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The cyclic q-clans with $q = 2^e$

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Dedicated to Professor Adriano Barlotti on the occasion of his 80th birthday

1 Introduction

Throughout this article $q = 2^e$, $e \geqslant 1$. The reader is assumed to have a general familiarity with GQ (see [10] for a thorough introduction), and in particular to be familiar with the construction of a GQ starting with a q-clan. (These GQ are often called *flock* GQ because of the connection between flocks of a quadratic cone and q-clans first pointed out in [13].) For a thorough introduction to this construction when $q = 2^e$, see the Subiaco Notebook [7], which is available on the web page of the second author. This unpublished "monograph" is based on several articles by a variety of authors, but we refer the reader to [7] for specific references.

S. E. Payne, T. Penttila and G. F. Royle [9] used a computer to generate several specific GQ of order (q^2, q), the largest with $q = 2^{16} = 65\,536$. Some of the smaller examples had already been discovered earlier. These GQ were called *cyclic* because they admit a group of collineations acting as a single cycle on the $q + 1$ lines through the point (∞). The classical GQ, the so-called FTWKB GQ, and the Subiaco GQ were already known to be cyclic in this sense, and the new ones seemed certain to belong to a new infinite family. W. E. Cherowitzo, C. M. O'Keefe and T. Penttila [4] discovered a new infinite family that appeared to include the examples given in [9], and they gave the name *Adelaide* to all the new associated geometries, i.e., the GQ, the flocks of the quadratic cone, the ovals, etc. Remarkably they gave a unified construction that included the three previously known infinite families as well as the new Adelaide family. (See [12] for a rather complete survey of the known flock GQ and, for q even, the associated herds of ovals, as well as much other material related to ovals.)

However, [4] leaves several questions unanswered. In particular, there is no proof that the unified construction always gives a cyclic GQ, i.e., includes the examples of [9]. In this paper we provide the proof (this material also appears in [8]) as well as showing that there arises just one new GQ and one new oval (up to isomorphism) for each q. We also provide a clarification of the relationship between the group of collineations of a GQ and the magic action (see [6]) used in [4] to establish the stabilizer of the Adelaide GQ. The stabilizer of the Adelaide oval, not completely specified in

[4], is shown to be the complete stabilizer in [11]. Finally, we address, in geometric terms, the relationship between cyclic GQ's and the flocks of the quadratic cone that they give rise to.

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2 Cyclic GQ

We recall the basic setup. A q-clan is a set

$$\mathscr{C} = \left\{ A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix} : t \in \mathrm{GF}(q) \right\}$$

of q 2×2 matrices over $\mathrm{GF}(q)$ such that the pairwise differences $A_s - A_t$ for distinct $s, t \in \mathrm{GF}(q)$ are all anisotropic. Here f and g are permutation polynomials over $\mathrm{GF}(q)$ normalized so that $f(0) = g(0) = 0$. And for $q = 2^e$ the anisotropic condition is exactly that for distinct $s, t \in \mathrm{GF}(q)$, if "trace" denotes the absolute trace function,

$$\mathrm{trace}\left(\frac{(f(s) + f(t))(g(s) + g(t))}{s + t} \right) = 1.$$

The steps leading from a q-clan to the associated geometries have so frequently been reviewed in the literature that we assume here that the reader is familiar with the general process. Indeed, the review in [4] is quite suitable for our purposes. Here we shall review only that part of the Fundamental Theorem for q-clan geometries that deals with the representation of collineations of the associated GQ as a tensor product of two simpler actions, one on the set of lines of the GQ through the special point (∞) and the other on the set of associated ovals.

The elation group G consists of all pairs $((\alpha, \beta), c)$ such that α and β are each arbitrary pairs of elements of $F = \mathrm{GF}(q)$, and $c \in F$. The binary operation in G is:

$$((\alpha_1, \beta_1), c_1) \circ ((\alpha_2, \beta_2), c_2) = ((\alpha_1 + \alpha_2, \beta_1 + \beta_2), c_1 + c_2 + \beta_1 \circ \alpha_2),$$

where, in general, $\alpha \circ \beta = \alpha P \beta^T$, $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Writing $\bar{0} = (0, 0)$, the center of the group G is

$$Z = \{(\bar{0}, \bar{0}, c) : c \in F\}.$$

For each $t \in F$, there is a subgroup $A(t)$ of G defined by

$$A(t) = \{((1, t^{1/2}) \otimes \alpha, g(\alpha, t)) : \alpha \in F^2\},$$

where, if $\alpha = (a, b)$, $g(\alpha, t) := \alpha A_t \alpha^T = a^2 f(t) + abt^{1/2} + b^2 g(t)$. (Note that in the literature both $g_\alpha(t)$ and $g_t(\alpha)$ have been used to denote this function. As we will have occasion to consider this function at different times for fixed α and for fixed t, we have decided not to use these specialized notations to avoid the necessity of having to switch notation in the middle of our arguments.) We also set,

$$A(\infty) = \{((\bar{0}, \beta), 0) : \beta \in F^2\}.$$

The family $\mathscr{J} = \{A(t) : t \in \tilde{F} = F \cup \{\infty\}\}$ of subgroups of G is the so-called Kantor family used to index the lines $[A(t)] : t \in \tilde{F}$ through the point (∞) in the generalized quadrangle $\mathscr{S}(\mathscr{C})$ associated with the given q-clan $\mathscr{C}$.

For $\bar{0} = (0,0) \neq \alpha \in F^2$, let $\mathscr{R}_\alpha = \{(\gamma \otimes \alpha, c) \in G^{\otimes} : \gamma \in F^2, c \in F\}$.

Theorem 2.1 (1.4.1 in [7]). *Each $\mathscr{R}_\alpha$ is an elementary abelian group of order q^3. For nonzero $\alpha, \gamma \in F^2$, $\mathscr{R}_\gamma = \mathscr{R}_\alpha$ iff $\{\alpha, \gamma\}$ is F-linearly dependent, so we may think of the groups $\mathscr{R}_\alpha$ as indexed by the points of* $\mathrm{PG}(1,q)$ *which in turn can be indexed by the elements of $\tilde{F}$.*

We use the elements of $\tilde{F}$ to index the points of $\mathrm{PG}(1,q)$ as follows: $\gamma_\infty = (0,1)$ and $\gamma_t = (1,t)$ for $t \in F$. So for $(0,0) \neq \gamma = (a,b)$, we have $\gamma \equiv \gamma_{b/a}$.

The scalar multiplication

$$d(\gamma \otimes \alpha, c) = (d\gamma \otimes \alpha, d^2 c) = (\gamma \otimes d\alpha, d^2 c) \tag{1}$$

turns each $\mathscr{R}_\alpha$ into a 3-dimensional vector space over F containing the center Z. Hence there are associated projective planes $\overline{\mathscr{R}}_\alpha$ isomorphic to $\mathrm{PG}(2,q)$. Since $g(d\alpha, t) = d^2 g(\alpha, t)$, we have

$$(\gamma_{t^{1/2}} \otimes d\alpha, g(d\alpha, t)) = d(\gamma_{t^{1/2}} \otimes \alpha, g(\alpha, t)),$$

and we see that

$$A(t) \cap \mathscr{R}_\alpha = \{d(\gamma_{t^{1/2}} \otimes \alpha, g(\alpha, t)) : d \in F\} \tag{2}$$

corresponds to a point of the plane $\overline{\mathscr{R}}_\alpha$. The elements of the GQ in such an equivalence class are called *o-points*. Moreover,

$$\overline{\mathscr{O}}_\alpha = \{A(t) \cap \mathscr{R}_\alpha : t \in \tilde{F}\} \tag{3}$$

is an oval in $\overline{\mathscr{R}}_\alpha$. (Note: $\infty^{1/2} = \infty$.)

Notice that $\overline{\mathscr{O}}_\alpha$ and $\overline{\mathscr{O}}_\beta$, where $\{\alpha, \beta\}$ is F-linearly independent, have no point in common, although they do have the same nucleus $\bar{Z}$.

For more details see the Subiaco Notebook [7].

According to the Fundamental Theorem of q-clan geometry (developed in [7] for q a power of 2), each collineation of $\mathscr{S}(\mathscr{C})$ that fixes the point $((\bar{0}, \bar{0}), 0)$ (and also the point (∞)) must be induced by an automorphism of the group G determined by a field automorphism $\sigma = 2^i$ and a 4×4 matrix that is the tensor product $A \otimes B$ of two 2×2 matrices

$$A = \begin{pmatrix} a_4 & a_2 \\ a_3 & a_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_4 & b_2 \\ b_3 & b_1 \end{pmatrix}.$$

Moreover, if $\theta = \theta(\sigma, A \otimes B)$ is such a collineation, then

$$\theta : [A(t)] \mapsto [A(\bar{t})], \quad \text{where } t \in \tilde{F} \text{ and } \bar{t} = \frac{a_1^2 t^\sigma + a_2^2}{a_3^2 t^\sigma + a_4^2}. \tag{4}$$

Analogously, the corresponding map on the ovals is given by

$$\theta : \overline{\mathcal{O}}_s \mapsto \overline{\mathcal{O}}_{\bar{s}}, \quad \text{where } s \in \tilde{F} \text{ and } \bar{s} = \frac{b_1^2 s^\sigma + b_2^2}{b_3^2 s^\sigma + b_4^2}. \tag{5}$$

Finally, we record the following useful formulas:

$$\theta(\sigma_1, A_1 \otimes B_1) \circ \theta(\sigma_2, A_2 \otimes B_2) = \theta(\sigma_1 \circ \sigma_2, A_1^{\sigma_2} A_2 \otimes B_1^{\sigma_2} B_2). \tag{6}$$

$$\theta(\sigma, A \otimes B)^{-1} = \theta(\sigma^{-1}, (A^{-1})^{\sigma^{-1}} \otimes (B^{-1})^{\sigma^{-1}}). \tag{7}$$

2.1 Preliminary notation and computations. Recall that we are assuming $q = 2^e$. Let $F = \mathrm{GF}(q) \subseteq \mathrm{GF}(q^2) = E$. Write $\bar{x} = x^q$ for $x \in E$. Thus $x = \bar{x}$ iff $x \in F$. Let λ be a primitive element of E, so the multiplicative order of λ is $|\lambda| = q^2 - 1$. Put $\beta = \lambda^{q-1}$, so $|\beta| = q + 1$. Then $\bar{\beta} = \beta^q = \beta^{-1}$. Define $\delta = \mathrm{Tr}_{E/F}(\beta) = \beta + \bar{\beta}$. More generally, for each rational number a with denominator relatively prime to $q + 1$ we use the following convenient notation for the relative trace function:

$$[a] := \beta^a + \bar{\beta}^a = \mathrm{Tr}_{E/F}(\beta^a). \tag{8}$$

Lemma 2.2. *The following have easy proofs*:

(1) $\delta = [1]$; $0 = [0]$.

(2) $[a] = [b]$ *iff* $a \equiv \pm b \pmod{q+1}$.

(3) $[a] \cdot [b] = [a+b] + [a-b]$; $[a] + [b] = \left[\frac{a+b}{2}\right] \cdot \left[\frac{a-b}{2}\right]$.

(4) $[a]^\sigma = [\sigma a]$ *for* $\sigma = 2^i \in \mathrm{Aut}(F)$.

(5) $\left[\frac{a+c}{2}\right]\left[\frac{a}{2}\right]\left[\frac{c}{2}\right] = [a+c] + [a] + [c]$.

(6) *The map* $\frac{[j+k]}{[j]} \mapsto \frac{[j+1+k]}{[j+1]}$, *for all* $j \in \mathbb{Z}_{q+1}$, *permutes the elements of* $\tilde{F}$ *in a cycle of length* $q + 1$. *If* $k = 1$, *the map* $\frac{[j+1]}{[j]} \mapsto \frac{[j+2]}{[j+1]}$, *for all* $j \in \mathbb{Z}_{q+1}$, *is the same as the map* $t \mapsto t^{-1} + \delta$. *More generally,* $\frac{[j+k]}{[j]} \mapsto \frac{[j+1+k]}{[j+1]}$ *is the map* $t \mapsto \frac{t[k+1]+[1]}{t[1]+[k-1]}$.

It will also be useful to have a generalization of the above notation. For each k, $1 \leqslant k \leqslant q$, put $\beta_k = \lambda^{k(q-1)}$, so $|\beta_k| = \frac{q+1}{\gcd(k,q+1)}$. Then $\bar{\beta}_k = \beta_k^q = \beta_k^{-1}$. Let $\delta_k = \beta_k + \bar{\beta}_k$ and in general define $[a]_k = \beta_k^a + \bar{\beta}_k^a$. When $k = 1$ we suppress the subscript. Properties of this generalized notation follow easily from Lemma 2.2 and the observation that $[a]_k = [ka]$.

Another notational device which we will find useful when discussing herds in terms of the q-clan functions is:

$$f_s^*(t) := g((1, s^{1/2}), t) = f(t) + s^{1/2}t^{1/2} + sg(t), \quad \text{for } s \in F, \tag{9}$$

and

$$f_\infty^*(t) := g(t) = g((0,1), t). \tag{10}$$

The matrix M of the following lemma will play a fundamental role in the sequel.

Lemma 2.3. *Let $M = \begin{pmatrix} 0 & 1 \\ 1 & \delta^{1/2} \end{pmatrix}$ with eigenvalues $\beta^{\pm(1/2)}$. Clearly $M = M^T$ and $\det(M) = 1$. The matrix M has multiplicative order $q+1$, since*

$$M^j = \frac{1}{\delta^{1/2}} \begin{pmatrix} \left[\frac{j-1}{2}\right] & \left[\frac{j}{2}\right] \\ \left[\frac{j}{2}\right] & \left[\frac{j+1}{2}\right] \end{pmatrix} \quad \textit{for all } j \textit{ modulo } q+1. \tag{11}$$

Furthermore, with $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$M^j P M^{-j} = \frac{1}{\delta^{1/2}} \begin{pmatrix} [j] & [j - \frac{1}{2}] \\ [j + \frac{1}{2}] & [j] \end{pmatrix}, \tag{12}$$

and it has unique eigenvector $\left(\left[\frac{2j+1}{4}\right], \left[\frac{2j-1}{4}\right]\right)$.

2.2 Some special linear collineations.

Lemma 2.4 ([9]). *The automorphism $\theta(\mathrm{id}, P \otimes P)$ of G induces a collineation of $\mathscr{S}(\mathscr{C})$ iff*

$$tf\left(\frac{1}{t}\right) = g(t) \quad \textit{and} \quad tg\left(\frac{1}{t}\right) = f(t). \tag{13}$$

When this condition holds, $\theta(\mathrm{id}, P \otimes P)$ is an involution fixing the line $[A(1)]$ and also fixing the oval $\overline{\mathscr{O}}_1$. More generally, $\theta : [A(t)] \mapsto [A(t^{-1})]$ and $\theta : \overline{\mathscr{O}}_t \mapsto \overline{\mathscr{O}}_{t^{-1}}$, for $t \in \tilde{F}$.

Let m be a positive integer (modulo $q+1$). We are interested in determining just when there is a collineation of $\mathscr{S}(\mathscr{C})$ of the form $\theta = \theta(\mathrm{id}, M \otimes M^{-m})$.

Lemma 2.5. *The automorphism* $\theta(\mathrm{id}, M \otimes M^{-m})$ *of* G *induces a collineation of* $\mathscr{S}(\mathscr{C})$ *if and only if*:

$$t(f(t^{-1}+\delta)+f(\delta)) = \frac{[m-1]}{\delta}\left\{ f(t) + \left(\frac{[m]}{[m-1]}\right)^{1/2} t^{1/2} + \frac{[m]}{[m-1]}g(t)\right\}$$

and

$$t(g(t^{-1}+\delta)+g(\delta)) = \frac{[m]}{\delta}\left\{ f(t) + \left(\frac{[m+1]}{[m]}\right)^{1/2} t^{1/2} + \frac{[m+1]}{[m]}g(t)\right\}.$$

Proof. Recall from the Subiaco Notebook [7] that as an automorphism of G, θ has the following effect:

$$\begin{aligned}\theta = \theta(\mathrm{id}, M\otimes M^{-m}):&\\ ((\alpha,\beta),c) \mapsto ((\alpha,\beta)(M\otimes M^{-m}), c+\alpha\circ\beta+\beta M^{-m}A_\delta M^{-m}\beta^T).&\end{aligned} \tag{14}$$

A little work shows that if $t = \frac{[k+1]}{[k]}$, then the typical element of $A(t)$ can be represented in the form:

$$\left(\left(\left[\frac{k}{2}\right], \left[\frac{k+1}{2}\right]\right)\otimes\alpha, [k]g(\alpha,t)\right), \quad \alpha\in F^2.$$

It follows that

$$\begin{aligned}\theta = \theta(\mathrm{id}, M\otimes M^{-m}) &: \left(\left(\left[\frac{k}{2}\right], \left[\frac{k+1}{2}\right]\right)\otimes\alpha, [k]g(\alpha,t)\right)\\ &\mapsto \left(\left(\left[\frac{k}{2}\right], \left[\frac{k+1}{2}\right]\right)M\otimes\alpha M^{-m}, [k]g(\alpha,t)+[k+1]\alpha M^{-m}A_\delta M^{-m}\alpha^T\right)\\ &= \left(\left(\left[\frac{k+1}{2}\right], \left[\frac{k+2}{2}\right]\right)\otimes\alpha M^{-m}, [k]g(\alpha,t)+[k+1]\alpha M^{-m}A_\delta M^{-m}\alpha^T\right).\end{aligned}$$

What does it mean for this to be in $A(\bar{t})$ where $\bar{t} = t^{-1}+\delta$? Since $t = \frac{[k+1]}{[k]}$, $\bar{t} = \frac{[k+2]}{[k+1]}$. So containment holds iff

$$\begin{aligned}&\left(\left(\left[\frac{k+1}{2}\right], \left[\frac{k+2}{2}\right]\right)\otimes\alpha M^{-m}, [k]g(\alpha,t)+[k+1]\alpha M^{-m}A_\delta M^{-m}\alpha^T\right)\\ &\quad= \left(\left(\left[\frac{k+1}{2}\right], \left[\frac{k+2}{2}\right]\right)\otimes\alpha M^{-m}, [k+1]g(\alpha M^{-m}, t^{-1}+\delta)\right)\end{aligned}$$

iff

$$\alpha\{[k+1]M^{-m}A_{t^{-1}+\delta}M^{-m}\}\alpha^T = \alpha\{[k]A_t + [k+1]M^{-m}A_\delta M^{-m}\}\alpha^T$$

iff

$$\frac{[k+1]}{[k]}A_{t^{-1}+\delta} \equiv M^m A_t M^m + \frac{[k+1]}{[k]}A_\delta \quad \text{for all } t \in F$$

iff

$$M^m A_t M^m \equiv t(A_{t^{-1}+\delta} + A_\delta).$$

A routine computation shows that

$$M^m A_t M^m \equiv \frac{1}{\delta}\begin{pmatrix} [m-1]f^*_{[m]/[m-1]}(t) & \delta t^{1/2} \\ 0 & [m]f^*_{[m+1]/[m]}(t) \end{pmatrix}. \tag{15}$$

From this the statement follows. □

Theorem 2.6. *If $\theta = \theta(\mathrm{id}, M \otimes M^{-m})$ is a collineation of $\mathcal{S}(\mathcal{C})$ then θ has order $q+1$ on the lines of $\mathcal{S}(\mathcal{C})$ through the point (∞). For each integer j modulo $q+1$ we have the following*:

(1) $\theta^j = \theta(\mathrm{id}, M^j \otimes M^{-mj}) : [A(t)] \mapsto [A(\bar{t})]$, *where* $\bar{t} = \frac{[j+1]t+[j]}{[j]t+[j-1]}$. *If* $t = \frac{[k+1]}{[k]}$, *then* $\bar{t} = \frac{[j+k+1]}{[j+k]} = \frac{(1+\delta s)t+s}{st+1}$, *where* $s = \frac{[j]}{[j-1]}$.

(2) $\theta : \overline{\mathcal{O}}_t \mapsto \overline{\mathcal{O}}_{\bar{t}}$ *where* $\bar{t} = \frac{[mj-1]t+[mj]}{[mj]t+[mj+1]}$. *If* $t = \frac{[k+1]}{[k]}$, *then* $\bar{t} = \frac{[k-mj+1]}{[k-mj]} = \frac{(1+\delta r)t+r}{rt+1}$, *where* $r = \frac{[mj]}{[mj+1]}$.

From now on we assume that the automorphisms of $G^{\otimes}$ given by $\theta = \theta(\mathrm{id}, M \otimes M^{-m})$ and $\varphi = \theta(\mathrm{id}, P \otimes P)$ really do induce collineations of $\mathcal{S}(\mathcal{C})$. Recall that φ is an involution interchanging $[A(t)]$ and $[A(t^{-1})]$ and interchanging $\overline{\mathcal{O}}_s$ and $\overline{\mathcal{O}}_{s^{-1}}$, for $t, s \in \tilde{F}$.

Theorem 2.7. *Let $\theta_j = \theta^j \circ \varphi \circ \theta^{-j} = \theta(\mathrm{id}, M^j P M^{-j} \otimes M^{-mj} P M^{mj})$. Then θ_j is an involution fixing*

$$[A(s)] \text{ with } s = \frac{\left[j-\frac{1}{2}\right]}{\left[j+\frac{1}{2}\right]},$$

and fixing

$$\overline{\mathcal{O}}_r \text{ with } r = \frac{\left[mj+\frac{1}{2}\right]}{\left[mj-\frac{1}{2}\right]}.$$

In general $\theta_j : [A(t)] \mapsto [A(\bar{t})]$ *with*

$$\bar{t} = \frac{[2j]t + [2j-1]}{[2j+1]t + [2j]} = \frac{t(s^2+1) + \delta s^2}{t\delta + s^2 + 1}.$$

Similarly, $\theta_j : \overline{\mathcal{O}}_t \mapsto \overline{\mathcal{O}}_{\bar{t}}$ *with*

$$\bar{t} = \frac{[2mj]t + [2mj+1]}{[2mj-1]t + [2mj]} = \frac{t(r^2+1) + \delta r^2}{t\delta + r^2 + 1}.$$

2.3 Some semi-linear collineations. Assume that $\mathscr{C}$ is a given q-clan for which the conditions in Lemmas 2.4 and 2.5 both hold. In the nonclassical case, each line through (∞) (resp., each oval $\overline{\mathcal{O}}_t$) is fixed by a *unique* involution. In the classical case this is not true. However, in the computations below we assume that an involution that turns up which fixes $[A(\infty)]$ is the one we know about. Then we ask for necessary and sufficient conditions on the functions f and g that give $\mathscr{C}$ so that there be a collineation of $\mathscr{S}(\mathscr{C})$ that fixes the line $[A(\infty)]$ and the points (∞) and $((\bar{0}, \bar{0}), 0)$ and belongs to the field automorphism $\sigma = 2$. We state the result here and the interested reader can find the details of the proof in [1] or [8].

Theorem 2.8. *There is a collineation of* $\mathscr{S}(\mathscr{C})$ *that fixes the line* $[A(\infty)]$ *and the two points* (∞) *and* $((\bar{0}), (\bar{0}), 0)$ *and which is semilinear with associated field automorphism* $\sigma = 2$ *if and only if the following two equations hold*:

$$f(\delta^{-1}t^2 + \delta^{-1}) + f(\delta^{-1}) = \frac{1}{\delta^2}\left(\left[\frac{m+3}{2}\right] f(t)^2 + \left[\frac{m+3}{4}\right]\left[\frac{m-1}{4}\right] t + \left[\frac{m-1}{2}\right] g(t)^2\right); \qquad (16)$$

$$g(\delta^{-1}t^2 + \delta^{-1}) + g(\delta^{-1}) = \frac{1}{\delta^2}\left(\left[\frac{m+1}{2}\right] f(t)^2 + \left[\frac{m+1}{4}\right]\left[\frac{m-3}{4}\right] t + \left[\frac{m-3}{2}\right] g(t)^2\right). \qquad (17)$$

This result is based on the following lemma.

Lemma 2.9. *If there is a collineation of* $\mathscr{S}(\mathscr{C})$ *that fixes the line* $[A(\infty)]$ *and the two points* (∞) *and* $((\bar{0}), (\bar{0}), 0)$ *and which is semilinear with associated field automorphism* $\sigma = 2$, *there are two possibilities*: $[A(0)]$ *is mapped to* $[A(t)]$ *with* $t = \delta^{-1}$ *or* $t = \delta^{-1} + \delta$. *If the first case holds, then the collineation must be the following*:

$$\theta\left(2, \frac{1}{\delta^{1/2}}\begin{pmatrix} \delta^{1/2} & 1 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\delta^{1/2}}\begin{pmatrix} \frac{\left[\frac{m-3}{4}\right]}{\left[\frac{1}{4}\right]} & \frac{\left[\frac{m-1}{4}\right]}{\left[\frac{1}{4}\right]} \\ \frac{\left[\frac{m+1}{4}\right]}{\left[\frac{1}{4}\right]} & \frac{\left[\frac{m+3}{4}\right]}{\left[\frac{1}{4}\right]} \end{pmatrix}\right).$$

The two equations of Theorem 2.8 hold if and only if the collineation of Lemma 2.9 actually is a collineation moving $[A(0)]$ to $[A(\delta^{-1})]$. By composing with the involution $I_\infty = \theta_{-1/2}$ which maps $[A(t)]$ to $[A(t+\delta)]$, we see that both possible collineations of Lemma 2.9 exist or both fail to exist.

The order of the collineation given in Lemma 2.9 can be determined, and the group it generates is $\tilde{G} \cong C_{q+1} \rtimes C_{2e}$. Moreover, by L. E. Dickson we know that $C_{q+1} \rtimes C_{2e}$ is a maximal subgroup of $\mathrm{P\Gamma L}(2,q)$.

3 Some cyclic q-clans

3.1 The unified construction of [4]. As before, λ is a primitive element of $E = \mathrm{GF}(q^2)$, $q = 2^e$, and let $\beta_k = \lambda^{k(q-1)}$, where $1 \leqslant k \leqslant q$. Recall our $[\dots]_k$ notation and especially, $\delta_k = \beta_k + \bar{\beta}_k$. Write $T(x) = x + \bar{x}$ for $x \in E$, $a(t) = \beta_k^{1/2} t + \bar{\beta}_k^{1/2}$, $v(t) = t + (\delta_k t)^{1/2} + 1$, and let $\mathrm{tr} : F \to \mathrm{GF}(2)$ be the absolute trace function.

Theorem 3.1 (W. E. Cherowitzo, C. M. O'Keefe and T. Penttila [4]). *Let m and k be nonzero residues modulo $q+1$, where $q+1$ does not divide km, such that*

$$\mathrm{tr}\left(\frac{[m]_k}{[1]_k}\right) \equiv 1 \pmod 2. \tag{18}$$

Let

$$F(t) = \frac{\left[\frac{m}{2}\right]_k}{\left[\frac{1}{2}\right]_k}(t+1) + \frac{T(a(t)^m)}{\left[\frac{1}{2}\right]_k (v(t))^{m-1}} + t^{1/2}. \tag{19}$$

$$G(t) = \frac{\left[\frac{m}{2}\right]_k}{\left[\frac{1}{2}\right]_k} t + \frac{T((\beta_k^{1/2} a(t))^m)}{\left[\frac{1}{2}\right]_k \left[\frac{m}{2}\right]_k (v(t))^{m-1}} + \frac{t^{1/2}}{\left[\frac{m}{2}\right]_k}. \tag{20}$$

Then $\mathscr{C}^{(k)} = \left\{ A_t = \begin{pmatrix} F(t) & t^{1/2} \\ 0 & G(t) \end{pmatrix} : t \in F \right\}$ *is a (normalized) q-clan.*

To see the connection with reference [4], replace their f with F, their ag with G, and their β with $\beta^{1/2}$. Notice that if either k or m is relatively prime to $q+1$, the condition that $q+1$ does not divide km is satisfied. In particular, we see that $k=1$ will always satisfy the condition. Then we have the following remarkable theorem.

Theorem 3.2 ([4]). *Four infinite families of q-clans arise*:

(1) *If $m \equiv \pm 1 \pmod{q+1}$, then $\mathscr{C}^{(k)}$ is the classical q-clan for all $q = 2^e$ and all k.* $(\mathscr{S}(\mathscr{C}^{(k)}) \cong H(3,q^2))$

(2) *If $m \equiv \pm\frac{q}{2} \pmod{q+1}$ and e is odd, then $\mathscr{C}^{(k)}$ is the FTWKB q-clan for all k, i.e., equivalent to having* $A_t = \begin{pmatrix} t^{1/4} & t^{2/4} \\ 0 & t^{3/4} \end{pmatrix}$.

(3) *If* $m \equiv \pm 5 \pmod{q+1}$, *then* $\mathscr{C}^{(k)}$ *is the Subiaco q-clan for all* $q = 2^e$. (*Note that when* $e \equiv 2 \pmod 4$, m *divides* $q+1$ *and not all values of* k *will satisfy the condition.*)

(4) *If* $m \equiv \pm\left(\frac{q-1}{3}\right) \pmod{q+1}$ *with e even, then for all k* $\mathscr{C}^{(k)}$ *is a new q-clan called an Adelaide q-clan.*

For small q there is some overlap in these four families, but for $q > 16$ these families are almost certainly disjoint.

Of course we would like to know that the Adelaide q-clans really do contain the cyclic q-clans obtained by S. E. Payne, T. Penttila and G. Royle [9] (with the help of a computer), for $q = 4^e$, $e \leqslant 8$. We will first show that the new q-clans are cyclic.

For this purpose it will be convenient to modify the description of the q-clans in the unified construction. Put $D = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a = \left(\frac{[m+1]}{[m][1]}\right)^{1/4}$, $b = d = 1$, and $c = \left(\frac{[m-1]}{[m][1]}\right)^{1/4}$. Let

$$\hat{D} = \begin{pmatrix} a^2 & ab & b^2 & 0 \\ 0 & 1 & 0 & 0 \\ c^2 & cd & d^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define f_k and g_k by

$$\begin{pmatrix} f_k(t) \\ t^{1/2} \\ g_k(t) \\ 1 \end{pmatrix} = \hat{D} \cdot \begin{pmatrix} F(t) \\ t^{1/2} \\ G(t) \\ 1 \end{pmatrix}.$$

After routine computation we find that (suppressing all k's in the notation for legibility)

$$f(t) = \frac{\left[\frac{m+1}{2}\right]}{[1]} + \frac{\left[\frac{m-1}{2}\right]}{[1]} t + \frac{\left[\frac{m+1}{2}\right] T(a(t)^m) + \left[\frac{1}{2}\right] T((\bar{\beta}^{1/2} a(t))^m)}{[1]\left[\frac{m}{2}\right] v(t)^{m-1}} + \left(\frac{t}{\delta}\right)^{1/2}$$

and

$$g(t) = \frac{\left[\frac{m-1}{2}\right]}{[1]} + \frac{\left[\frac{m+1}{2}\right]}{[1]} t + \frac{\left[\frac{m-1}{2}\right] T(a(t)^m) + \left[\frac{1}{2}\right] T((\beta^{1/2} a(t))^m)}{[1]\left[\frac{m}{2}\right] v(t)^{m-1}} + \left(\frac{t}{\delta}\right)^{1/2}.$$

With a little more routine computation we can show that

$$f(t) = f_k(t) = \frac{\left[\frac{m+1}{2}\right]_k}{[1]_k} + \frac{\left[\frac{m-1}{2}\right]_k}{[1]_k} t + \frac{T(a(t)^m \bar{\beta}_k^{1/2})}{[1]_k v(t)^{m-1}} + \left(\frac{t}{\delta_k}\right)^{1/2}, \tag{21}$$

and

$$g(t) = g_k(t) = \frac{\left[\frac{m-1}{2}\right]_k}{[1]_k} + \frac{\left[\frac{m+1}{2}\right]_k}{[1]_k} t + \frac{T(a(t)^m \beta_k^{1/2})}{[1]_k v(t)^{m-1}} + \left(\frac{t}{\delta_k}\right)^{1/2}. \tag{22}$$

At this point we would like to express $f_k(t)$ and $g_k(t)$ in terms of the square bracket function. This involves representing $t \in \tilde{F}$ with square brackets. The choice of $t = \frac{[j+1]_k}{[j]_k}$ for some j modulo $q+1$, while tempting, only works when $\gcd(k, q+1) = 1$. To obtain formulas that are valid for all k we use $t = \frac{[j+k]}{[j]}$ for all $j \in \mathbb{Z}_{q+1}$ (see Lemma 2.2(6).) Our results will be written in terms of $[\ldots]_1$.

Lemma 3.3. *For any positive $k \in \mathbb{Z}_{q+1}$*

$$f_k(t) = f_k\left(\frac{[j+k]}{[j]}\right) = \frac{\left[(k+j)\left(\frac{m-1}{2}\right)\right]\left[j\left(\frac{m+1}{2}\right)\right]}{[j]} + \left(\frac{[j+k]}{[j][k]}\right)^{1/2} \tag{23}$$

and

$$g_k(t) = g_k\left(\frac{[j+k]}{[j]}\right) = \frac{\left[(j+k)\left(\frac{m+1}{2}\right)\right]\left[j\left(\frac{m-1}{2}\right)\right]}{[j]} + \left(\frac{[j+k]}{[j][k]}\right)^{1/2}. \tag{24}$$

Proof. The details, although somewhat routine, are rather lengthy. Since they are similar for f_k and g_k we just barely sketch them and only for f_k.

Observe that $v(t) = t + (\delta_k t)^{1/2} + 1 = (t + \beta_k)^{1/2}(t + \bar{\beta}_k)^{1/2}$. Then, considering Equation 21, first rewrite the expression

$$\begin{aligned}\frac{T(a(t)^m \bar{\beta}_k^{1/2})}{[k]v(t)^{m-1}} &= \frac{(\beta_k^{1/2} t + \bar{\beta}_k^{1/2})^m \bar{\beta}_k^{1/2} + (\bar{\beta}_k^{1/2} t + \beta_k^{1/2})^m \beta_k^{1/2}}{[k](t+\beta_k)^{(m-1)/2}(t+\bar{\beta}_k)^{(m-1)/2}} \\ &= \frac{\beta_k^{(m-1)/2}(t+\bar{\beta}_k)^{(m+1)/2}}{[k](t+\beta_k)^{(m-1)/2}} + \frac{\bar{\beta}_k^{(m-1)/2}(t+\beta_k)^{(m+1)/2}}{[k](t+\bar{\beta}_k)^{(m-1)/2}} \\ &= \frac{\beta_k^{(m-1)/2}(t+\bar{\beta}_k)}{[k]}\left(\frac{t+\bar{\beta}_k}{t+\beta_k}\right)^{(m-1)/2} + \frac{\bar{\beta}_k^{(m-1)/2}(t+\beta_k)}{[k]}\left(\frac{t+\beta_k}{t+\bar{\beta}_k}\right)^{(m-1)/2}.\end{aligned}$$

It is routine to show that for $t = \frac{[j+k]}{[j]}$, $\frac{t+\bar{\beta}_k}{t+\beta_k} = \beta^{2j}$ and $\frac{t+\beta_k}{t+\bar{\beta}_k} = \bar{\beta}_1^{2j}$. Then after a few steps

$$\frac{T(a(t)^m \bar{\beta}_k^{1/2})}{[k]v(t)^{m-1}} = \frac{\left[jm + k\left(\frac{m-1}{2}\right)\right]}{[j]}.$$

Now return to Equation 21 and put the pieces together over a common denominator (except for the piece involving $t^{1/2}$), using the fact that a product of brackets is easily written as a sum, and vice versa, to obtain the desired result. □

3.2 Collineations for the unified construction. The following theorem shows that every GQ arising from the unified construction is cyclic in the sense of [9] i.e., it admits a collineation $\theta(\mathrm{id}, M \otimes M^{-m})$ which permutes the lines through (∞) in a cycle of length $q+1$.

Theorem 3.4. *The q-clan functions given by Equations* 21 *and* 22 *satisfy all six conditions of Lemmas* 2.4 *and* 2.5 *and Theorem* 2.8.

Proof. The conditions in Lemma 2.4 are easy to check. Those of Lemma 2.5 are routine but more tedious and similar to each other. Here we give the details for the first one. Verifying the two conditions of Theorem 2.8 is also routine, but very lengthy. The interested reader can again find the details of these computations in [7] or [8].

For the first condition of Lemma 2.5 we need to show that

$$\begin{aligned}&[1]_k t\{f_k(t^{-1}+\delta_k) + f_k(\delta_k)\}\\ &\quad = [m-1]_k f_k(t) + \left[\frac{m-1}{2}\right]_k \left[\frac{m}{2}\right]_k t^{1/2} + [m]_k g_k(t).\end{aligned} \tag{25}$$

Throughout this computation we shall suppress the k's of the notation for clarity. Start with the left hand side and do some special pieces of the computation first: $a(\delta)^m \bar{\beta}^{1/2} = \beta^{(3m-1)/2}$; $v(\delta) = 1$; $\mathrm{Tr}(a(\delta)^m \bar{\beta}^{1/2}) = \left[\frac{3m-1}{2}\right]$. So $[1]f(\delta) = \left[\frac{m+1}{2}\right] + \left[\frac{m-1}{2}\right][1] + \left[\frac{3m-1}{2}\right] + 1$.

Next, compute $v(t^{-1}+\delta) = v(t)t^{-1}$; $a(t^{-1}+\delta) = \beta a(t)t^{-1}$, so

$$\frac{T(a(t^{-1}+\delta)^m \bar{\beta}^{1/2})}{v(t^{-1}+\delta)^{m-1}} = \frac{T(a(t)^m \beta^{m-1/2})}{tv(t)^{m-1}}.$$

Then with several terms cancelling we find:

$$\begin{aligned}&[1]t\{f(t^{-1}+\delta) + f(\delta)\}\\ &\quad = \left[\frac{m-1}{2}\right] + \left[\frac{3m-1}{2}\right]t + \frac{T(a(t)^m \beta^{m-1/2})}{v(t)^{m-1}} + (t\delta)^{1/2}.\end{aligned} \tag{26}$$

Now we begin with the right hand side of Equation 25 which is

$$\frac{[m-1]}{[1]}\left\{\left[\frac{m+1}{2}\right]+\left[\frac{m-1}{2}\right]t+\frac{T(a(t)^m\bar{\beta}^{1/2})}{v(t)^{m-1}}+(t\delta)^{1/2}\right\}+\left[\frac{m-1}{2}\right]\left[\frac{m}{2}\right]t^{1/2}$$
$$+\frac{[m]}{[1]}\left\{\left[\frac{m-1}{2}\right]+\left[\frac{m+1}{2}\right]t+\frac{T(a(t)^m\beta^{1/2})}{v(t)^{m-1}}+(t\delta)^{1/2}\right\}.$$

Consider the summands one at a time. First, the constant term equals

$$\frac{\left[\frac{2m-2}{2}\right]\left[\frac{m+1}{2}\right]+\left[\frac{2m}{2}\right]\left[\frac{m-1}{2}\right]}{[1]}=\left[\frac{m-1}{2}\right].$$

The coefficient of t equals

$$\frac{\left[\frac{2m-2}{2}\right]\left[\frac{m-1}{2}\right]+\left[\frac{2m}{2}\right]\left[\frac{m+1}{2}\right]}{[1]}=\left[\frac{3m-1}{2}\right].$$

The coefficient of $t^{1/2}$ is

$$\frac{[m-1]+\left[\frac{m-1}{2}\right]\left[\frac{m}{2}\right]\left[\frac{1}{2}\right]+[m]}{\left[\frac{1}{2}\right]}=\left[\frac{1}{2}\right]=\delta^{1/2}.$$

Finally, we consider the term

$$\begin{aligned}&\frac{[m-1]T(a(t)^m\bar{\beta}^{1/2})+[m]T(a(t)^m\beta^{1/2})}{[1]v(t)^{m-1}}\\&=\frac{T\{a(t)^m(\beta^{m-1}+\bar{\beta}^{m-1})\bar{\beta}^{1/2}+a(t)^m(\beta^m(\beta^m+\bar{\beta}^m))\beta^{1/2}\}}{[1]v(t)^{m-1}}\\&=\frac{T(a(t)^m\beta^{m-1/2})}{v(t)^{m-1}}.\end{aligned}$$

When we compare this with Equation 26, we see that we have equality. □

4 The isomorphism theorem

The following theorem shows that the GQ's obtained from the unified construction for different values of k are isomorphic.

Theorem 4.1. *Let* $\mu=\left(\frac{[k]}{[1]}\right)^{1/2}=\det(A)$ *where we put*

$$A=\begin{pmatrix}1 & \left(\frac{[k-1]}{[1]}\right)^{1/2}\\ 0 & \left(\frac{[k]}{[1]}\right)^{1/2}\end{pmatrix}\quad \textit{and}\quad B=\frac{1}{\left[\frac{1}{4}\right]\left[\frac{k}{4}\right]}\begin{pmatrix}\left[\frac{(k-1)(m-1)+2k}{4}\right] & \left[\frac{(k-1)(m-1)}{4}\right]\\ \left[\frac{(k-1)(m+1)}{4}\right] & \left[\frac{(k-1)(m-1)-2}{4}\right]\end{pmatrix}.$$

Then $\theta(\mathrm{id}, A \otimes B)$ *is an isomorphism from* $GQ(\mathscr{C}^{(1)})$ *to* $GQ(\mathscr{C}^{(k)})$ *mapping* $[A(\infty)]$ *to* $[A(\infty)]_k$ *and in general mapping* $[A(t)]$ *to* $[A(\bar{t})]_k$ *where* $\bar{t} = \frac{[k]}{[1]}t + \frac{[k-1]}{[1]}$. *In particular, if* $t = \frac{[j+1]}{[j]}$, *then* $\bar{t} = \frac{[j+k]}{[j]}$.

Proof. By the Fundamental Theorem of [7] we must show that if

$$A'_t = \begin{pmatrix} f_k(t) & t^{1/2} \\ 0 & g_k(t) \end{pmatrix},$$

and A_t is the corresponding q-clan matrix when $k = 1$, then

$$A'_{\bar{t}} + A'_{\bar{0}} \equiv \mu B^{-1} A_t B^{-T}. \tag{27}$$

First check that $\det(B) = 1$, so that

$$B^{-1} = \frac{1}{\left[\frac{1}{4}\right]\left[\frac{k}{4}\right]} \begin{pmatrix} \left[\frac{(k-1)(m-1)-2}{4}\right] & \left[\frac{(k-1)(m-1)}{4}\right] \\ \left[\frac{(k-1)(m+1)}{4}\right] & \left[\frac{(k-1)(m-1)+2k}{4}\right] \end{pmatrix}.$$

Using Equation 1.1 of [7] we easily compute that $\mu B^{-1} A_t B^{-T} \equiv \frac{1}{[1]}\begin{pmatrix} x & y \\ z & w \end{pmatrix}$, where

$$\begin{aligned} x &= \left[\frac{(k-1)(m-1)-2}{2}\right] f(t) + \left[\frac{(k-1)(m-1)-2}{4}\right]\left[\frac{(k-1)(m-1)}{4}\right] t^{1/2} \\ &\quad + \left[\frac{(k-1)(m-1)}{2}\right] g(t), \\ y &= \left[\frac{k}{2}\right]\left[\frac{1}{2}\right] t^{1/2}, \\ z &= 0, \\ w &= \left[\frac{(k-1)(m+1)}{2}\right] f(t) + \left[\frac{(k-1)(m+1)}{4}\right]\left[\frac{(k-1)(m-1)+2k}{4}\right] t^{1/2} \\ &\quad + \left[\frac{(k-1)(m-1)+2k}{2}\right] g(t). \end{aligned}$$

It is quite easy to check that the entry in the $(1,2)$ position on left hand side of Equation 27 equals the entry in the $(1,2)$ position on the right hand side. It is not so easy to check the entries in the $(1,1)$ positions and the $(2,2)$ positions, but the details are quite similar for the two positions. We give fairly complete details for the $(1,1)$ positions and leave the details for the other case as an exercise for the reader.

First consider the left hand side. So

$$\begin{aligned}
f_k(\bar{t}) + f_k(\bar{0}) &= f_k\left(\frac{[j+k]}{[j]}\right) + f_k\left(\frac{[-1+k]}{[-1]}\right) \\
&= \frac{\left[(k+j)\left(\frac{m-1}{2}\right)\right]\left[j\left(\frac{m+1}{2}\right)\right]}{[j]} + \left(\frac{[j+k]}{[j][k]}\right)^{1/2} \\
&\quad + \frac{\left[(k-1)\left(\frac{m-1}{2}\right)\right]\left[\frac{m+1}{2}\right]}{[1]} + \left(\frac{[k-1]}{[1][k]}\right)^{1/2} \\
&= \frac{\left[(k+j)\left(\frac{m-1}{2}\right)\right]\left[j\left(\frac{m+1}{2}\right)\right][1] + \left[(k-1)\left(\frac{m-1}{2}\right)\right]\left[\frac{m+1}{2}\right][j]}{[j][1]} \\
&\quad + \left(\frac{1}{[k]}\left(\frac{[j+k][1]+[k-1][j]}{[j][1]}\right)\right)^{1/2} \\
&= \frac{\left[k\left(\frac{m-1}{2}\right)+jm+1\right] + \left[k\left(\frac{m-1}{2}\right)+jm-1\right]}{[j][1]} \\
&\quad + \frac{\left[k\left(\frac{m-1}{2}\right)\right][j+1] + \left[k\left(\frac{m-1}{2}\right)-m+j\right] + \left[k\left(\frac{m-1}{2}\right)-m-j\right]}{[j][1]} \\
&\quad + \left(\frac{[j+1]}{[j][1]}\right)^{1/2}.
\end{aligned}$$

Now for the entry x in the $(1,1)$ position of the right hand side we obtain

$$\begin{aligned}
&\frac{1}{[1]}\left[\frac{(k-1)(m-1)-2}{2}\right]\left\{\frac{\left[(j+1)\left(\frac{m-1}{2}\right)\right]\left[j\left(\frac{m+1}{2}\right)\right]}{[j]} + \left(\frac{[j+1]}{[j][1]}\right)^{1/2}\right\} \\
&\quad + \frac{\left[\frac{(k-1)(m-1)-2}{4}\right]\left[\frac{(k-1)(m-1)}{4}\right]}{[1]}\left(\frac{[j+1]}{[j]}\right)^{1/2} \\
&\quad + \frac{\left[\frac{(k-1)(m-1)}{2}\right]}{[1]}\left\{\frac{\left[(j+1)\left(\frac{m+1}{2}\right)\right]\left[j\left(\frac{m-1}{2}\right)\right]}{[j]} + \left(\frac{[j+1]}{[j][1]}\right)^{1/2}\right\} \\
&= \frac{\left[\frac{(k-1)(m-1)-2}{2}\right]\left[(j+1)\left(\frac{m-1}{2}\right)\right]\left[j\left(\frac{m+1}{2}\right)\right]}{[1][j]} \\
&\quad + \frac{\left[\frac{(k-1)(m-1)}{2}\right]\left[(j+1)\left(\frac{m+1}{2}\right)\right]\left[j\left(\frac{m-1}{2}\right)\right]}{[1][j]} \\
&\quad + W\left\{\frac{\left[\frac{(k-1)(m-1)-2}{2}\right] + \left[\frac{(k-1)(m-1)-2}{4}\right]\left[\frac{(k-1)(m-1)}{4}\right]\left[\frac{1}{2}\right] + \left[\frac{(k-1)(m-1)}{2}\right]}{[1]}\right\},
\end{aligned}$$

where $W = \left(\frac{[j+1]}{[j][1]}\right)^{1/2}$. Consider the coefficient of W. If we write $a = \frac{(k-1)(m-1)}{2}$, then this coefficient has the form $\frac{[a-1]+\left[\frac{a-1}{2}\right]\left[\frac{a}{2}\right]\left[\frac{-1}{2}\right]+[a]}{[1]} = 1$, which equals the corresponding coefficient on the left hand side. For the remaining terms, continue to use $a = \frac{(k-1)(m-1)}{2}$, so $\frac{k(m-1)}{2} = a + \frac{m}{2} - \frac{1}{2}$. Expand out the products on the right hand side and multiply through by the denominator $[1][j]$. Two terms cancel, leaving the following six terms:

$$\left[a+\frac{m}{2}+jm+\frac{1}{2}\right]+\left[a+\frac{m}{2}+jm-\frac{3}{2}\right]+\left[a+\frac{m}{2}-j-\frac{3}{2}\right]$$
$$+\left[a+\frac{m}{2}+j+\frac{1}{2}\right]+\left[a-\frac{m}{2}-j-\frac{1}{2}\right]+\left[a-\frac{m}{2}+j-\frac{1}{2}\right].$$

When the terms on the left hand side above are expanded in the same fashion, exactly the same terms appear, completing this part of the proof. □

5 The herd cover and the magic action of [6]

5.1 Definition of the herd cover. Recall the planes $\overline{\mathscr{R}}_\alpha$ and their ovals

$$\overline{\mathcal{O}}_\alpha = \{\langle(\gamma_{t^{1/2}} \otimes \alpha, g(\alpha, t))\rangle_1 : t \in \tilde{F}\},$$

where $\langle\ldots\rangle_1$ denotes an equivalence class with respect to the scalar multiplication in the GQ.

For each $\lambda \in F^*$ and a fixed $\alpha \neq (0,0)$ the mapping $\pi_{\lambda\alpha} : \overline{\mathscr{R}}_\alpha \to \mathrm{PG}(2,q) : \langle(\gamma \otimes \lambda\alpha, \lambda^2 c)\rangle_1 \mapsto \langle(\gamma^{(2)}, \lambda^2 c)\rangle_2$ is an isomorphism of planes, where $\langle\ldots\rangle_2$ denotes an equivalence class of the scalar multiplication in the vector space underlying $\mathrm{PG}(2,q)$. $\pi_{\lambda\alpha}$ maps $\overline{\mathcal{O}}_\alpha$ to the oval

$$\begin{aligned}\mathcal{O}_{\lambda\alpha} &= \{(\gamma_t, g(\lambda\alpha, t)) : t \in \tilde{F}\}\\ &= \{(1, t, \lambda^2[a^2 f(t) + abt^{1/2} + b^2 g(t)]) : t \in F\} \cup \{(0,1,0)\}\end{aligned} \tag{28}$$

provided that $\alpha = (a,b) \neq (0,0)$. Unfortunately, $\pi_{\lambda\alpha} \neq \pi_{\nu\alpha}$ if $\lambda \neq \nu$. This leads to the situation where, corresponding to the oval $\overline{\mathcal{O}}_\alpha = \overline{\mathcal{O}}_{\lambda\alpha}$ in $\overline{\mathscr{R}}_\alpha$ there are $q-1$ projectively equivalent ovals in $\mathrm{PG}(2,q)$, namely, $\{\mathcal{O}_{\lambda\alpha} : \lambda \in F^*\}$. There are two approaches that we may take in working with this correspondence. We could associate the oval $\overline{\mathcal{O}}_\alpha$ with the set $\{\mathcal{O}_{\lambda\alpha} : \lambda \in F^*\}$, which we refer to as an *oval cover*, or we can select a representative from the set to associate with $\overline{\mathcal{O}}_\alpha$ (this is referred to as *normalizing*). The first approach more accurately reflects the relations in the GQ, but the second approach is easier to deal with computationally.

For the remainder of this paper we shall adopt the following convention. α will always indicate an element of $\mathrm{PG}(1,q)$ of the form $(1, s^{1/2})$, $s \in F$ or $(0,1)$. In particular, we will always have $\alpha \neq (0,0)$. An arbitrary element of F^2 will be denoted by $\lambda\alpha$ with $\lambda \in F$.

Normalization is equivalent to choosing, for each $\alpha \in \mathrm{PG}(1,q)$, one specific value $\lambda_\alpha \in F^*$ and using only the isomorphisms $\pi_{\lambda_\alpha \alpha}$. The set of $q+1$ ovals of $\mathrm{PG}(2,q)$ obtained in this way is called a *herd of ovals* corresponding to the q-clan $\mathscr{C}$. A priori there appears to be no reason to choose one normalization over another. In fact, from the literature it can be seen that one of the authors is fond of the normalization $\lambda_\alpha = 1$, for all α, while the other author has often chosen λ_α so that $g(\lambda_\alpha \alpha, 1) = 1$, for all α. These different normalizations produce different sets of ovals to be called herds of ovals for the same q-clan. From this point of view, a herd of ovals, depending as it does on the normalization, is not a fundamental object. Furthermore, as we shall see in the following sections, normalization can have a confounding effect on the computation of automorphism groups.

For these reasons we prefer to take the first approach and define the *herd cover*, $\mathscr{H}(\mathscr{C})$, of the q-clan $\mathscr{C}$ as the set of $q+1$ oval covers in $\mathrm{PG}(2,q)$ associated to the ovals $\overline{\mathscr{O}}_\alpha$ of the planes $\overline{\mathscr{R}}_\alpha$. That is,

$$\mathscr{H}(\mathscr{C}) = \{[\overline{\mathscr{O}}_\alpha] : \alpha \in \mathrm{PG}(1,q)\} \tag{29}$$

where

$$[\overline{\mathscr{O}}_\alpha] = \{\mathscr{O}_{\lambda\alpha} : \lambda \in F^*\}. \tag{30}$$

This definition is consistent with the terminology introduced in [2] for a more general setting provided the indices α are interpreted as points of a conic in $\mathrm{PG}(2,q)$. In the current context this distinction is not relevant.

An *o-permutation for* $\mathrm{PG}(2,q)$ is a permutation polynomial f over $F = \mathrm{GF}(q)$ of degree at most $q-2$ with $f(0) = 0$ satisfying the condition that $v(x) = \frac{f(x+t)+f(t)}{x}$, $x \neq 0$, $v(0) = 0$ is a permutation for each $t \in F$. The set of points

$$\{(1,t,f(t)) : t \in F\} \cup \{(0,1,0)\}$$

forms an oval in $\mathrm{PG}(2,q)$ passing through the origin and having nucleus $(0,0,1)$ if and only if f is an o-permutation. If $f(1) = 1$ then an o-permutation f is called an *o-polynomial*.

It was first proved in [3] that if there are o-permutations f and g with $f(0) = g(0) = 0$ for which $f_s^*(t) = f(t) + (st)^{1/2} + sg(t)$ is an o-permutation for all $s \in F$, then

$$\mathscr{C} = \left\{ A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix} : t \in F \right\}$$

is a q-clan. We point out the obvious statement that for $s \in F$, $f_s^*(t)$ is an o-permutation if and only if $\lambda f_s^*(t)$ is an o-permutation for all $\lambda \in F^*$.

Since herd covers and q-clans are equivalent objects in some sense, there ought to be a close connection between the automorphism group of the herd cover and the automorphism group of the associated GQ. Since the members of a herd cover are the projections of the oval sets which live in $G^{\otimes}$, there also ought to be a direct connection with $\mathrm{Aut}(G^{\otimes})$.

We define the *automorphism group G_0 of the herd cover $\mathscr{H}(\mathscr{C})$* to be the subgroup of $\mathrm{P\Gamma L}(3, q)$ that induces a permutation of the oval covers of $\mathscr{H}(\mathscr{C})$. Under this definition, the automorphism group of a herd of ovals would naturally be the group induced on a herd of ovals by the automorphism group of the corresponding herd cover. We note that the induced group of the herd of ovals depends on the normalization used to define the herd of ovals.

The set

$$\overline{\mathscr{H}}(\mathscr{C}) = \{\overline{\mathcal{O}}_{\alpha} : \alpha \in \mathrm{PG}(1, q)\} \tag{31}$$

of $q + 1$ ovals of the planes $\overline{\mathscr{R}}_{\alpha}$ is called the *profile of the herd cover $\mathscr{H}(\mathscr{C})$*. Clearly, the automorphism group of a herd cover is isomorphic to the automorphism group of the profile of that herd cover which in turn is induced by a subgroup of $\mathrm{Aut}(G^{\otimes})$. We frequently use this correspondence in the computations which follow. We have chosen to define the automorphism group in terms of the herd cover rather than the profile of the herd cover for a reason that is not central to this work. Namely, herd covers exist in a more general context but the profiles do not and we prefer to keep our definitions generalizable.

Put

$$T = \{(\gamma \otimes \alpha, c) \in G^{\otimes}; \gamma, \alpha \in F^2, c \in F\}.$$

Clearly

$$T = \bigcup\{\mathscr{L}_{\gamma} : \gamma \in \mathrm{PG}(1, q)\} = \bigcup\{\mathscr{R}_{\alpha} : \alpha \in \mathrm{PG}(1, q)\}.$$

It is easy to show that any subgroup of $G^{\otimes}$ spanned by any two distinct "points" of T and lying entirely in T must either lie in some $\mathscr{L}_{\gamma}$ or in some $\mathscr{R}_{\alpha}$. Recall that the elements of the GQ which form the equivalence classes that are the points of an oval in $\overline{\mathscr{R}}_{\alpha}$ are called *o-points*. Specifically, the set $\{\gamma_{y_t} \otimes d\alpha, g(d\alpha, t)) : d \in F^*\}$ of non-zero vectors in $A_{\alpha}(t) = A(t) \cap \mathscr{R}_{\alpha}$ which correspond to a point of the oval $\overline{\mathcal{O}}_{\alpha}$ are o-points. Let p_1 and p_2 be any two distinct o-points lying in some $\mathscr{L}_{\gamma}$. Then $p_1 \cdot p_2 = p_3$ also lies in $\mathscr{L}_{\gamma}$. Hence any $\theta \in \mathrm{Aut}(G^{\otimes})$ that induces a permutation of the o-points maps all three points p_1, p_2, p_3 to o-points for which the third is the "sum" of the first two. Hence the three images also lie in the same $\mathscr{L}_{\gamma'}$. This implies that θ must also permute the $\mathscr{L}_{\gamma}$ among themselves. Also, the points of T that are o-points and lie in some $\mathscr{L}_{\gamma}$ lie on the line that is the member of the 4-gonal family in $\mathscr{L}_{\gamma}$. Hence θ must permute the members of the 4-gonal family, i.e., θ induces a collineation of the GQ $\mathscr{S}(\mathscr{C})$. As θ clearly fixes $(\bar{0}, \bar{0}, 0)$, the Fundamental Theorem tells us what form θ must have. We have shown that

Theorem 5.1. *The automorphism group G_0 of the herd cover $\mathscr{H}(\mathscr{C})$ is isomorphic to the group of automorphisms of $\mathscr{S}(\mathscr{C})$ that fix the points* (∞) *and* $(\bar{0}, \bar{0}, 0)$.

Note: In the above argument we did not use the assumption that $\theta \in \mathrm{Aut}(G^{\otimes})$ permutes the ovals, but we only used the fact that it permutes the o-points among themselves. But of course since such a θ induces an automorphism of the GQ, it will have to permute the ovals among themselves.

Let $\theta = \theta(\sigma, A \otimes B) \in G_0$. Suppose $A = \begin{pmatrix} a_4 & a_2 \\ a_3 & a_1 \end{pmatrix} = \begin{pmatrix} a^{1/2} & c^{1/2} \\ b^{1/2} & d^{1/2} \end{pmatrix}$, $u = \det(A)$, $1 = \det(B)$. By specializing the general form of a collineation of $\mathscr{S}(\mathscr{C})$ (see [7], Theorem 1.9.1), and writing our vectors as column vectors, we see, after a little routine computation, that the action of θ on the points of T is given by:

$$\theta = \theta(\sigma, A \otimes B):$$
$$(\gamma \otimes \alpha, r) \mapsto (A^T \gamma^\sigma \otimes B^T \alpha^\sigma, ur^\sigma + \gamma^{2\sigma}((\alpha^\sigma)^T BCB^T \alpha^\sigma, (\alpha^\sigma)^T BEB^T \alpha^\sigma)^T), \tag{32}$$

where $C = a_4^2 A_{(a_2/a_4)^2} = aA_{c/a}$ and $E = a_3^2 A_{(a_1/a_3)^2} = bA_{d/b}$.

In general, $\theta : \mathscr{R}_\alpha \to \mathscr{R}_{B^T\alpha^\sigma}$, so the oval $\bar{\mathscr{O}}_\alpha$ in $\bar{\mathscr{R}}_\alpha$ is mapped to the oval $\bar{\mathscr{O}}_{B^T\alpha^\sigma}$ in $\bar{\mathscr{R}}_{B^T\alpha^\sigma}$. (The herd cover, however, is contained in one plane $\mathrm{PG}(2, q)$.)

Now θ induces a map $\hat{\theta} = \hat{\theta}(\sigma, A \otimes B)$ on the herd cover. After a little computation we can write the effect of $\hat{\theta}$ on the points of an oval $\mathscr{O}_{\lambda\alpha}$ as a matrix equation. This equation will turn out to be quite useful a little later.

$$\hat{\theta} : \begin{pmatrix} 1 \\ t \\ g(\lambda\alpha, t) \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ w \\ \lambda^{2\sigma} g(B^T\alpha^\sigma, w) \end{pmatrix}$$
$$= \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ a\lambda^{2\sigma} g\left(B^T\alpha^\sigma, \frac{c}{a}\right) & b\lambda^{2\sigma} g\left(B^T\alpha^\sigma, \frac{d}{b}\right) & u \end{pmatrix} \begin{pmatrix} 1 \\ t \\ g(\lambda\alpha, t) \end{pmatrix}^\sigma, \tag{33}$$

where $w = \frac{c+dt^\sigma}{a+bt^\sigma}$.

5.2 The magic action of O'Keefe & Penttila. Let $\mathscr{F} = \{f : F \to F \mid f(0) = 0\}$. Each element of $\mathscr{F}$ can be expressed as a polynomial in one variable of degree at most $q - 1$, and $\mathscr{F}$ is naturally a vector space over F. For $f(t) = \sum a_i t^i \in \mathscr{F}$ and $\sigma \in \mathrm{Aut}(F)$, put $f^\sigma(t) = \sum a_i^\sigma t^i = (f(t^{1/\sigma}))^\sigma$. Start with the group $\mathrm{P\Gamma L}(2, q)$ acting on the projective line $\mathrm{PG}(1, q)$,

$$\mathrm{P\Gamma L}(2, q) = \{x \mapsto Ax^\sigma : A \in \mathrm{GL}(2, q), \sigma \in \mathrm{Aut}(F)\}.$$

We are going to construct an action $\mathscr{M} : \mathrm{P\Gamma L}(2, q) \to \mathrm{Sym}_{\mathscr{F}}$ of $\mathrm{P\Gamma L}(2, q)$ on the set $\mathscr{F}$.

For each $f \in \mathscr{F}$ and each $\psi \in \mathrm{P\Gamma L}(2,q)$, where $\psi : x \mapsto Ax^\sigma$ for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,q)$ and $\sigma \in \mathrm{Aut}(F)$, let the image of f under $\mathscr{M}(\psi)$ be the function $\psi f : F \to F$ such that

$$\psi f(t) = |A|^{-1/2}\left\{(bt+d)f^\sigma\left(\frac{at+c}{bt+d}\right) + btf^\sigma\left(\frac{a}{b}\right) + df^\sigma\left(\frac{c}{d}\right)\right\}. \tag{34}$$

Lemma 5.2. *$\mathscr{M}$ is an action (called the* magic action) *of* $\mathrm{P\Gamma L}(2,q)$ *on the set $\mathscr{F}$ whose kernel contains (so equals) the set of scalar matrices $aI : 0 \neq a \in F$ that form the center of* $\mathrm{P\Gamma L}(2,q)$.

Lemma 5.3. *The function $f \in \mathscr{F}$ defined by $f(t) = t^{1/2}$ is fixed by every $\psi \in \mathrm{P\Gamma L}(2,q)$ under the magic action.*

Lemma 5.4. *The magic action of* $\mathrm{P\Gamma L}(2,q)$ *on $\mathscr{F}$ is (projective) semi-linear and the magic action of the subgroup* $\mathrm{PGL}(2,q)$ *is (projective) linear.*

O'Keefe and Penttila [6] proceed to use the magic action in the study of ovals and herds of ovals. However, if we shift to the examination of oval covers and herd covers, the $\mathscr{F}$ setting for the magic action is not optimal. Since $\mathscr{F}$ is a vector space over F, we can pass to the projective space $\hat{\mathscr{F}}$ whose points $\langle f \rangle$ are the 1-dimensional subspaces of $\mathscr{F}$ spanned by a nonzero $f \in \mathscr{F}$. It is easy to see that for all $\lambda \in F^*$, $\psi(\lambda f) = \lambda^\sigma \psi f$, and so, the magic action can be lifted to $\hat{\mathscr{F}}$. By a fairly common abuse of notation we will continue to use ψ to denote the result of the magic action on the points of $\hat{\mathscr{F}}$, i.e., $\psi\langle f \rangle := \langle \psi f \rangle$. We will now state further results of [6] in the $\hat{\mathscr{F}}$ setting. This has the additional advantage of permitting a more natural phrasing of these results.

We will use the following notation. Let $\mathscr{D}(f) = \{(1, t, f(t)) : t \in F\} \cup \{(0,1,0)\}$ for any o-permutation f. That is, $\mathscr{D}(f)$ is an oval in $\mathrm{PG}(2,q)$. For any primitive element ξ of F define

$$\mathscr{D}_\xi(f) = \{(1, t, f(t), \xi f(t), \xi^2 f(t), \ldots, \xi^{q-2} f(t)) : t \in F\} \cup \{(0,1,0,\ldots,0)\} \tag{35}$$

a set of points in $\mathrm{PG}(q,q)$ for any o-permutation f. Then $\mathscr{D}_\xi(f)$ is a coordinate representation of an oval cover. The oval cover can be recovered by projecting the coordinates x_0, x_1, x_i of $\mathscr{D}_\xi(f)$ into the same $\mathrm{PG}(2,q)$ for each i, $2 \leqslant i \leqslant q$. Notice that if we rearrange, in any way, the last $q-1$ coordinates of $\mathscr{D}_\xi(f)$, these projections will give the same oval cover. We should therefore consider two of these sets to be equivalent if one can be obtained from the other by such a rearrangement. The equivalence classes may be described by

$$\mathscr{D}(\langle f \rangle) = \{(1, t, \overline{f(t)}B)\colon B \text{ is a permutation matrix}, t \in F\},$$

where $\overline{f(t)} = (f(t), \xi f(t), \xi^2 f(t), \ldots, \xi^{q-2} f(t))$ for any fixed primitive element $\xi \in F$.

Note that, as a special case, $\mathscr{D}_\xi(f)$ and $\mathscr{D}_\mu(f)$ will be equivalent for different primitive elements ξ and μ.

The next theorem is what makes the magic action useful to us.

Theorem 5.5. *Let $f \in \mathscr{F}$ be an o-permutation for* $\mathrm{PG}(2,q)$ *and let $\psi \in \mathrm{P\Gamma L}(2,q)$ be $\psi : x \mapsto Ax^\sigma$ for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,q)$ and $\sigma \in \mathrm{Aut}(F)$. Then $f' \in \psi\langle f\rangle$ is also an o-permutation. In fact, for any o-permutation f, $\mathscr{D}(\psi f) = \bar{\psi}_f \mathscr{D}(f)$ where $\bar{\psi}_f : x \mapsto \bar{A}_f x^\sigma$ $(x \in F^3)$ for*

$$\bar{A}_f = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ a\psi f\left(\frac{c}{a}\right) & b\psi f\left(\frac{d}{b}\right) & |A|^{1/2} \end{pmatrix}. \tag{36}$$

In terms of oval covers, we have $\mathscr{D}_{\xi^\sigma}(\psi f) = \bar{\psi}_{\langle f\rangle} \mathscr{D}_\xi(f)$ where $\bar{\psi}_{\langle f\rangle} \in \mathrm{P\Gamma L}(q+1,q)$ is such that $\bar{\psi}_{\langle f\rangle} : x \mapsto \bar{A}_{\langle f\rangle} x^\sigma$ $(x \in F^{q+1})$ for

$$\bar{A}_{\langle f\rangle} = \begin{pmatrix} A & \mathbf{0}_{2\times q-1} \\ D & |A|^{1/2}\mathbf{I}_{q-1\times q-1} \end{pmatrix} \tag{37}$$

where

$$D = \begin{pmatrix} a\psi f\left(\frac{c}{a}\right) & b\psi f\left(\frac{d}{b}\right) \\ a\psi\xi f\left(\frac{c}{a}\right) & b\psi\xi f\left(\frac{d}{b}\right) \\ a\psi\xi^2 f\left(\frac{c}{a}\right) & b\psi\xi^2 f\left(\frac{d}{b}\right) \\ \vdots & \vdots \\ a\psi\xi^{q-2} f\left(\frac{c}{a}\right) & b\psi\xi^{q-2} f\left(\frac{d}{b}\right) \end{pmatrix}.$$

Proof. We only consider the oval cover extension of this result. By the first part of the theorem, for ξ a primitive element of F and $0 \leqslant i < q-1$, $\mathscr{D}(\psi\xi^i f) = \bar{\psi}_{\xi^i f}\mathscr{D}(\xi^i f)$. Since $\mathscr{D}(\psi\xi^i f) = \mathscr{D}(\xi^{\sigma i}\psi f)$ the result follows. $\square$

Corollary 5.6. *Let $f \in \mathscr{F}$ be an o-permutation, and let $\psi \in \mathrm{P\Gamma L}(2,q)$. If $\psi\langle f\rangle = \langle f\rangle$, then $\bar{\psi}_{\lambda f}$ is in the stabilizer of $\mathscr{D}(\lambda f)$ in $\mathrm{P\Gamma L}(3,q)$ for each $\lambda \in F^*$. Thus, $\bar{\psi}_{\langle f\rangle}$ is in the stabilizer of $\mathscr{D}(\langle f\rangle)$.*

The next theorem is crucial to the further development of the theory. Unfortunately, the proof given in [6] is flawed, so we will provide a corrected proof of this result.

Theorem 5.7. *Let f and g be o-permutations for which $\mathscr{D}(f)$ and $\mathscr{D}(g)$ are equivalent under $\mathrm{P\Gamma L}(3,q)$. Then there exists $\psi \in \mathrm{P\Gamma L}(2,q)$ such that $\psi\langle f\rangle = \langle g\rangle$.*

Proof. For $0 \neq k \in F$, $\mathscr{D}(kf)$ and $\mathscr{D}(g)$ are also equivalent. Suppose $\eta : x \mapsto Bx^\sigma$

($\in \mathrm{P\Gamma L}(3,q)$) satisfies $\eta\mathscr{D}(g) = \mathscr{D}(kf)$. Since η fixes $(0,0,1)$, $\eta(1,0,0) \in \mathscr{D}(kf)$, $\eta(0,1,0) \in \mathscr{D}(kf)$, it follows easily that B has the form

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ akf\left(\frac{c}{a}\right) & bkf\left(\frac{d}{b}\right) & z \end{pmatrix}$$

for some $a, b, c, d, z \in \mathrm{GF}(q)$ with $ad + bc \neq 0$ and $z \neq 0$. From

$$\eta\begin{pmatrix} 1 \\ t \\ g(t) \end{pmatrix} = \begin{pmatrix} a + bt^\sigma \\ c + dt^\sigma \\ akf\left(\frac{c}{a}\right) + bkt^\sigma f\left(\frac{d}{b}\right) + zg(t)^\sigma \end{pmatrix}$$

we can compute that

$$g(t)^\sigma = \left(\frac{k}{z}\right)\left[(a + bt^\sigma)f\left(\frac{c + dt^\sigma}{a + bt^\sigma}\right) + bt^\sigma f\left(\frac{d}{b}\right) + af\left(\frac{c}{a}\right)\right].$$

Thus, $g(t)^\sigma = \left(\frac{k}{z}\right)|A'|^{1/2}\psi_0 f(t^\sigma)$ where $\psi_0 : x \mapsto A'x$ with $A' = \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right) \in \mathrm{GL}(2,q)$. We then have $g(t) = \left(\frac{k}{z}\right)^{1/\sigma}|A'|^{1/(2\sigma)}\psi f(t)$ where $\psi : x \mapsto (A')^{1/\sigma}x^{1/\sigma}$. Hence, $\left(\frac{z^{1/\sigma}}{|A'|^{1/(2\sigma)}}\right)g = \psi kf$, and so $\psi\langle f\rangle = \langle g\rangle$. $\square$

Comment. It should be noted that the above proof shows that the result is more general than stated. No properties of o-permutations were used. Besides the inclusion of the points $(1,0,0)$ and $(0,1,0)$ in $\mathscr{D}(f)$ and $\mathscr{D}(g)$, the only requirement is that the projectivity between them fixes the point $(0,0,1)$ (whether or not this point is considered to be in the sets or even related to them). This comment also applies to the following corollary.

Corollary 5.8. *Let f be an o-permutation in $\mathscr{F}$. Then each element of the stabilizer of $\mathscr{D}(\langle f\rangle)$ is of the form $\bar{\psi}_{\langle f\rangle}$ for some $\psi \in \mathrm{P\Gamma L}(2,q)$ such that $\psi\langle f\rangle = \langle f\rangle$.*

Lemma 5.9. *Let $\psi : x \mapsto Ax^\sigma$ for $A \in \mathrm{GL}(2,q)$ and $\sigma \in \mathrm{Aut}(F)$. If*

$$\mathscr{C} = \left\{\begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix} : t \in F\right\} \quad \textit{is a q-clan,}$$

then so is

$$\mathscr{C} = \left\{\begin{pmatrix} \psi f(t) & t^{1/2} \\ 0 & \psi g(t) \end{pmatrix} : t \in F\right\}.$$

Remark 5.10. This can also be stated in terms of herd covers or herds of ovals. O'Keefe and Penttila normalize the herd cover by insisting that the ovals of the herd always contain the point $(1,1,1)$. So they write their q-clans in the form

$$\mathscr{C} = \left\{ \begin{pmatrix} f_0(t) & t^{1/2} \\ 0 & \kappa f_\infty(t) \end{pmatrix} : t \in F \right\},$$

where $f_0(1) = f_\infty(1) = 1$ and κ is some fixed element with $\mathrm{tr}(\kappa) = 1$. In their formulas for $f_0'(t)$, $f_\infty'(t)$ and $f_s'(t)$, they should have written (in their notation and correcting the last two so that $f_\infty'(1) = 1$).

$$\begin{aligned} f_0'(t) &= \psi f_0(t)/\psi f_0(1) \\ f_\infty'(t) &= \psi f_\infty(t)/\psi f_\infty(1) \end{aligned} \tag{38}$$

$$f_s'(t) = \frac{f_0'(t) + s^{1/2}t^{1/2} + \kappa' s f_\infty'(t)}{1 + s^{1/2} + \kappa' s}, \quad \text{where } \kappa' = \psi f_0(1)\psi f_\infty(1)\kappa^\gamma. \tag{39}$$

Their lemma then states that if $\{\mathscr{D}(f_s) : s \in \tilde{F}\}$ is a herd of ovals, so is $\{\mathscr{D}(f_s') : s \in \tilde{F}\}$. It then follows that if $\{\mathscr{D}(\langle f_s \rangle) : s \in \tilde{F}\}$ represents a herd cover, so does $\{\mathscr{D}(\langle f_s' \rangle) : s \in \tilde{F}\}$.

5.3 The automorphism group of $\hat{\mathscr{H}}(\mathscr{C})$. Let $\hat{\mathscr{H}}(\mathscr{C}) = \{\mathscr{D}(f_s) : s \in \tilde{F}\}$ and $\hat{\mathscr{H}}(\mathscr{C}') = \{\mathscr{D}(f_t') : t \in \tilde{F}\}$ be herds of two herd covers $\mathscr{H}(\mathscr{C})$ and $\mathscr{H}(\mathscr{C}')$. O'Keefe and Penttila [6] *define* an isomorphism $\theta : \hat{\mathscr{H}}(\mathscr{C}) \to \hat{\mathscr{H}}(\mathscr{C}')$ to be a pair $\theta = (\psi, \pi)$ where $\psi \in \mathrm{P\Gamma L}(2,q)$ and π is a permutation of the elements of $\tilde{F}$ such that $\psi f_s \in \langle f_{\pi(s)}' \rangle$ for all $s \in \tilde{F}$.

Put $\hat{G}_0 = \mathrm{Aut}(\hat{\mathscr{H}}(\mathscr{C})) = \{\theta : \hat{\mathscr{H}}(\mathscr{C}) \to \hat{\mathscr{H}}(\mathscr{C}) \mid \theta \text{ is an isomorphism}\}$.

Remark 5.11. O'Keefe and Penttila observe that $\hat{G}_0$ is the stabilizer of the herd cover $\{\langle f_s \rangle : s \in \tilde{F}\}$ in $\mathrm{P\Gamma L}(2,q)$ under the magic action. We are not happy with this approach to defining the automorphism group since it presupposes that all "isomorphisms" are magic actions without providing any justification for that assumption. We believe that they were led to this because of the difficulty of working with herds of ovals. By passing to herd covers, these difficulties evaporate and it is possible to provide the more natural definition that we have given.

Remark 5.12. They go on to calculate the group $\hat{G}_0$ for the known herds of ovals. For both the classical and the FTWKB herds of ovals they get the same answer, the group $\mathrm{P\Gamma L}(2,q)$. We find this unsatisfactory. In the classical case the group G_0 of the profile of the herd cover (which is also the group of the GQ fixing the points $(0,0,0)$ and (∞)) is larger than the group $\hat{G}_0$ of the herd of ovals. And the induced stabilizer of an oval (i.e., a conic) in the classical case is smaller than the full stabilizer of that conic. This would be impossible to detect under the "magic action" definition since $\hat{G}_0 = \mathrm{P\Gamma L}(2,q)$ in the classical case is the largest possible group permitted by that definition. The classical situation also provides an example of the difference between working with herd covers and working with herds of ovals. In the classical case, the automorphism $\theta = \theta(\mathrm{id}, I \otimes P)$ of $G^{\otimes}$ is a collineation of the GQ $\mathscr{S}(\mathscr{C})$ that permutes the ovals $\overline{\mathscr{O}}_s$ according to the rule $s \mapsto s^{-1}$, and so, may be considered as an element of G_0, the automorphism group of the herd cover. The herd cover $\{[\overline{\mathscr{O}}_s] : s \in \tilde{F}\}$ of the classical herd has $[\overline{\mathscr{O}}_s] = \{\mathscr{D}(\lambda t^{1/2}) : \lambda \in F^*\}$ for each $s \in \tilde{F}$. Now, if we normalize so that $[\overline{\mathscr{O}}_s]$ is represented by $\mathscr{D}(st^{1/2})$ for $s \in F^*$ and $\mathscr{D}(t^{1/2})$ for $s = 0, \infty$, then θ will be expressed faithfully in the automorphism group of this herd of ovals. On the other hand, if we normalize so that $[\overline{\mathscr{O}}_s]$ is represented by $\mathscr{D}(t^{1/2})$ for all $s \in \tilde{F}$ (which is the normalization used by O'Keefe and Penttila) then θ induces the identity on this herd of ovals. This cannot happen in a nonclassical situation, since then the automorphisms of the herd cover faithfully induce automorphisms of any herd of ovals and $\hat{G}_0$ and G_0 are the same (see below). Thus, although the two definitions generally lead to the same groups, we believe that our definition properly handles the classical case. This is also consistent with the belief that the classical case should be distinguishable from the FTWKB case, where the ovals are nonconical translation ovals.

5.4 Aut($\mathscr{S}(\mathscr{C})$), G_0 and $\hat{G}_0$. The fact that $\mathscr{M}$ is an action implies that $(\mathscr{M}(\psi))^{-1} = \mathscr{M}(\psi^{-1})$. Hence for each nonzero $g \in \mathscr{F}$ there is a nonzero $f \in \mathscr{F}$ for which $\langle g \rangle = \psi \langle f \rangle$.

Recall the notation of Equation 33. Then put $f(t) = \psi^{-1} g(B^T \alpha^\sigma, t)$, i.e., $g(B^T \alpha^\sigma, t) = \psi f$ for $\psi : x \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) x^\sigma$. Also, let $\mu = \lambda^2$.

We can then use $\hat{\theta}$ to define

$$\hat{\theta}_{\langle f \rangle} : \begin{pmatrix} 1 \\ t \\ g(\alpha, t) \\ \mu g(\alpha, t) \\ \vdots \\ \mu^{q-2} g(\alpha, t) \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 & 0 & \cdots & 0 \\ c & d & 0 & 0 & \cdots & 0 \\ a\psi f\left(\frac{c}{a}\right) & b\psi f\left(\frac{d}{b}\right) & u & 0 & \cdots & 0 \\ a\psi \mu f\left(\frac{c}{a}\right) & b\psi \mu f\left(\frac{d}{b}\right) & 0 & u & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a\psi \mu^{q-2} f\left(\frac{c}{a}\right) & b\psi \mu^{q-2} f\left(\frac{d}{b}\right) & 0 & 0 & \cdots & u \end{pmatrix} \begin{pmatrix} 1 \\ t \\ g(\alpha, t) \\ \mu g(\alpha, t) \\ \vdots \\ \mu^{q-2} g(\alpha, t) \end{pmatrix}^{\sigma}. \tag{40}$$

Comparing this with Equation 37 we see that

$$\hat{\theta}_{\langle f \rangle} = \bar{\psi}_{\langle f \rangle}. \tag{41}$$

By Theorem 5.5 f is an o-permutation and

$$\bar{\psi}_{\langle f \rangle} : \mathscr{D}_\mu(f) \mapsto \mathscr{D}_{\mu^\sigma}(\psi f) = \mathscr{D}_{\mu^\sigma}(g(B^T \alpha^\sigma, t)).$$

Since

$$\bar{\psi}_{\langle f\rangle} = \hat{\theta}_{\langle f\rangle} : \mathscr{D}_\mu(g(\alpha, t)) \mapsto \mathscr{D}_{\mu^\sigma}(g(B^T\alpha^\sigma, t)), \tag{42}$$

it must be that $\mathscr{D}_\mu(f) = \mathscr{D}_\mu(g(\alpha, t))$. It follows that $\langle f(t)\rangle = \langle g(\alpha, t)\rangle$.

This proves the following important theorem.

Theorem 5.13. *If $\mathscr{C}$ is a q-clan, then*

$$\theta(\sigma, (A^T)^{(1/2)} \otimes (B^T)^{(1/2)}) \in \mathrm{Aut}(\mathscr{S}(\mathscr{C})) \quad \textit{iff } \psi\langle g(\alpha, t)\rangle = \langle g(B\alpha^\sigma, t)\rangle$$

for all $\alpha \in \mathrm{PG}(1, q)$, where $\psi : x \mapsto Ax^\sigma$.

In [6] $\pi(\mathscr{C})$ is defined to be the subspace of $\mathscr{F}$ spanned by $f(t)$, $t^{1/2}$ and $g(t)$, the q-clan functions. In the classical case $\pi(\mathscr{C})$ is 1-dimensional and for any $A \in \mathrm{GL}(2, q)$, $\sigma \in \mathrm{Aut}(F)$ and any $B \in \mathrm{SL}(2, q)$, if $\psi : x \mapsto (A^{(2)})^T x^\sigma$, automatically $\psi\langle g(\alpha, t)\rangle = \langle g((B^{(2)})^T\alpha^\sigma, t)\rangle$. However, following Lemma 13 of [6] it is shown (see also [2]) that if $\mathscr{C}$ is nonclassical, then $\pi(\mathscr{C})$ is 3-dimensional and no two herd functions are even in the same 1-dimensional space. Hence if ψ is given, and σ is given, clearly only one B can exist. (This also follows from Theorem 1.10.3 of [7].) This then implies

Corollary 5.14. *If $\mathscr{C}$ is a non-classical q-clan, then $G_0 = \hat{G}_0$.*

6 The flock model

The material in this section is not new, it is implicit in [7] and has been cited as such in [6] (see Theorem 12). We take this opportunity to make explicit the connections between some automorphisms of the GQ, magic actions and derivation, especially in the context of cyclic GQ.

In odd characteristic there is a geometric construction of a flock GQ due to Knarr. In this construction a set of $q + 1$ points called a *BLT-set* is used. To each of these points there is associated a cone having the point as vertex. These $q + 1$ cones have the property that, fixing any one of them, the intersections of the other cones with the fixed cone form a flock of the fixed cone. These flocks are called *derived flocks*. The derived flocks need not be projectively equivalent, but they do not give rise to different flock GQ's (just reparameterizations of the same flock GQ.)

Knarr's construction does not work in even characteristic, and BLT sets don't exist. However, if one describes the derived flocks algebraically, the description carries over to the even characteristic case, giving new flocks (but again, no new GQ's.) Given a flock, a derived flock (with respect to an element $s \in \mathrm{GF}(q)$) is obtained by applying the following operator to each function that defines the flock:

$$D_s f(t) = t\left(f\left(\frac{1}{t} + s\right) - f(s)\right).$$

We first observe that derivation is a magic action.

Lemma 6.1. *If* $A = \left(\begin{smallmatrix} s & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $\sigma = \mathrm{id}$ *then the image of* $f \in \mathscr{F}$ *under* $\mathscr{M}(\psi_s)$, *where* $\psi_a : x \mapsto Ax$, *is the function* $\psi_s f(t) = t(f(t^{-1} + s) + f(s))$.

For each $s \in F$, there is an automorphism (a *shift-flip*) of $G^{\otimes}$ defined by

$$i_s = \theta\left(\mathrm{id}, \begin{pmatrix} s^{1/2} & 1 \\ 1 & 0 \end{pmatrix} \otimes I\right) : ((\alpha, \beta), c) \mapsto ((s^{1/2}\alpha + \beta, \alpha), c + g(\alpha, s) + \alpha \circ \beta). \quad (43)$$

Also put $i_\infty = \mathrm{id} : G^{\otimes} \to G^{\otimes}$.

As an automorphism of $G^{\otimes}$, certainly i_s maps the 4-gonal family $\mathscr{J}(\mathscr{C})$ to another 4-gonal family. Fortunately, we can recognize the new 4-gonal family as arising from a q-clan which we denote by $\mathscr{C}^{i_s} = \{A_t^{i_s} : t \in F\}$. For $u \in F$, consider the image under i_s, $s \in F$, of the typical element of $A(u)$.

$$\begin{aligned} i_s : (\gamma_{u^{1/2}} \otimes \alpha, g(\alpha, u)) &= (\alpha, u^{1/2}\alpha, g(\alpha, u)) \mapsto ((s^{1/2} + u^{1/2})\alpha, \alpha, g(\alpha, u) + g(\alpha, s)) \\ &= ((1, (s+u)^{-1/2}) \otimes (s^{1/2} + u^{1/2})\alpha, g(\alpha, u) + g(\alpha, s)) \quad (\text{if } u \neq s). \\ &= (\gamma_{(s+u)^{-1/2}} \otimes \beta, (s+u)^{-1}(g(\beta, u) + g(\beta, s))), \quad \beta = (s+u)^{1/2}\alpha. \end{aligned} \quad (44)$$

This element of $G^{\otimes}$ must be in $A^{i_s}((s+u)^{-1})$. Hence the matrix $A^{i_s}_{(s+u)^{-1}}$ must equal $(s+u)^{-1}(A_u + A_s)$. Put $t = (s+u)^{-1}$, so $u = s + t^{-1}$. Then

$$A_t^{i_s} = t(A_{s+t^{-1}} + A_s), \quad \text{for all } t, s \in F. \quad (45)$$

Note: If $u = s$ in Equation 44 above, it is clear that

$$i_s : A(s) \to A^{i_s}(\infty).$$

Thus, if

$$\mathscr{C} = \left\{ A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix} : t \in \mathrm{GF}(q) \right\}$$

Equation 45 can be stated as

$$\mathscr{C}^{i_s} = \left\{ A_t^{i_s} = \begin{pmatrix} \psi_s f(t) & t^{1/2} \\ 0 & \psi_s g(t) \end{pmatrix} : t \in \mathrm{GF}(q) \right\}. \quad (46)$$

In terms of flocks, we see that $\mathscr{F}(\mathscr{C}^{i_s})$ is a derived flock of $\mathscr{F}(\mathscr{C})$.

The following result is basically a corollary of the Fundamental Theorem.

Theorem 6.2 ([7], Theorem 1.10.4). *Let* $\mathscr{C}$ *be a normalized q-clan. Then there is an automorphism* θ *of* $\mathscr{S}(\mathscr{C})$ *mapping* $[A(s)]$ *to* $[A(u)]$, $s, u \in \tilde{F}$, *iff the flocks* $\mathscr{F}(\mathscr{C}^{i_s})$ *and* $\mathscr{F}(\mathscr{C}^{i_u})$ *are projectively equivalent.*

We now turn specifically to the cyclic GQ case.

The conditions of Lemma 2.5 imply that $\mathscr{F}(\mathscr{C})$ and its derived flock $\mathscr{F}(\mathscr{C}^{i_\delta})$ are projectively equivalent. Thus, by Theorem 6.2, the collineation $\theta = \theta(\mathrm{id}, M \otimes M^{-m})$ of $\mathscr{S}(\mathscr{C})$ maps $[A(\infty)]$ to $[A(\delta)] = [A([1])]$. By Theorem 2.6 θ has order $q+1$ on the lines of $\mathscr{S}(\mathscr{C})$ through the point (∞) mapping $\left[A\left(\frac{[k+1]}{[k]}\right)\right]$ to $\left[A\left(\frac{[j+k+1]}{[j+k]}\right)\right]$. Since by Lemma 2.6 (6) as j runs through the values $0, 1, \ldots, q$, $\frac{[j+1]}{[j]}$ runs through the elements of $\tilde{F}$, we see that all the derived flocks of $\mathscr{F}(\mathscr{C})$ are projectively equivalent, and θ induces a cyclic action on them.

We conclude with a slightly amusing comment, considering the title of this paper. As we have shown, the cyclic collineation of $\mathscr{S}(\mathscr{C})$, which defines this class of GQ's, expresses itself in several ways in the corresponding flock model. Above, we have shown how it induces a cyclic action on the derived flocks. In the previous section we have shown that it induces a cyclic action on the herd cover. This is also reflected in the various herds of ovals which can be obtained from this herd cover, but one must work modulo scalar multiplication in those settings. The cyclic action in the GQ also induces a cyclic action on the generators of the quadratic cone in $\mathrm{PG}(3, q)$. This follows immediately from the correspondence between the generators and the herd cover expounded upon in [2], but can also be seen from Theorem 1.14.1 of [7]. About the only place where you do not see this cyclic action expressed is in the q-clan associated with $\mathscr{S}(\mathscr{C})$!

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