceptional isomorphism $PGL(4,2) \cong A_8$, we see that ρ_p corresponds to an outer automorphism of order 2 of A_8, i.e. an involution in $S_8 \backslash A_8$; it cannot be an involution using three disjoint transpositions since there are 420 of them, and this is too many compared with the number of symplectic spaces in $PG(3,2)$, which is 28. Hence ρ_p corresponds to a transposition in S_8, and hence to a unique point of $U_{2,3}(8)$, and conversely each point of $U_{2,3}(8)$ corresponds to a unique point of $NQ^+(5,2)$. If two points of $NQ^+(5,2)$ are collinear, then the corresponding quadrics $Q(4,2)$ meet in an elliptic quadric $Q^-(3,2)$; hence the corresponding symplectic spaces meet in a spread of $PG(3,2)$ (see for example [5] p. 109). If the corresponding polarities (i,j) and (k,l) would centralize each other, then their product would be an involution σ fixing the spread elementwise. σ would obviously fix one point per line, and so a plane pointwise. Therefore σ would be a perspectivity. Since the only lines fixed by a perspectivity are the lines in the axis or through the center, σ could not fix a spread. This contradiction proves that the polarities corresponding to collinear points do not commute (i.e. $i = k$). An easy counting argument shows that any point p of $NQ^+(5,2)$

has exactly 12 neighbours. If p corresponds to $(1,2)$, any neighbour of p must be of the form $(1,i)$ or $(2,i)$ with $i \in \{3,4,5,6,7,8\}$. Hence two points corresponding to (i,j) and (i,k) must be collinear. If the third point of the line containing the points (i,j) and (i,k) is of the form (i,l) (where i,j,k,l are pairwise distinct), then the point (i,m) (with m distinct from i,j,k,l) is collinear with all the points of this line, which is not possible. Therefore, the third point must be of the form (j,k), and so every line corresponds to a 3-subset of $\{1,2,3,4,5,6,7,8\}$. This shows the isomorphism $NQ^+(5,2) \cong U_{2,3}(8)$. □

Proposition 5.2. *The only* 4*-homogeneous connected spaces* $NQ^{\pm}(2n+1,2)$ *are* $NQ^-(3,2) \cong U_{2,3}(5)$ *and* $NQ^+(5,2) \cong U_{2,3}(8)$.

Proof. The spaces $NQ^{\pm}(1,2)$ are trivially non-connected. The space $NQ^+(3,2)$, consisting of two disjoint lines, is also non-connected.

Suppose that S is a 4-homogeneous connected copolar space $NQ^-(2n+1,2)$ for $n \geqslant 2$ or $NQ^+(2n+1,2)$ for $n \geqslant 3$, and let Q be the corresponding quadric of $\mathrm{PG}(2n+1,2)$. Let p be a point of Q and let Q_p be the tangent hyperplane at p. There is a subspace W of $\mathrm{PG}(2n+1,2)$ such that $p \notin W$ and such that the subspace generated by p and W is Q_p.

It is well known (see for example [1]) that $W \cap Q$ is a non-degenerate quadric of the same type (i.e. hyperbolic or elliptic) as Q, in the subspace $W \simeq \mathrm{PG}(2n-1,2)$. Hence the semilinear structure induced by S on the points of W not on Q is an $NQ^{\pm}(2n-1,2)$. It is easy to show (for instance by induction on n) that with the prescribed conditions on n, the points of $NQ^{\pm}(2n-1,2)$ have degree at least 2. Hence W contains two lines L_1 and L_2 intersecting in a point t and disjoint from Q. Let x be a point on one of these two lines. Since $x \in Q_p$, the line px is tangent to Q, and so meets Q in a single point, namely p. Therefore, the plane π_i generated by p and L_i $(i=1,2)$ meets Q only in p. The semilinear structure induced by S on $\pi_i \backslash \{p\}$ consists of the 6 points of π_i different from p and of the 4 lines not passing through p. Let t' be the third point of the line pt, which is the intersection of π_1 and π_2. Denote the other four points of π_i by a_i, b_i, c_i, d_i in such a way that $L_i = a_ib_i$, $a_ib_i \cap c_id_i = t$ and $a_ic_i \cap b_id_i = t'$.

Now consider the two subsets $A = \{a_1, b_1, c_1, d_1\}$ and $B = \{a_1, b_1, a_2, c_2\}$ of S. Since S is copolar and since a_2 is collinear with t, a_2 is collinear with exactly one other point of L_1, say a_1 without loss of generality. In the antiflag (b_1, a_2c_2t'), b_1 is collinear with t' and non-collinear with a_2, and so b_1 is collinear with c_2. In the antiflag (c_2, L_1), c_2 is collinear with t and b_1, and so is non-collinear with a_1. Hence the semilinear structure induced by S on B is isomorphic to the one induced on A. But the lines a_1b_1 and a_2c_2 are disjoint in S, while $a_1b_1 \cap c_1d_1 \neq \emptyset$ and $a_1c_1 \cap b_1d_1 \neq \emptyset$. This contradicts the 4-homogeneity of S.

Hence S must be $NQ^-(3,2)$ or $NQ^+(5,2)$. The isomorphisms with $U_{2,3}(5)$ and $U_{2,3}(8)$ have been described in Proposition 5.1. □

Proposition 5.3. *The only* 4*-homogeneous space* $\overline{W(2n+1,\mathbb{K})}$ *is* $\overline{W(3,2)}$, *which is isomorphic to* $U_{2,3}(6)$.

Proof. $\overline{W(2n+1, \mathbb{K})}$ contains two lines L and L' intersecting in a point o. Let x and y be two points of L distinct from o. There are unique distinct points x', y' of L' non-collinear with x, y, respectively. Let $A = \{x, y, x', y'\}$. In a suitable coordinate system, the symplectic polarity maps a point $[a_1, a_2, \dots, a_{2n+1}, a_{2n+2}]$ to the hyperplane with equation $a_2x_1 - a_1x_2 + \cdots + a_{2n+2}x_{2n+1} - a_{2n+1}x_{2n+2} = 0$.

Assume first that $\mathbb{K}$ contains at least 3 elements. Let X be the set of points of $\mathrm{PG}(2n+1, \mathbb{K})$ all of whose coordinates are 0, except the first four. The semilinear structure induced by $\overline{W(2n+1, \mathbb{K})}$ on X is a subspace of $\overline{W(2n+1, \mathbb{K})}$ which is clearly isomorphic to $\overline{W(3, \mathbb{K})}$. On the other hand, X together with its totally isotropic lines is isomorphic to $W(3, \mathbb{K})$, which is a generalized quadrangle. Let M and M' be two disjoint totally isotropic lines in X and let a, b be two points of M. In $W(3, \mathbb{K})$, a (resp. b) is collinear with exactly one point of M', say a' (resp. b'). Since the lines of $\mathrm{PG}(2n+1, \mathbb{K})$ have size at least 4, there are two points of M' distinct from a' and b', say c and d. Let $B = \{a, b, c, d\}$. The semilinear structure induced by $\overline{W(2n+1, \mathbb{K})}$ on B consists of exactly 4 lines of size 2, namely ac, ad, bc and bd. Note that $ac \cap bd = \varnothing$ and $ad \cap bc = \varnothing$ in X, otherwise B would be contained in a plane of $\mathrm{PG}(2n+1, \mathbb{K})$, which is impossible since M and M' are disjoint.

If $\mathbb{K} = \mathbb{F}_2$ and $n \geqslant 2$, we define B as the set consisting of the following 4 points of $\mathrm{PG}(2n+1, 2)$:

$$a = [0,0,0,0,1,0,\underbrace{0,\dots,0}_{2n-4}], \quad b = [1,0,0,0,1,0,\underbrace{0,\dots,0}_{2n-4}],$$
$$c = [0,0,0,0,0,1,\underbrace{0,\dots,0}_{2n-4}], \quad d = [0,0,1,0,0,1,\underbrace{0,\dots,0}_{2n-4}].$$

In both cases, the semilinear structures induced on A and B are isomorphic, contradicting the 4-homogeneity of $\overline{W(2n+1, \mathbb{K})}$ (because $xy \cap x'y' \neq \varnothing$, while $ac \cap bd = \varnothing$ and $ad \cap bc = \varnothing$). The only possibility left is $\overline{W(3,2)}$, which is isomorphic to $U_{2,3}(6)$ by Proposition 5.1. □

Figure 3 is a representation of this ultrahomogeneous space.

Proposition 5.4. *The only* 4*-homogeneous proper Moore space is* $\overline{M(3)} \cong U_{2,3}(5)$.

Proof. $\overline{M(2)}$ is isomorphic to the graph C_5 of the pentagon, and so is not proper. $M(3)$ is the Petersen graph and $\overline{M(3)}$ is is easily seen to be isomorphic to $U_{2,3}(5)$, the Desargues configuration.

Suppose now that $k \geqslant 4$. If k is finite, then $M(k)$ has exactly $k^2 + 1$ vertices, and so $M(k)$ is finite. By the result of Hoffman and Singleton [17], we know that $k \geqslant 7$. Of course, if k is infinite, we also have $k \geqslant 7$. Denote by L_p the line of $\overline{M(k)}$ which corresponds to the neighbourhood of p in the graph $M(k)$. It follows easily from the properties of $M(k)$ that any antiflag (p, L_p) has collinearity index 0 and that any antiflag (q, L_p) with $p \neq q$ has non-collinearity index 1.

We claim that if (p, L) is an antiflag of $\overline{M(k)}$ with non-collinearity index 1, then there is exactly one line through p which is disjoint from L. Indeed, let q be the only

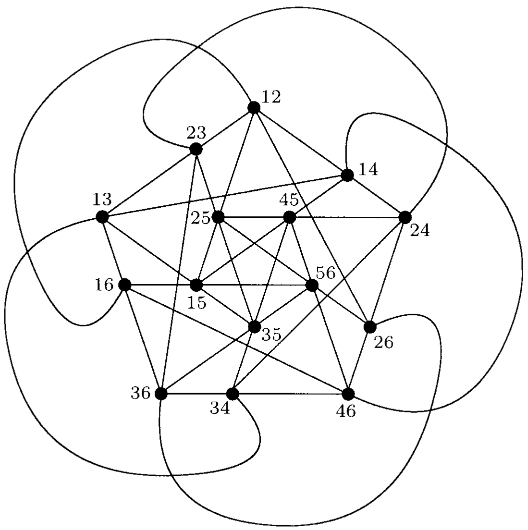

Figure 3. $U_{2,3}(6) \cong \overline{W(3,2)}$

point of L which is non-collinear with p. Since p and q are adjacent in $M(k)$, $p \in L_q$. The line L_q is disjoint from L, otherwise the antiflag (q, L_q) would not have collinearity index 0. Suppose that there exists another line L_r through p disjoint from L $(r \neq q)$. The point r is non-collinear with p, and so r is on the line L_p (which contains also q). Consider any line joining p to a point of L. Such a line cannot be equal to L_r, and so r must be non-collinear with exactly one point of this line, namely p. Hence r is collinear with all the points of L, contradicting the fact that $\overline{M(k)}$ is a copolar space. This proves our claim.

Let p be a point of $\overline{M(k)}$ and let L_q be a line through p. Obviously L_q is disjoint from L_p and $q \in L_p$. Let r and s be two points of L_q distinct from p. Each of the antiflags (r, L_p) and (s, L_p) has non-collinearity index 1, and so r and s are collinear with all the points of L_p except q. Let t be one of these points. The antiflag (s, rt) has non-collinearity index 1, and so (as we have seen before) there is exactly one line through s which is disjoint from rt. This line must intersect L_p in some point u, otherwise there would be two lines (namely this one and L_q) passing through s and disjoint from L_p, which is impossible since the antiflag (s, L_p) has non-collinearity index 1. Let $A = \{r, s, t, u\}$. The semilinear structure induced on A consists of 4 points and 6 lines of size 2, and at least two pairs of disjoint lines of size 2 are induced by disjoint lines in $\overline{M(k)}$ (namely $rs \cap tu = \phi$ and $rt \cap su = \phi$).

Let L and L' be two lines of $\overline{M(k)}$ intersecting in a point o. Each point of L (except o) is non-collinear with exactly one point of L', and conversely. Let a, b, c be three points of L, and let a' (resp. b', c') be the unique point of L' which is non-collinear with a (resp. b, c). Let E be the set $L' \backslash \{o, a', b', c'\}$. E contains at least 3 points (because the lines of $\overline{M(k)}$ have size at least 7) and c is collinear with all the points of E. The antiflag (c, ab') has non-collinearity index 1, and so there is exactly one line through c which is disjoint from ab'. Hence there exists a point d' of E such that the line cd' meets the line ab' (actually there are at least two such points). Let $B = \{a, b', c, d'\}$. The semilinear structure induced on B consists of 4 points and 6 lines of size 2, and at least two pairs of disjoint lines of size 2 are induced by intersecting lines in $\overline{M(k)}$ (namely $ac \cap b'd' \neq \phi$ and $ab' \cap cd' \neq \phi$).

The semilinear structures induced on A and on B are isomorphic, but there is obviously no automorphism mapping A onto B. Therefore $\overline{M(k)}$ is not 4-homogeneous for $k \geqslant 4$. □

A copolar space S is said to be *reduced* if it is connected and if distinct points have distinct neighbourhoods. A *reduced tower of length m* in a reduced copolar space S is a set $\{S_i \mid i = 0, 1, \dots, m\}$ of connected subspaces of S such that $S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_m$. The *reduced rank* of S is the supremum of all cardinal numbers m for which S contains a reduced tower of length m. S has *finite reduced rank* if it has reduced rank m for some integer m.

We now prove the following theorem, which yields Case (viii) and part of (vii) and (c) in Theorem 1.1.

Theorem 5.5. *Let S be a* 4*-homogeneous proper connected copolar space. If S is reduced and of finite reduced rank, then S is isomorphic to* $U_{2,3}(n)$ *for some integer* $n \geqslant 4$, *and if S is not reduced, then S is isomorphic to a transversal design* $TD(m, n)$ *as in* 3.4. *Moreover,* $U_{2,3}(n)$ *is homogeneous for any cardinal number n.*

Proof. According to Proposition 2.2.1 in [11], every 2-homogeneous connected copolar space which is not reduced is a transversal design. By Hall [16], any proper reduced connected copolar space S of finite reduced rank is included in the following list: $NQ^{\pm}(2n+1, 2)$, $\overline{W(2n+1, q)}$, a Moore space, or $U_{2,3}(n)$. The first three cases are covered by 5.2, 5.3 and 5.4, and so S must be a $U_{2,3}(n)$. We have proved in [11] that every space $U_{2,3}(n)$ is ultrahomogeneous (and so a fortiori homogeneous) for any cardinal number n. $U_{2,3}(n)$ is not proper for $1 \leqslant n \leqslant 3$, and $U_{2,3}(4)$ is not reduced (it is isomorphic to $TD(3, 2)$ and also to $T(4)$). □

6 Two types of antiflags, concluded

In this section we complete the treatment of semilinear spaces with exactly two types of antiflags.

Lemma 6.1. *If S is a* 4*-homogeneous proper connected semilinear space having exactly two isomorphism types of antiflags, one with collinearity index* 2 *and one with non-collinearity index* 0, *then all the lines of S have size* 3.

Proof. Suppose on the contrary that the lines of S have $k \geqslant 4$ points. Let (p, L) be an antiflag with collinearity index 2 and (p', L') an antiflag with non-collinearity index 0. There are two points a, b of L (resp. a', b' of L') collinear with p (resp. p'). The semilinear structures induced on $\{a, b, p\}$ and $\{a', b', p'\}$ are isomorphic. Using the 3-homogeneity of S, we deduce that either (b, ap) or (a, bp) must have non-collinearity index 0. Without loss of generality, we may assume that it is (b, ap).

Let $c \in L \setminus \{a, b\}$. Since c is non-collinear with p, the antiflag (c, ap) has collinearity index 2. Let d be the unique point of ap distinct from a and collinear with c. The antiflag (d, L) has non-collinearity index 0. Let $f \in ap \setminus \{a, p, d\}$ and $g \in L \setminus \{a, b, c\}$ (f and g exist since $k \geqslant 4$). Since f is non-collinear with c, the antiflag (f, L) has collinearity index 2, and so f is non-collinear with g. Let $A = \{c, f, g, p\}$.

Since the degree of p' is at least k and since all the points of S have the same degree, there exists a line M through p which is disjoint from L. The point c is non-collinear with p, and so the antiflag (c, M) has collinearity index 2. Let r, s be the two points of M collinear with c. Let $t \in M \setminus \{p, r, s\}$. The point t is non-collinear with c, and so the antiflag (t, L) has collinearity index 2.

If there exists a point $u \in L \setminus \{a, b, c\}$ which is non-collinear with t, then the semilinear structure induced on $B = \{c, p, t, u\}$ is isomorphic to the one induced on A. This contradicts the 4-homogeneity of S because $pf \cap cg \neq \emptyset$ and $pt \cap cu = L \cap M = \emptyset$. Therefore such a point u does not exist.

If $k \geqslant 6$, the set $L \setminus \{a, b, c\}$ has cardinality at least 3 and, since t is collinear with exactly two points of L, such a point u would exist. Therefore $k = 4$ or 5.

If $k = 5$, t is collinear with g and h, which are the only points of $L \setminus \{a, b, c\}$. Let t' be the only point of $M \setminus \{p, r, s, t\}$. Since t' is non-collinear with c, the antiflag (t', L) has collinearity index 2. If $u = g$ or h is non-collinear with t', we get a contradiction by using the same argument as above with A and $B' = \{c, u, p, t'\}$. Therefore t' must be collinear with g and h, and non-collinear with the other points of L. It follows that t and t' are both non-collinear with b and c, and we get a contradiction by using the same argument as above with A and $B'' = \{b, c, t, t'\}$.

Hence $k = 4$ and t is collinear with g, the only point of $L \setminus \{a, b, c\}$. Since g is non-collinear with p and collinear with t, the antiflag (g, M) has collinearity index 2, and so we may assume, without loss of generality, that g is collinear with r. Let $C = \{c, g, p, r\}$ and $D = \{c, g, d, f\}$. The semilinear structures induced on C and D are isomorphic, and any automorphism of S mapping C onto D must fix the line $cg = L$ and map the line $pr = M$ onto df. This contradicts the 4-homogeneity of S, because $L \cap M = \emptyset$ and $L \cap df \neq \emptyset$.

We can now conclude that all the lines of S must have size 3. □

Lemma 6.2. *Let S be a 4-homogeneous proper connected semilinear space all of whose lines have size 3, having at least one antiflag with collinearity index 2 and at least one antiflag with collinearity index 3, but no antiflag with collinearity index 1. Then S is the punctured affine plane* AG(2, 3).

Proof. Let (g, L) be an antiflag of S having collinearity index 2; let a, b be the two points of L collinear with g, and let c be the unique point of L non-collinear with g.

S contains at least one antiflag (p', L') with collinearity index 3; let a', b' be two points of L' collinear with p'. The semilinear structures induced on $\{a, b, g\}$ and $\{a', b', p'\}$ are isomorphic and any automorphism of S mapping the first set onto the second one must map a or b onto p'; without loss of generality, we may assume that it is b. Hence, by the 3-homogeneity of S, the antiflag (b, ag) has collinearity index 3, and so b is collinear with h, the third point of the line ag. Since c is collinear with a and non-collinear with g, and since S contains no antiflag with collinearity index 1, the antiflag (c, ag) has collinearity index 2, and so c is collinear with h.

The semilinear structures induced on $\{a, b, g\}$ and $\{a, b, h\}$ are isomorphic. Since the antiflag (g, ab) has collinearity index 2 and since the antiflags (h, ab) and (b, ah) have collinearity index 3, the 3-homogeneity of S implies that the antiflag (a, bh) must have collinearity index 2. It follows that a is non-collinear with e, the third point of bh. Hence the antiflags (e, ag) and (e, ac) have collinearity index 2, and so e is collinear with g and c. The two lines eg and ec are distinct since c and g are non-collinear. Therefore d, the third point of ce, is distinct from f, the third point of eg. The antiflag (a, ce) has also collinearity index 2, and so a is collinear with d.

Let $A = \{a, b, g, e\}$ and $B = \{b, c, e, g\}$. The semilinear structures induced on A and B are isomorphic. Note that a and e (resp. c and g) is the only pair of non-collinear points in A (resp. in B). Note also that the lines ab and ge are disjoint in S, and that the lines ag and be intersect in h. By the 4-homogeneity of S and since the lines bc and ge are disjoint in S, the lines bg and ce must meet in S. Therefore the third point of the line bg is necessarily d.

The same type of argument, applied to the set $C = \{c, e, g, h\}$ (resp. $D = \{a, d, e, g\}$), shows that the lines ge and ch (resp. ge and ad) meet in S, and so f is the third point of the line ch (resp. ad).

The semilinear structure induced on $\{b, e, g\}$ (resp. $\{a, d, g\}$) is the same as the one induced on $\{g, a, b\}$. By the 3-homogeneity of S and since the antiflag (g, ab) has collinearity index 2 while the antiflags (g, eb) and (e, bg) (resp. (a, dg) and (g, ad)) have collinearity index 3, we see that the antiflag (b, eg) (resp. (d, ag)) must have collinearity index 2. Therefore b and f (resp. d and h) are non-collinear.

We conclude that the semilinear structure induced on the set $S' = \{a, b, c, d, e, f, g, h\}$ is the punctured affine plane AG$(2, 3)$ (see Figure 4).

Suppose by way of contradiction that S is larger than S'. Since S is connected, there is a point p of S outside of S', which is collinear with a point of S' (without

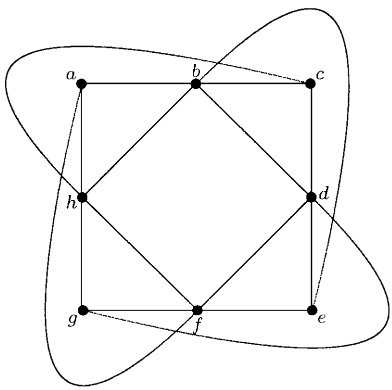

Figure 4. The punctured AG$(2, 3)$

loss of generality, we may assume that p is collinear with a). The antiflag (p, ab) has collinearity index at least 2 (because S contains no antiflag with collinearity index 1). Since the automorphism group of the punctured affine plane $\mathrm{AG}(2,3)$ is transitive on the ordered pairs of collinear points, we may assume without loss of generality that p is also collinear with b.

Suppose that p is non-collinear with h. The semilinear structure induced on $\{a, b, h, p\}$ is isomorphic to the one induced on $\{a, b, g, e\}$. By the 4-homogeneity of S, either the lines ap and bh or the lines ah and bp must intersect in S. This implies that either $e \in ap$ or $g \in bp$. The first case is impossible since a and e are non-collinear; the second case is also impossible since the third point of the line bg is d and not p. Hence p is collinear with h.

If we suppose that p is non-collinear with f, we get a contradiction by applying a similar argument to the sets $\{a, f, h, p\}$ and $\{a, b, g, e\}$. Hence p is also collinear with f.

Consider now the set $\{a, b, f, p\}$. The semilinear structure induced on this set is isomorphic to the one induced on $\{a, b, g, e\}$. As before, this implies that ab meets pf or that af meets pb. In the first case, c would be on the line pf, which is impossible since the third point of the line cf is h and not p. In the second case, d would be on the line pb, which is impossible since the third point of the line bd is g and not p.

This shows that S has no point outside of S', and so S is the punctured affine plane $\mathrm{AG}(2,3)$. It is easily checked that the punctured affine plane $\mathrm{AG}(2,3)$ is indeed homogeneous. □

The following result follows from Lemmas 6.1 and 6.2 and yields part of Case (v) in Theorem 1.1.

Corollary 6.3. *Let S be a 4-homogeneous proper connected semilinear space having exactly two isomorphism types of antiflags, one with collinearity index 2 and one with non-collinearity index 0. Then S is the punctured affine plane $\mathrm{AG}(2,3)$, which is homogeneous.*

7 Three types of antiflags

Let S be a 4-homogeneous connected proper semilinear space with at least three isomorphism types of antiflags. From Section 4 we know that the lines of S have size 3, and S has exactly three isomorphism types of antiflags, whose collinearity indices are in $\{0, 2, 3\}$ or in $\{1, 2, 3\}$. Both cases are impossible:

Theorem 7.1. *There is no 4-homogeneous proper connected semilinear space having exactly three isomorphism types of antiflags, whose collinearity indices are 0, 2 and 3.*

Proof. By Lemma 6.2 such a semilinear space is isomorphic to the punctured affine plane $\mathrm{AG}(2,3)$, but the punctured $\mathrm{AG}(2,3)$ does not contain an antiflag with collinearity index 0. □

Theorem 7.2. *There is no 3-homogeneous proper connected semilinear space having exactly three isomorphism types of antiflags, whose collinearity indices are 1, 2 and 3.*

Proof. Suppose on the contrary that such a semilinear space S exists. All the lines of S have size 3. Let (d, L) be an antiflag of S having collinearity index 1. Let a, b be the two points of L which are non-collinear with d, and let c be the unique point of L collinear with d.

Since S contains an antiflag (p', L') with collinearity index 2, there is a point a' of L' non-collinear with p' and a point c' of L' collinear with p'. The semilinear structures induced on $E = \{a, c, d\}$ and $E' = \{a', c', p'\}$ are isomorphic. Since the antiflags (d, L) and (p', L') are non-isomorphic and since S is 3-homogeneous, the antiflag (a, cd) must have collinearity index 2, and so a is collinear with e, the third point of the line cd. The same argument, applied to the sets $\{b, c, d\}$ and E', implies that b is also collinear with e.

Since the semilinear structures induced on the sets E and $\{a, d, e\}$ are isomorphic, we may use the same type of argument as above to conclude that d is non-collinear with f, the third point of the line ae. Comparing the semilinear structures induced on the sets $\{d, e, f\}$ and E', we get that f is collinear with c.

Suppose that f is non-collinear with b. Consider the set $\{a, b, f\}$. Each of the antiflags (f, ab) and (b, af) has collinearity index 2. Hence there is no automorphism of S mapping $\{a, b, f\}$ onto E, contradicting the fact that the semilinear structures induced on E and $\{a, b, f\}$ are isomorphic. Therefore, b must be collinear with f.

The semilinear structures induced on $\{b, c, e\}$ and $\{a, b, e\}$ are isomorphic. Since the antiflag (b, ce) has collinearity index 2 and the antiflags (e, ab), (b, ae) have collinearity index 3 and since S is 3-homogeneous, the antiflag (a, be) must have collinearity index 2, and so a is non-collinear with g, the third point of the line be.

Comparing the sets $\{a, b, g\}$ and E, an argument similar to the one used above shows that g must be non-collinear with c.

The semilinear structures induced on $\{a, c, e\}$ and $\{b, c, e\}$ are isomorphic. Among the 3 antiflags defined by $\{a, c, e\}$, two have collinearity index 3 and one has collinearity index 2. On the other hand, among the 3 antiflags defined by $\{b, c, e\}$, one has collinearity index 3 and two have collinearity index 2. This contradicts the 3-homogeneity of S and therefore the existence of S. □

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