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Real analytic projective planes with large automorphism groups

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Abstract. We prove that there exist non-classical projective planes whose point space and line space are real analytic (or Nash) manifolds such that the geometric operations of joining points and intersecting lines are real analytic (even Nash) maps on their respective domains. Our examples have the dimensions 2, 4, or 8. These planes are the first examples of non-classical smooth projective planes with large automorphism groups. In dimension 2, they correspond to a class of projective planes discovered by Segre.

Key words. Smooth/real analytic/Nash projective plane, automorphism group.

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1 Introduction

In [11], B. Segre constructed examples of non-desarguesian smooth projective planes, whose lines are real algebraic curves in the real projective plane with its usual real algebraic structure. The construction of these planes was motivated by a prize-question posed by *Het Wiskundig Genootschap* in 1955. However, as mentioned in [10], 75.6, he did 'not consider the question whether the planes are, for example, real analytic or algebraic planes, that is, whether the geometric operations belong to one of these categories'. In this paper we show that the geometric operations of joining points and intersecting lines are in fact real analytic and even Nash maps. For the definition of Nash functions and maps, see [1], 2.9.3 and 2.9.9, cf. also Section 8.1 in that book. Furthermore, we present the first examples of non-desarguesian projective planes with these properties in dimensions 4 and 8. Recall that by [10], 75.1, or by [7], every holomorphic projective plane is isomorphic to $\mathscr{P}_2\mathbb{C}$ with its usual holomorphic structure, and by [12] or [8], every algebraic projective plane over an algebraically closed field is pappian. Our approach also yields a new proof for Segre's result that the incidence structures constructed by him are projective planes. Note in this context that finite-dimensional, compact, connected projective planes always have dimension 2, 4, 8 or 16, cf. [10], 52.5. It should be possible to prove an analogous result in the 16-dimensional setting by using Veronese coordinates instead of homogeneous coordinates (see [10], 16.1). Homogeneous coordinates cannot be used in this case because of the non-associativity of the octonions.

The projective planes considered in this paper are constructed as follows: the point space P and the line space $\mathcal{L}$ are copies of the point space and the line space of $\mathcal{P}_2\mathbb{K}$ with their standard smooth, real analytic, and real algebraic structure ($\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). Hence, points and lines may be described by means of homogeneous coordinates in the usual way. A point $(x, y, z)^t \in P$ (where t denotes transposition) and a line $(a, b, c) \in \mathcal{L}$ are called *incident* if

$$(|a|^2 + |b|^2 + |c|^2)(ax + by + cz)(|x|^2 + |y|^2 + |z|^2) + \lambda|c|^2 cz|z|^2 = 0.$$

Here, $\lambda \in \mathbb{R}$ is a fixed parameter. For $\lambda = 0$ we get the incidence relation of the classical projective plane $\mathcal{P}_2\mathbb{K}$. The flag space $\mathcal{F}_\lambda$ is the set of incident point-line-pairs. The incidence structures $\mathcal{P}_\lambda = (P, \mathcal{L}, \mathcal{F}_\lambda)$ defined in this way are self-dual. A polarity is given by the map $P \times \mathcal{L} \to P \times \mathcal{L} : ((x, y, z)^t, (a, b, c)) \mapsto ((\bar{a}, \bar{b}, \bar{c})^t, (\bar{x}, \bar{y}, \bar{z}))$, where $^-$ denotes conjugation. Of course, the incidence structures $\mathcal{P}_\lambda$ cannot be expected to be projective planes in general. In this paper we prove that they are real analytic and even Nash projective planes for $|\lambda|$ sufficiently small. To be more precise, our proof yields that $|\lambda| < \frac{1}{9}$ is sufficient. In [11], Segre proves in two different ways that the planes $\mathcal{P}_\lambda$ are non-desarguesian for $\lambda \neq 0$ and $\mathbb{K} = \mathbb{R}$. In Section 1 (pp. 36/37) he shows this by a theoretical argument, and in Section 4 (pp. 39/40) he verifies directly that Desargues' theorem fails in $\mathcal{P}_\lambda$ for $\lambda \neq 0$ sufficiently small. A projective plane $\mathcal{P}_\lambda$ with $\mathbb{K} = \mathbb{C}, \mathbb{H}$ has a 2-dimensional subplane equal to the projective plane constructed by Segre with the same parameter λ and hence is not desarguesian for $\lambda \neq 0$.

Before we proceed, let us first recall some basic results on automorphisms of compact or smooth projective planes. The *automorphism group* Σ of a compact (smooth) projective plane $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$ is the group of all automorphisms of $\mathcal{P}$ as an incidence structure which induce homeomorphisms (diffeomorphisms) on P and $\mathcal{L}$. These automorphisms are called continuous (smooth) automorphisms. Note that by [4], 4.7, a continuous automorphism of a smooth projective plane is smooth. The automorphism group Σ of $\mathcal{P}$ is endowed with the compact-open topology derived from its action on P or $\mathcal{L}$, respectively. These two topologies coincide by [10], 44.2. In this way, Σ becomes a locally compact topological group with a countable basis, see [10], 44.3. Hence, the dimension of Σ is defined, compare [10], 93.5 and 6. By a group of automorphisms of $\mathcal{P}$ we mean a closed subgroup of Σ endowed with the induced topology.

The projective planes $\mathcal{P}_\lambda$ presented in this paper are the first examples of non-classical smooth projective planes with large automorphism groups. They admit Lie groups of smooth automorphisms of dimension 1, 4 or 13 for $l = 1$, 2 or 4, respectively. By [2], the dimension of the automorphism group of a $2l$-dimensional non-classical smooth projective plane is at most 2, 6 or 16 for $l = 1$, 2 or 4, respectively. These bounds are 2 less than the corresponding bounds in the case of compact projective planes, but it is not known if they are sharp. Our examples show that the bounds found by Bödi are not far from the truth. The Lie groups of smooth automorphisms of the projective planes $\mathscr{P}_\lambda$ mentioned above are in fact *compact* groups. This shows that, in contrast to the automorphism groups of smooth projective planes, the bounds for the dimensions of compact groups of automorphisms of non-classical compact projective planes are the same as those in the smooth setting for $l \in \{1,2,4\}$, see Theorem 2.9.

2 Proofs and details

Let $\mathscr{I} = (P, \mathscr{L}, \mathscr{F})$ be an incidence structure. The elements of P, $\mathscr{L}$, and $\mathscr{F}$ are called *points*, *lines*, and *flags*, respectively. For $p \in \mathscr{P}$, $L \in \mathscr{L}$ we call $P_L = \pi_P(\pi_{\mathscr{L}}^{-1}(L))$ the *point row* associated with L and $\mathscr{L}_p = \pi_{\mathscr{L}}(\pi_P^{-1}(p))$ the *line pencil* through p, where $\pi_P : \mathscr{F} \to P$ and $\pi_{\mathscr{L}} : \mathscr{F} \to \mathscr{L}$ denote the canonical projections. A projective plane $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ is called a *smooth projective plane* if P and $\mathscr{L}$ are smooth manifolds such that the two geometric operations of joining distinct points and of intersecting distinct lines are smooth, i.e. differentiable in the sense of C^∞. *Real analytic* or *Nash projective planes* are defined analogously. The next theorem is essential for the proof of the main result of this paper, for a proof see [3], 1.5, or [6], 4.5.

Theorem 2.1. *Let $\mathscr{I} = (P, \mathscr{L}, \mathscr{F})$ be an incidence structure which satisfies the following conditions*:

(SGP1) *There is a positive integer l such that P and $\mathscr{L}$ are compact, connected smooth $2l$-dimensional manifolds.*

(SGP2) *The flag space $\mathscr{F}$ is a closed smooth $3l$-dimensional submanifold of $P \times \mathscr{L}$, and the canonical projections π_P and $\pi_{\mathscr{L}}$ are submersions.*

(SGP3) *Any two distinct lines intersect transversally in P and any two line pencils associated with distinct points intersect transversally in $\mathscr{L}$.*

Then there are positive integers m, n such that any two distinct points are joined by exactly m lines and any two distinct lines intersect in exactly n points.

We add some comments on this theorem. The canonical projections $\pi_{\mathscr{P}}$ and $\pi_{\mathscr{L}}$ are surjective since $\mathscr{F}$ is compact, submersions are open maps, and P, $\mathscr{L}$ are connected. Hence, point rows and line pencils are smooth l-dimensional submanifolds of P and $\mathscr{L}$, respectively, by (SGP1) and (SGP2), see [3], 1.1, or [6], 4.1. Thus the transversality condition in (SGP3) makes sense. Recall that two lines L, K of an incidence structure satisfying conditions (SGP1) and (SGP2) are said to intersect *transversally* in some point p, if the associated point rows P_L and P_K intersect transversally in p as submanifolds of P, i.e. their tangent spaces in p span the tangent space $\mathrm{T}_p P$, or, equivalently, the intersection of their tangent spaces in p is trivial. They are said to intersect transversally if they intersect transversally in each common point. Note that two lines which intersect transversally need not have a common point. Transversal intersection of line pencils is defined dually.

The proof of the main result of this paper is based on the following corollary (see [3], 1.6, or [6], 4.6).

Corollary 2.2. *Assume that $\mathcal{I} = (P, \mathcal{L}, \mathcal{F})$ satisfies the conditions of Theorem* 2.1. *If there are two lines whose intersection consists of at most one point, or if there are two points which are joined by at most one line, then $\mathcal{I}$ is a smooth projective plane.*

We want to show now that the incidence structures $\mathcal{P}_\lambda$ in general ($\lambda \in \mathbb{R}$ arbitrary) admit non-trivial groups of smooth automorphisms which are compact Lie groups.

Lemma 2.3. *For $\mathbb{K} = \mathbb{R}$, the orthogonal group $O_2\mathbb{R}$ acts on $\mathcal{P}_\lambda$ as a group of smooth automorphisms. For $\mathbb{K} = \mathbb{C}$, the incidence structure $\mathcal{P}_\lambda$ admits a group of smooth automorphisms isomorphic to the unitary group $U_2\mathbb{C}$. Moreover, also complex conjugation induces a smooth automorphism of $\mathcal{P}_\lambda$. For $\mathbb{K} = \mathbb{H}$, the incidence structure $\mathcal{P}_\lambda$ admits a group of smooth automorphisms isomorphic to the product of* $\mathrm{Spin}_3\mathbb{R}$ *and* $\mathrm{Spin}_5\mathbb{R}$ *with amalgamated centers.*

Proof. For $\mathbb{K} = \mathbb{R}$, let Γ be the subgroup of $O_3\mathbb{R}$ which fixes $(0,0,1) \in \mathbb{R}^3$. This subgroup is isomorphic to $O_2\mathbb{R}$. The standard action of Γ on $\mathbb{R}^3$ induces an effective smooth action of Γ on the line space $\mathcal{L}$. Analogously, we define an effective smooth action of Γ on P by $\Gamma \times P \to P : (\gamma, (x,y,z)^t) \mapsto \gamma^{-1}(x,y,z)^t$. By definition of the incidence relation in $\mathcal{P}_\lambda$ we see that the induced action of Γ on $P \times \mathcal{L}$ leaves $\mathcal{F}_\lambda$ invariant, i.e. Γ acts on $\mathcal{P}_\lambda$ as a group of smooth automorphisms.

For $\mathbb{K} = \mathbb{C}$, an analogous proof shows that the unitary group $U_2\mathbb{C}$ acts on $\mathcal{P}_\lambda$ as a group of smooth automorphisms. The fact that complex conjugation induces a smooth automorphism of $\mathcal{P}_\lambda$ also follows directly from the definition of the incidence relation.

For $\mathbb{K} = \mathbb{H}$, let Γ be the subgroup of the unitary group $U_3\mathbb{H}$ isomorphic to $U_2\mathbb{H} \times U_1\mathbb{H}$, which acts on the first two components of $(x,y,z) \in \mathbb{H}^3$ as $U_2\mathbb{H}$ and on the last component as $U_1\mathbb{H}$. Note that $U_1\mathbb{H}$ is isomorphic to $\mathrm{Spin}_3\mathbb{R}$ and that $U_2\mathbb{H}$ is isomorphic to $\mathrm{Spin}_5\mathbb{R}$, cf. [10], 95.10. The action of Γ on $\mathbb{H}^3$ induces an effective smooth action of the product of $U_2\mathbb{H}$ and $U_1\mathbb{H}$ with amalgamated centers on the line space $\mathcal{L}$. As before, we see that this group acts on $\mathcal{P}_\lambda$ by smooth automorphisms. □

The preceding lemma will enable us to choose appropriate coordinates in the proof of the main result of this paper.

Theorem 2.4. *The incidence structures $\mathcal{P}_\lambda = (P, \mathcal{L}, \mathcal{F}_\lambda)$ are smooth projective planes for $|\lambda| < \frac{1}{9}$.*

The next lemma presents the most difficult part of the proof of this theorem. In the sequel, we will use the description of the point space P and the line space $\mathcal{L}$ of the incidence structure $\mathcal{P}_\lambda = (P, \mathcal{L}, \mathcal{F}_\lambda)$ by means of the standard charts: for the point space P, the corresponding open sets U_1, U_2, and U_3 are given by $x \neq 0$, $y \neq 0$, and $z \neq 0$, respectively, and these sets are identified with $\mathbb{K}^2$ in the usual way. In the latter case, for example, we use the map $U_3 \to \mathbb{K}^2 : (x,y,z)^t \mapsto (x/z, y/z)$. Analogously we define open sets V_1, V_2, and V_3 by $a \neq 0$, $b \neq 0$, and $c \neq 0$, respectively, which cover the line space $\mathscr{L}$. Sometimes it will be convenient to identify $\mathbb{K}$ with $\mathbb{R}^l$ by choosing $\{1\}$, $\{1, i\}$ or $\{1, i, j, k\}$, respectively, as a basis of $\mathbb{K}$ over $\mathbb{R}$. In this way, left multiplication by some element $c \in \mathbb{K}$ gives rise to a linear map $L_c : \mathbb{R}^l \to \mathbb{R}^l$, and right multiplication by c induces a linear map $R_c : \mathbb{R}^l \to \mathbb{R}^l$.

In order to avoid cumbersome notation we will sometimes use the same names for different variables in the following two proofs, if such a choice is natural, facilitates reading, and no confusion is possible.

Lemma 2.5. *Let $|\lambda| < \frac{1}{9}$. Then the set $\mathscr{F}^\circ = \mathscr{F} \cap (U_3 \times V_3)$ is a smooth $3l$-dimensional submanifold of $\mathscr{P} \times \mathscr{L}$. The restrictions of the natural projections π_P and $\pi_{\mathscr{L}}$ to $\mathscr{F}^\circ$ are submersions. The sets $P_L \cap U_3$ and $\mathscr{L}_p \cap V_3$ with $p \in \pi_P(\mathscr{F}^\circ)$ and $L \in \pi_{\mathscr{L}}(\mathscr{F}^\circ)$ are smooth l-dimensional submanifolds of P and $\mathscr{L}$, respectively. If two distinct lines $L, L' \in V_3$ intersect in a point $p \in U_3$ then the submanifolds $P_L \cap U_3$ and $P_{L'} \cap U_3$ intersect transversally in p. Also the dual statement holds.*

Proof. We identify the open subsets $U_3 \subseteq P$ and $V_3 \subseteq \mathscr{L}$ with two distinct copies of $\mathbb{K}^2$. By means of these identifications, the set $\mathscr{F}^\circ$ corresponds to

$$\{(x, y, a, b) \in \mathbb{K}^2 \times \mathbb{K}^2 \mid (|a|^2 + |b|^2 + 1)(ax + by + 1)(|x|^2 + |y|^2 + 1) + \lambda = 0\}.$$

For any $(a, b) \in V_3$ we define

$$g_{(a,b)} : \mathbb{K}^2 \to \mathbb{K} : (x, y) \mapsto (|a|^2 + |b|^2 + 1)(ax + by + 1)(|x|^2 + |y|^2 + 1) + \lambda.$$

We want to prove the technical result that the kernels of the differentials $\mathrm{D}_{(x,y)} g_{(a,b)}$ and $\mathrm{D}_{(x,y)} g_{(a',b')}$ have trivial intersection for any two distinct quadruples (x, y, a, b), $(x, y, a', b') \in \mathscr{F}^\circ$. Then the claims above will follow easily. By using transitivity properties of the group of smooth automorphisms of $\mathscr{P}_\lambda$ (see Lemma 2.3) we may assume that $y = 0$, $x \in \mathbb{R}$, and $b \in \mathbb{R}$. The above incidence relation then shows that $ax \in \mathbb{R} \backslash \{0\}$ (because of $|\lambda| < 1$) and hence that $a \in \mathbb{R}$. Analogously we see that $a' \in \mathbb{R}$. For the sake of simplicity we will assume in the following that $\mathbb{K} = \mathbb{H}$. Sometimes we will identify $\mathbb{H}$ with $\mathbb{R}^4$ and associate to any element $w \in \mathbb{H}$ a vector $(w_1, w_2, w_3, w_4) \in \mathbb{R}^4$. In this way, the differential of the map $\theta : \mathbb{H} \to \mathbb{H} : t \mapsto |t|^2$ at a point $t \in \mathbb{H}$ corresponds to the map $\mathrm{D}_t\theta : \mathbb{R}^4 \to \mathbb{R}^4 : (w_1, w_2, w_3, w_4) \mapsto (2(w_1 t_1 + w_2 t_2 + w_3 t_3 + w_4 t_4), 0, 0, 0)$. Now let $(u, v) \in \ker \mathrm{D}_{(x,0)} g_{(a,b)} \cap \ker \mathrm{D}_{(x,0)} g_{(a',b')}$ and assume that $(u, v) \neq (0, 0)$. By differentiating $g_{(a,b)}$ at $(x, 0)$ we get

$$\begin{aligned}(R_{x^2+1} L_a + L_{ax+1} \mathrm{D}_x\theta)u + (R_{x^2+1} L_b + L_{ax+1} \mathrm{D}_0\theta)v \\ = (x^2 + 1)au + (ax + 1)2xu_1 + (x^2 + 1)bv = 0. \qquad (1)\end{aligned}$$

Here we have considered u and v as elements of $\mathbb{R}^4$ in the first line and as elements of $\mathbb{H}$ in the second line. Analogously, we get

$$(x^2 + 1)a'u + (a'x + 1)2xu_1 + (x^2 + 1)b'v = 0. \qquad (2)$$

We multiply Equation (1) by b' from the left and Equation (2) by b. Subtracting the two equations obtained in this way yields

$$(ab' - a'b)(x^2+1)u + ((ax+1)b' - (a'x+1)b)2xu_1 = 0$$

and hence

$$(ab' - a'b)((x^2+1)u + 2x^2u_1) + (b' - b)2xu_1 = 0. \tag{3}$$

As a next step we want to prove that $b \neq b'$. If we have $b' = b \in \mathbb{R}$, then a and a' are zeros of the real polynomial function $p : \mathbb{R} \to \mathbb{R} : s \mapsto (s^2 + b^2 + 1)(sx+1) \cdot (x^2+1) + \lambda$. Let $s \in \mathbb{R}$ with $p'(s) = 0$ (if it exists). Then we have $2s(sx+1) + (s^2 + b^2 + 1)x = 0$ which implies that

$$sx + 1 = 1 - \frac{2s^2}{3s^2 + b^2 + 1} > \frac{1}{3}.$$

Hence, we get $p(s) = (s^2 + b^2 + 1)(sx+1)(x^2+1) + \lambda > \frac{1}{3} + \lambda > 0$ because of $|\lambda| < \frac{1}{9}$. Since p is a real polynomial function of degree 3, this shows that p has precisely one real zero. We conclude that $a = a'$, a contradiction. So, we have $b \neq b'$ and therefore also $u \neq 0$ by Equations (1) and (2). Equation (3) then yields

$$(b' - b)^{-1}(ab' - a'b) = -2xu_1((x^2+1)u + 2x^2u_1)^{-1} \tag{4}$$

and

$$\begin{aligned} \frac{|ab' - a'b|^2}{|b' - b|^2} &= \frac{(2x)^2u_1^2}{(3x^2+1)^2u_1^2 + (x^2+1)^2(u_2^2 + u_3^2 + u_4^2)} \\ &\leqslant \frac{(2x)^2}{(x^2+1)^2} \frac{u_1^2}{u_1^2 + u_2^2 + u_3^2 + u_4^2} \\ &\leqslant \left(\frac{2x}{x^2+1}\right)^2 \leqslant 1. \end{aligned}$$

Thus we get

$$|(b' - b)^{-1}(ab' - a'b)| \leqslant 1. \tag{5}$$

On the other hand, Equation (4) implies that

$$\begin{aligned} (b' - b)^{-1}(ab' - a'b)x + 1 &= 1 - 2x^2u_1((x^2+1)u + 2x^2u_1)^{-1} \\ &= (x^2+1)u((x^2+1)u + 2x^2u_1)^{-1}. \end{aligned}$$

We conclude that

$$|(b'-b)^{-1}(ab'-a'b)x+1|^2 = \frac{(x^2+1)^2|u|^2}{(3x^2+1)^2u_1^2+(x^2+1)^2(u_2^2+u_3^2+u_4^2)}$$
$$\geqslant \frac{(x^2+1)^2}{(3x^2+1)^2}\frac{|u|^2}{u_1^2+u_2^2+u_3^2+u_4^2}.$$

Because of $\frac{x^2+1}{3x^2+1} > \frac{1}{3}$, we get the inequality

$$|(b'-b)^{-1}(ab'-a'b)x+1| > \frac{1}{3}. \tag{6}$$

Since $(x,0,a,b)\in\mathscr{F}^\circ$ implies $(a^2+b^2+1)(ax+1)(x^2+1)+\lambda=0$, we have $ax+1 = -\lambda(a^2+b^2+1)^{-1}(x^2+1)^{-1}$ and, analogously, $a'x+1 = -\lambda(a'^2+|b'|^2+1)^{-1}\cdot(x^2+1)^{-1}$. We multiply the first of these two equations by b' and the second by b. After subtracting these two equations we obtain

$$(ab'-a'b)x+(b'-b) = -\frac{\lambda}{x^2+1}\frac{(a'^2+|b'|^2+1)b'-(a^2+b^2+1)b}{(a^2+b^2+1)(a'^2+|b'|^2+1)}. \tag{7}$$

We want to multiply Equation (7) by $(b'-b)^{-1}$ in order to combine it with (6). We have

$$(a'^2+|b'|^2+1)b'-(a^2+b^2+1)b = (a'^2b'-a^2b)+(|b'|^2b'-b^3)+(b'-b),$$

where $a'^2b'-a^2b = (b'-b)(a^2+aa'+a'^2)-(ab'-a'b)(a+a')$, and $|b'|^2b'-b^3 = (b'-b)(|b'|^2+\overline{b'}b+b^2)-(b'-\overline{b'})b^2$ with

$$\frac{|b'-\overline{b'}|^2}{|b'-b|^2} = \frac{4(b_2'^2+b_3'^2+b_4'^2)}{(b_1'-b)^2+b_2'^2+b_3'^2+b_4'^2} \leqslant 4.$$

Hence, we get

$$|(b'-b)^{-1}(|b'|^2b'-b^3)| \leqslant |b'|^2+|b'b|+b^2+|(b'-b)^{-1}(b'-\overline{b'})|b^2$$
$$\leqslant |b'|^2+|b'b|+3b^2$$

and

$$|(b'-b)^{-1}(a'^2b'-a^2b)|$$
$$\leqslant a^2+|aa'|+a'^2+|(b'-b)^{-1}(ab'-a'b)|(|a|+|a'|)$$
$$\leqslant a^2+|aa'|+a'^2+|a|+|a'|$$

by inequality (5). By combining these inequalities with (6) and (7), we obtain

$$\begin{aligned}\frac{1}{3} &< |(b'-b)^{-1}(ab'-a'b)x+1| \\ &\leqslant |\lambda| \frac{a^2+|aa'|+a'^2+|a|+|a'|+|b'|^2+|b'b|+3b^2+1}{(a^2+b^2+1)(a'^2+|b'|^2+1)}. \end{aligned} \tag{8}$$

Obviously, we have

$$\frac{a^2+a'^2+|b'|^2+b^2+1}{(a^2+b^2+1)(a'^2+|b'|^2+1)} \leqslant 1.$$

Using that $\left(s-\frac{1}{2}\right)^2 \geqslant 0$, and hence $s \leqslant s^2+\frac{1}{4}$, for every $s \in \mathbb{R}$, we get

$$\begin{aligned}|a|+|a'|+|aa'|+|bb'|+b^2 &\leqslant a^2+a'^2+|aa'|^2+|b'b|^2+1+b^2 \\ &\leqslant (a^2+b^2+1)(a'^2+|b'|^2+1).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&(a^2+a'^2+|b'|^2+b^2+1)+(|a|+|a'|+|aa'|+|b'b|+b^2)+b^2 \\ &\quad \leqslant 3(a^2+b^2+1)(a'^2+|b'|^2+1),\end{aligned}$$

which shows together with inequality (8) that $\frac{1}{3} < 3|\lambda|$, in contradiction to $|\lambda| < \frac{1}{9}$. Thus the kernels of the differentials $\mathrm{D}_{(x,y)}g_{(a,b)}$ and $\mathrm{D}_{(x,y)}g_{(a',b')}$ intersect trivially for any two distinct quadruples $(x,y,a,b), (x,y,a',b') \in \mathscr{F}^\circ$.

We want to show next that there are infinitely many lines in V_3 through any point $(x,y) \in \pi_P(\mathscr{F}^\circ)$. Using the transitivity properties of the automorphism group of $\mathscr{P}_\lambda$ we may arrange again that $(x,y)=(x,0)$ with $x \in \mathbb{R}$. Because of $(x,0) \in \pi_P(\mathscr{F}^\circ)$ there is a line $(a',b') \in V_3$ incident with the point $(x,0)$. We then have $(|a'|^2+|b'|^2+1)(a'x+1)(x^2+1)+\lambda=0$, which shows that $x \neq 0$. Hence the real polynomial function $q_b : \mathbb{R} \to \mathbb{R} : s \mapsto (s^2+b^2+1)(sx+1)(x^2+1)+\lambda$ has degree 3 for every $b \in \mathbb{R}$. Thus, for any $b \in \mathbb{R}$ there exists $a \in \mathbb{R}$ such that $(a^2+b^2+1)(ax+1)\cdot(x^2+1)+\lambda=0$, i.e. such that $(x,0,a,b) \in \mathscr{F}^\circ$.

By (x,y) we denote again an arbitrary point of $\pi_P(\mathscr{F}^\circ)$. Choose two distinct lines $(a,b), (a',b') \in V_3$ through (x,y). By definition of $g_{(a,b)}$ and $g_{(a',b')}$, the dimensions of the kernels of the two differentials $\mathrm{D}_{(x,y)}g_{(a,b)}$ and $\mathrm{D}_{(x,y)}g_{(a',b')}$ are at least l. Since they intersect trivially, their dimension is precisely l and hence these differentials are surjective. In particular, also the total differential of the map

$$f : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K} : (x,y,a,b) \mapsto (|a|^2+|b|^2+1)(ax+by+1)(|x|^2+|y|^2+1)+\lambda$$

is surjective at every point of $\mathscr{F}^\circ$. Therefore $\mathscr{F}^\circ$ is a $3l$-dimensional submanifold of $U_3 \times V_3$ and hence of $P \times \mathscr{L}$.

Now we want to show that the restriction of the natural projection $\pi_{\mathscr{L}}$ to $\mathscr{F}^\circ$ is a submersion. Choose $(x, y, a, b) \in \mathscr{F}^\circ$ arbitrarily. An element of $\ker \mathrm{D}_{(x,y,a,b)}\pi_{\mathscr{L}}$ has the form $(u, v, 0, 0)$ with $(u, v) \in \mathbb{K}^2$. By definition of $\mathscr{F}^\circ$ we have $\mathrm{D}_{(x,y)}g_{(a,b)}(u, v) = \mathrm{D}_{(x,y,a,b)}f(u, v, 0, 0) = 0$. Since the kernel of $\mathrm{D}_{(x,y)}g_{(a,b)}$ is l-dimensional, we conclude that the dimension of $\ker \mathrm{D}_{(x,y,a,b)}\pi_{\mathscr{L}}$ is at most l. Thus the differential $\mathrm{D}_{(x,y,a,b)}\pi_{\mathscr{L}}$ is surjective. Hence the restriction of $\pi_{\mathscr{L}}$ to $\mathscr{F}^\circ$ and, for reasons of symmetry, also the restriction of π_P to $\mathscr{F}^\circ$ are submersions. By [3], 1.1, or [6], 4.1, it follows that the sets $P_L \cap U_3$ and $\mathscr{L}_p \cap V_3$ are smooth l-dimensional submanifolds of P and $\mathscr{L}$, respectively, for any $p \in \pi_P(\mathscr{F}^\circ)$, $L \in \pi_{\mathscr{L}}(\mathscr{F}^\circ)$.

It remains to show that any two distinct lines $L = (a, b)$ and $L' = (a', b')$ in V_3 which intersect in a point $p = (x, y) \in U_3$ intersect transversally in p. The dual statement then follows by symmetry. Choose (u, v) in the intersection of the tangent spaces of the point rows P_L and $P_{L'}$ in p. Since $g_{(a,b)}$ vanishes on $P_L \cap U_3$, we conclude that $\mathrm{D}_{(x,y)}g_{(a,b)}(u, v) = 0$. Analogously, we get $\mathrm{D}_{(x,y)}g_{(a',b')}(u, v) = 0$ and hence $(u, v) = (0, 0)$, since the kernels of $\mathrm{D}_{(x,y)}g_{(a,b)}(u, v) = 0$ and $\mathrm{D}_{(x,y)}g_{(a,b)}(u, v) = 0$ have trivial intersection. This completes the proof. □

Proof of Theorem 2.4. As in the classical projective plane $\mathscr{P}_0 = \mathscr{P}_2\mathbb{K}$, the point rows of the lines $(1, 0, 0), (0, 1, 0) \in \mathscr{L}$ intersect precisely in the point $(0, 0, 1)^t \in P$. Hence, by Corollary 2.2, it suffices to verify the conditions of Theorem 2.1. We first show that the flag space $\mathscr{F}_\lambda$ is a $3l$-dimensional submanifold of $P \times \mathscr{L}$ and that $\pi_{\mathscr{L}}$ is a submersion. Then also the natural projection π_P is a submersion for reasons of symmetry. By the previous lemma, it remains to prove these properties in neighbourhoods of flags (p, L) in $P \times \mathscr{L}$, where the last coordinate of p or L is 0. By using transitivity properties of the group of smooth automorphisms of $\mathscr{P}_\lambda$ (see Lemma 2.3), we see that it is sufficient to consider the following cases:

(F1) $p = (x, y, 1)^t$, $L = (1, 0, 0)$,

(F2) $p = (x, 1, 0)^t$, $L = (1, 0, 0)$,

(F3) $p = (1, 0, 0)^t$, $L = (a, b, 1)$.

Note that the point $(1, 0, 0)^t$ and the line $(1, 0, 0)$ are not incident. Moreover, the condition that (p, L) is a flag implies that $x = 0$ in the first two cases and that $a = 0$ in (F3). As in the proof of the previous lemma we introduce appropriate inhomogeneous coordinates. In the Case (F1) we identify U_3 and V_1 with two copies of $\mathbb{K}^2$. In this way, the point p corresponds to $(0, y) \in \mathbb{K}^2$ and the line L corresponds to $(0, 0) \in \mathbb{K}^2$. The set $\mathscr{F}_\lambda \times (U_3 \times V_1)$ is then given by $(f^{(1)})^{-1}(\{0\})$, where $f^{(1)}$ is defined by

$$f^{(1)} : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K} :$$
$$(x, y, b, c) \mapsto (1 + |b|^2 + |c|^2)(x + by + c)(|x|^2 + |y|^2 + 1) + \lambda|c|^2 c.$$

For any $(b, c) \in \mathbb{K}^2$ we define $g^{(1)}_{(b,c)} : \mathbb{K}^2 \to \mathbb{K} : (x, y) \mapsto f^{(1)}(x, y, b, c)$. We have $\mathrm{D}_{(0,y)}g^{(1)}_{(0,0)} : \mathbb{K}^2 \to \mathbb{K} : (u, v) \mapsto (|y|^2 + 1)u$, which shows that $\mathrm{D}_{(0,y)}g^{(1)}_{(0,0)}$ is surjective.

Hence, the total differential of $f^{(1)}$ in $(0, y, 0, 0)$ is also surjective. Thus there exists an open neighbourhood W of (p, L) in $P \times \mathscr{L}$ such that $\mathscr{F}_\lambda \cap W$ is a $3l$-dimensional submanifold of W. Moreover, we see as in the proof of Lemma 2.5 that the restriction of the natural projection $\pi_{\mathscr{L}}$ to $\mathscr{F}_\lambda \cap W$ is a submersion if the neighbourhood W of (p, L) is small enough so that the differential of $g^{(1)}_{(0,0)}$ is surjective at all points of W.

For (F2) we identify U_2 and V_1 with $\mathbb{K}^2$ such that (p, L) corresponds to $(0,0,0,0) \in \mathbb{K}^2 \times \mathbb{K}^2$. We define

$$f^{(2)} : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K} :$$
$$(x, z, b, c) \mapsto (1 + |b|^2 + |c|^2)(x + b + cz)(|x|^2 + 1 + |z|^2) + \lambda |c|^2 cz|z|^2$$

such that $\mathscr{F}_\lambda \cap (U_2 \times V_1)$ is identified with the set $(f^{(2)})^{-1}(\{0\})$. The differential of

$$g^{(2)}_{(0,0)} : \mathbb{K}^2 \to \mathbb{K} : (x, z) \mapsto x(|x|^2 + 1 + |z|^2),$$

defined analogously to $g^{(1)}_{(b,c)}$, at $(0,0)$ is given by $\mathrm{D}_{(0,0)} g^{(2)}_{(0,0)} : \mathbb{K}^2 \to \mathbb{K} : (u, v) \mapsto u$ and hence is surjective. As in the previous case we conclude that there is an open neighbourhood W of (p, L) in $P \times \mathscr{L}$ such that $\mathscr{F}_\lambda \cap W$ is a submanifold of W and $\pi_{\mathscr{L}}$ restricted to $\mathscr{F}_\lambda \cap W$ is a submersion.

In (F3) we identify $U_1 \times V_3$ with $\mathbb{K}^2 \times \mathbb{K}^2$ such that the flag (p, L) corresponds to $(0,0,0,b) \in \mathbb{K}^2 \times \mathbb{K}^2$. The set $\mathscr{F}_\lambda \cap (U_1 \times V_3)$ is then identified with $(f^{(3)})^{-1}(\{0\})$, where

$$f^{(3)} : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K} :$$
$$(y, z, a, b) \mapsto (|a|^2 + |b|^2 + 1)(a + by + z)(1 + |y|^2 + |z|^2) + \lambda z|z|^2.$$

We define $g^{(3)}_{(0,b)} : \mathbb{K}^2 \to \mathbb{K} : (y, z) \mapsto f^{(3)}(y, z, 0, b)$. Then we have

$$\mathrm{D}_{(0,0)} g^{(3)}_{(0,b)} : \mathbb{K}^2 \to \mathbb{K} : (u, v) \mapsto (|b|^2 + 1)(bu + v),$$

which shows that $\mathrm{D}_{(0,0)} g^{(3)}_{(0,b)}$ is surjective. The Case (F3) is then completed as the previous two cases above. Hence, $\mathscr{F}_\lambda$ is a $3l$-dimensional submanifold of $P \times \mathscr{L}$ and the natural projections π_P and $\pi_{\mathscr{L}}$ are submersions. Note that $\mathscr{F}$ is obviously closed in $P \times \mathscr{L}$. Thus π_P and $\pi_{\mathscr{L}}$ are surjective since $\mathscr{F}$ is compact, submersions are open maps, and $\mathscr{P}$, $\mathscr{L}$ are connected. It follows that point rows and line pencils are l-dimensional submanifolds of P and $\mathscr{L}$, respectively, see [3], 1.1, or [6], 4.1.

In order to complete this proof, it suffices for reasons of symmetry to show that any two distinct lines L, L' intersect transversally. By using transitivity properties of the group of smooth automorphisms of $\mathscr{P}_\lambda$, the different possibilities of pairs $(L, L') \in \mathscr{L} \times \mathscr{L}$ reduce to the following three cases:

(L1) $L = (a, b, 1)$, $L' = (a', b', 1)$, (L2) $L = (a, b, 1)$, $L' = (1, 0, 0)$,

(L3) $L = (a, 1, 0)$, $L' = (1, 0, 0)$.

In the first case, we may use the group of smooth automorphisms of $\mathscr{P}_\lambda$ in order to choose appropriate coordinates for possible intersection points of L and L'. We may assume that these two lines intersect in the point $(1, 0, 0)^t$ or in a point $(x, y, 1)^t$. Since the second case has been treated already in Lemma 2.5, we assume that the intersection point of L and L' is $(1, 0, 0)^t$. Then we have $a, a' = 0$ and hence $b \neq b'$ since L and L' are distinct. We identify the open sets U_1 and V_3 with two disjoint copies of $\mathbb{K}^2$ such that L and L' are identified with $(0, b)$ and $(0, b')$, respectively, and $(1, 0, 0)^t$ is identified with $(0, 0)$. The map $g^{(3)}_{(0,b)}$ of the previous paragraph vanishes on $P_L \cap U_1$. Thus the differential

$$\mathrm{D}_{(0,0)} g^{(3)}_{(0,b)} : \mathbb{K}^2 \to \mathbb{K} : (u, v) \mapsto (|b|^2 + 1)(bu + v)$$

vanishes on the tangent space of $P_L \cap U_1$ in $(0, 0)$, and an analogous statement holds for the line L'. Since the kernels of the differentials $\mathrm{D}_{(0,0)} g^{(3)}_{(0,b)}$ and $\mathrm{D}_{(0,0)} g^{(3)}_{(0,b')}$ have trivial intersection, we conclude that L and L' intersect transversally in $(1, 0, 0)^t$.

In the Case (L2), let $(x, y, z)^t$ denote an intersection point of L and L'. Then we have $x = 0$ and hence $(x, y, z)^t = (0, y, 1)^t$ or $(x, y, z)^t = (0, 1, 0)^t$. Let us first assume that $(0, y, 1)^t$ is an intersection point of L and L'. By using the transitivity properties of the group of smooth automorphisms acting on $\mathscr{P}_\lambda$ we may assume that $y \in \mathbb{R}$. After identifying U_3 with $\mathbb{K}^2$, the intersection point corresponds to $(0, y)$ and the submanifolds $P_L \cap U_3$ and $P'_L \cap U_3$ correspond to $g^{-1}_{(a,b)}(0)$ and $\{0\} \times \mathbb{K}$ with $g_{(a,b)}$ as in the proof of Lemma 2.5. Choose (u, v) in the intersection of the tangent spaces of $P_L \cap U_3$ and $P'_L \cap U_3$ in $(0, y)$. Then we get

$$(y^2 + 1)au + (y^2 + 1)bv + (by + 1)2yv_1 = 0$$

by differentiating $g_{(a,b)}$ (compare the proof of Lemma 2.5) and $u = 0$. Thus we have $(y^2 + 1)bv + (by + 1)2yv_1 = 0$ and hence

$$|b| = 2|by + 1| \frac{|y|}{y^2 + 1} \frac{|v_1|}{|v|}$$

provided that $v \neq 0$. We obtain that

$$|by| \leqslant 2|by + 1| \frac{y^2}{y^2 + 1} \leqslant 2|by + 1|.$$

Because of

$$g_{(a,b)}(0, y) = (|a|^2 + |b|^2 + 1)(by + 1)(y^2 + 1) + \lambda = 0$$

we have $|by + 1| \leqslant |\lambda|$, which implies that $1 \leqslant |by| + |by + 1| \leqslant 3|\lambda|$, a contradiction.

Thus we have $v = 0$. This proves the transversal intersection of L and L' in $(0, y, 1)^t$. Let us now consider the case that L and L' intersect in the point $(0,1,0)^t$. Then we have $b = 0$. We identify U_2 with $\mathbb{K}^2$ such that $(0,1,0)^t$ corresponds to $(0,0) \in \mathbb{K}^2$. The submanifolds $P_L \cap U_3$ and $P_{L'} \cap U_3$ are identified with the submanifolds $(g^{(4)}_{(a,0)})^{-1}(\{0\})$ and $\{0\} \times \mathbb{K}$, respectively, where

$$g^{(4)}_{(a,0)} : \mathbb{K}^2 \to \mathbb{K} : (x,z) \mapsto (|a|^2 + 1)(ax + z)(|x|^2 + 1 + |z|^2) + \lambda z|z|^2.$$

The differential $\mathrm{D}_{(0,0)} g^{(4)}_{(a,0)} : (u,v) \mapsto (|a|^2 + 1)(au + v)$ vanishes on the tangent space of $P_L \cap U_3$ in $(0,0)$. This proves the transversal intersection of L and L' in $(0,1,0)^t$, since $\ker \mathrm{D}_{(0,0)} g^{(4)}_{(a,0)}$ and $\{0\} \times \mathbb{K}$ have trivial intersection.

In the third case, both point rows P_L and $P_{L'}$ are equal to point rows of the classical projective plane $\mathscr{P}_0 = \mathscr{P}_2\mathbb{K}$. Hence they intersect transversally. □

For the projective planes $\mathscr{P}_\lambda$, where $|\lambda| < \frac{1}{9}$, the join map $\vee$ and the intersection map $\wedge$ are not only smooth but real analytic and even Nash maps, i.e. the $\mathscr{P}_\lambda$ are real analytic or Nash projective planes, respectively. This will be obtained from the following general fact:

Proposition 2.6. *Let $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ be a projective plane which satisfies the following conditions*:

(APP1) *There is a positive integer l such that P and $\mathscr{L}$ are real analytic* (*or Nash*) *$2l$-dimensional manifolds.*

(APP2) *The flag space $\mathscr{F}$ is a real analytic* (*or Nash*) *$3l$-dimensional submanifold of $P \times \mathscr{L}$, and the canonical projections π_P and $\pi_{\mathscr{L}}$ are submersions.*

Suppose, *moreover*, *that any two distinct point rows and any two distinct line pencils intersect transversally*. *Then the join map $\vee$ and the intersection map $\wedge$ are real analytic* (*or Nash maps*, *respectively*).

This proposition can be proved by simply copying the proof of [3], 1.4, or [6], 4.4, and using a real analytic or Nash version, respectively, of the implicit function theorem, see, e.g., [9], 1.8.3, and [1], 2.9.8.

It remains to check the conditions of the above proposition for the projective planes $\mathscr{P}_\lambda$ with $|\lambda| < \frac{1}{9}$. For simplicity we concentrate on the Nash setting in the sequel. First, the point space P and the line space $\mathscr{L}$ are copies of the point space and the line space of the classical projective plane $\mathscr{P}_2\mathbb{K}$ with their usual algebraic structure. Hence, P and $\mathscr{L}$ are Nash manifolds. In the proofs of Theorem 2.4 and Lemma 2.5 we have shown that for each flag there is an open neighbourhood W in $P \times \mathscr{L}$ (identified with an open subset of $\mathbb{K}^2 \times \mathbb{K}^2$) and a real polynomial submersion $f_W : W \to \mathbb{K}$ such that $\mathscr{F}_\lambda \cap W = f_W^{-1}(0)$. By a Nash version of the standard result on preimages of regular values we conclude that $\mathscr{F}_\lambda$ is a Nash submanifold of $P \times \mathscr{L}$, cf. [6], the end of Chapter 3, or [5], 5.1–5.9, [9], 1.8.1, [1], 2.9.7. The other conditions required in Proposition 2.6 have already been verified above. Hence, the join map $\vee$

and the intersection map $\wedge$ are Nash maps and, in particular, they are real analytic. So, we have proved the following

Theorem 2.7. *For* $|\lambda| < \frac{1}{9}$, *the incidence structures* $\mathscr{P}_\lambda$ *are Nash projective planes and, in particular, real analytic projective planes.*

The following theorem contains results on the dimensions of the automorphism groups of the planes $\mathscr{P}_\lambda$, which are direct consequences of Lemma 2.3.

Theorem 2.8. *The smooth projective planes* $\mathscr{P}_\lambda$ *admit groups of smooth automorphisms which are compact Lie groups of dimension* 1, 4 *or* 13 *for* $l = 1$, 2 *or* 4, *respectively.*

By the main result of [2], the dimension of the automorphism group of a $2l$-dimensional, non-classical smooth projective plane is at most 2, 6 or 16 for $l = 1$, 2 or 4, respectively. By Theorem 2.8, the smooth projective planes $\mathscr{P}_\lambda$ admit Lie groups of smooth automorphisms whose dimensions are close to these bounds. The dimensions of automorphism groups of non-classical compact projective planes of dimension $2l$ can be higher than in the smooth case, see [10], Section 65. The maximal dimensions of *compact* groups of automorphisms of non-classical compact projective planes (with $l = 1$, 2 or 4), however, are the *same* as the dimensions of the Lie groups in Theorem 2.8, i.e. in this respect there is no difference between compact projective planes and smooth projective planes. Indeed, by [10], 32.21 and 22 a compact group of automorphisms of a 2-dimensional, non-classical compact projective plane is a Lie group of dimension at most 1. In the 4-dimensional case, 71.9 and 72.6 in [10] imply that the dimension of a compact group of automorphisms acting on a non-classical compact projective plane is at most 4. Finally, in dimension 8 a compact group of automorphisms acting on a non-classical compact projective plane is at most 13-dimensional, see [10], 84.9. Even more, the identity connected component of such a group is necessarily isomorphic to $\mathrm{SO}_2\mathbb{R}$ for $l = 1$, to $\mathrm{U}_2\mathbb{C}$ for $l = 2$, and to the product of $\mathrm{Spin}_3\mathbb{R}$ and $\mathrm{Spin}_5\mathbb{R}$ with amalgamated centers for $l = 4$. The following theorem summarizes the general information obtained in this way.

Theorem 2.9. *The maximal dimensions of compact groups of automorphisms of* $2l$-*dimensional, non-classical smooth projective planes are the same as those in the case of* $2l$-*dimensional, non-classical compact projective planes for* $l \in \{1, 2, 4\}$.

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