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The classification of spreads in PG(3, q) admitting linear groups of order $q(q+1)$, II. Even order

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Dedicated to Adriano Barlotti on the occasion of his 80th birthday

Abstract. A classification is given of all spreads in $\mathrm{PG}(3,q)$, $q=2^r$, whose associated translation planes admit linear collineation groups of order $q(q+1)$.

Key words. Spread, conical flock, regulus-inducing group, Baer group.

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1 Introduction

In this article, we completely classify the even order spreads in $\mathrm{PG}(3,q)$ that admit a linear collineation group of order $q(q+1)$. In what should be considered a companion article to the present one, the authors have previously classified the odd order spreads in $\mathrm{PG}(3,q)$ that admit a linear collineation group of order $q(q+1)$ with the added hypothesis that a subgroup of order q must fix a line and act nontrivially on it. In the odd order case, the planes are all related to flocks of quadratic cones in $\mathrm{PG}(3,q)$. However, it will turn out in the even order case, there are no associated flocks of quadratic cones apart from the linear flock (the associated plane is Desarguesian). So we have decided to separate the even and odd order parts, for this reason and also due to the fact that the arguments for the even and odd order cases are probably as dissimilar as they are similar. We do, however, use various results connecting the theories of flocks of quadratic cones and translation planes whose spreads in $\mathrm{PG}(3,q)$ are unions of reguli sharing a common line.

By the work of Gevaert and Johnson [6], and Gevaert, Johnson and Thas [7], it is also true that within the collineation group of the translation plane associated with a so-called 'conical flock' (flock of a quadratic cone) is an elation group E of order q. In general, we shall use the term 'regulus-inducing' to denote such an elation group. Since each orbit of a regulus-inducing group union the axis is a regulus, we may derive any such regulus, turning the elation group into a group which fixes a Baer

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subplane pointwise. That is, we now have an associated Baer group of order q. Conversely, the existence of a Baer group of order q in a translation plane of order q^2 and spread in $\mathrm{PG}(3,q)$ implies that the plane is a derived conical flock plane.

Theorem 1 (Johnson [20], Payne–Thas [26]). *If π is a translation plane of order q^2 with spread in $\mathrm{PG}(3,q)$ that admits a Baer group of order q then π is a derived conical flock plane.*

Indeed, Baer groups of order q, by their very nature, are 'regulus-inducing' so if there is a Baer group of order q of a translation plane with spread in $\mathrm{PG}(3,q)$, it is shown in Johnson [20] that there is an corresponding 'partial flock of deficiency one' of a quadratic cone and conversely such partial flocks produce translation planes admitting Baer groups of order q.

It is also known that the partial flock may be extended to a flock if and only if the net of degree $q+1$ containing the Baer subplane fixed pointwise by the Baer group is derivable, and this implies that the net is, in fact, a regulus net. In this setting, derivation then does not change the fact that both spreads remain within the same projective space isomorphic to $\mathrm{PG}(3,q)$.

Furthermore, it is known by Payne and Thas [26] that every partial flock of deficiency one may, in fact, be uniquely extended to a flock. Putting all of this together means that Baer groups of order q in translation planes of order q^2 are essentially equivalent to flocks of quadratic cones.

Theorem 1 is crucial to our main classification theorem, which we now state.

Theorem 2. *Let π be a translation plane of even order q^2 with spread in $\mathrm{PG}(3,K)$, K isomorphic to $\mathrm{GF}(q)$, that admits a linear collineation group G of order $q(q+1)$ (i.e. in $\mathrm{GL}(4,q)$). Then π is one of the following types of planes:*

(1) *Desarguesian,*

(2) *Hall,*

(3) *a translation plane obtained from a Desarguesian plane by multiple derivation of a set of $q/2$ mutually disjoint regulus nets that are in an orbit under an elation group of order $q/2$.*

2 Background

In this background section, we list most of the results that we shall be using in the proof of our main theorems. For convenience of reading, we shall collect the results involving similar ideas. In the statements of the results, q is a prime power p^r, even though we shall be using $q = 2^r$ when we use the material in the body of the proof of the main theorem. The material has been cross-referenced so that this section could be skipped initially and then referred back to when a particular result is required.

2.1 General group-spread theoretic.

Theorem 3 (see Johnson [18], Theorem (2.3)). *Let V be a vector space of dimension $2t$ over F isomorphic to $\mathrm{GF}(p^r)$, p a prime, $q = p^r$. Let T be a linear transformation of*

V over F which fixes three mutually disjoint r-dimensional subspaces. Assume that $|T|$ *divides* $q^t - 1$ *but does not divide* $\mathrm{LCM}(q^s - 1; s < t, s \mid t)$. *Then*:

(1) *all T-invariant t-dimensional subspaces are mutually disjoint and the set of all such subspaces defines a Desarguesian spread*;

(2) *the normalizer of* $\langle T \rangle$ *in* $\mathrm{GL}(2t, q)$ *is a collineation group of the Desarguesian plane* Σ *defined by the spread of* (1);

(3) Σ *may be thought of as a* $2t$*-dimensional vector space over F*; *that is*, *the field defining* Σ *is an extension of F*.

Theorem 4 (see Lüneburg [23], (49.4), (49.5)). *Let* τ *be a linear mapping of order p of a vector space V of characteristic p and dimension* 4 *over* $\mathrm{GF}(p^r)$ *and leaving invariant a spread*.

(1) *Then the minimal polynomial of* τ *is* $(x-1)^2$ *or* $(x-1)^4$. *If the minimal polynomial is* $(x-1)^4$ *then* $p \geqslant 5$. *In the latter case*, τ *is said to be a 'quartic' element*.

(2) *The minimal polynomial of* τ *is* $(x-1)^2$ *if and only if* τ *is an affine elation* (*shear*) *or a Baer p-collineation* (*fixes a Baer subplane pointwise*).

Theorem 5 (Zassenhaus [29]). *Let G be a sharply* 2*-transitive permutation group of order* $t(t-1)$. *Then there is an associated nearfield plane of order t admitting G as a collineation group*.

Theorem 6 (Johnson [19]). *Let* π *be a translation plane of order* q^2, $q = p^t$, *p a prime*, *admitting a collineation group isomorphic to* $\mathrm{SL}(2, p^a)$. *If* $p^a > \sqrt{q}$ *then* $\mathrm{SL}(2, q)$ *is a collineation group of* π.

Theorem 7 (Gleason [8]). *Let G be a finite group operating on a set* Ω *and let p be a prime*. *If* Ψ *is a subset of* Ω *such that for every* $\alpha \in \Psi$, *there is a p-subgroup* Π_α *of G fixing* α *but no other point of* Ω *then* Ψ *is contained in an orbit*.

Theorem 8 (see e.g. Foulser [3]). *Let G be a subgroup of* $\mathrm{PGL}(2, q)$, *for* $q = p^r$, *p a prime*, *that does not admit p-elements*. *Then one of the following occurs*:

(1) *G is a subgroup of a dihedral group of order* $2(q \pm 1)$,

(2) *G is isomorphic to* A_4, S_4 *or* A_5.

Theorem 9 (see Johnson [17] or [15]). *There are exactly three translation planes of order* 16 *with kernel containing* $\mathrm{GF}(4)$. *These are the Desarguesian plane*, *the Hall plane*, *and a plane coordinatized by a semifield with all semi-nuclei equal*. *In the last case*, *there is a component that is invariant under the full translation complement*.

Theorem 10 (Lüneburg [23]). *Let* π *be a Lüneburg–Tits plane of order* q^2. *Then the translation complement of* π *does not contain an element of order a* 2*-primitive divisor of* $q^2 - 1$.

Theorem 11 (Foulser [2]). *If π is a non-Desarguesian generalized André plane then π does not admit affine elations.*

2.2 Combinatorial field theory.

Theorem 12 (Zsigmondy [30]). *Let $h = p^t$, where p is a prime. Then there is a prime divisor u of $h - 1$ but not of $p^s - 1$ for $s < t$ unless $t = 2$ and $p + 1 = 2^a$ or $p^t = 2^6$.*

Theorem 13 (also, see Ribenboim [27], and note the correction to Ganley, Jha, Johnson [6], (3.5)). *Consider the following equation*:

$$\frac{w^n - 1}{w - 1} = p^r,$$

where w is a prime power and p is a prime.

(a) *If $n > 2$ and r is even then $(w, n, p, r) = (3, 5, 11, 2)$.*

(b) *If $n = 2$ then either*
  (i) $r = 1$,
  (ii) $(w, p, r) = (8, 3, 2)$ *or*
  (iii) $(w, p) = (2^r - 1, 2)$, *where r is a prime.*

Proof. If $n > 2$ and r is even, we may apply the results of Ribenboim [27]. Hence, assume that $n = 2$. First assume that r is >2 and p is odd. Then $p^r - 1$ has an odd prime t-primitive divisor by Theorem 12. Since $p^r - 1$ cannot have a p-primitive divisor, thus it must be that $r = 1$ or 2. If $r = 2$ then $p + 1 = 2^b$, for some integer b. Since $(p - 1, p + 1) = 2$, it follows that $p - 1 = 2$, so $b = 2$. Hence, either $r = 1$ or $w = 8$, $p = 3$ and $r = 2$.

Now assume that $p = 2$. Then, since $2^6 - 1$ is not a prime power, then $2^r - 1$ admits a prime 2-primitive divisor u, where $u^c = w$. However, this says that r is a prime to ensure that u is a 2-primitive divisor. □

2.3 Elation groups of translation planes.

Theorem 14 (Hering [9], Ostrom [24], [25]). *Let π be a translation plane of order p^r, p a prime, and let G be a collineation group of π in the translation complement. Further, let E denote the collineation group generated by all elations in G.*

Then one of the following situations apply:

(i) *E is elementary Abelian*,

(ii) *E has order $2k$, where k is odd, and $p = 2$*,

(iii) *E is isomorphic to* $\mathrm{SL}(2, p^t)$,

(iv) *E is isomorphic to* $\mathrm{SL}(2, 5)$ *and $p = 3$*,

(v) *E is isomorphic to* $\mathrm{Sz}(2^{2s+1})$ *and $p = 2$.*

Theorem 15 (Johnson and Ostrom [22], (3.4)). *Let π be a translation plane of even order q^2 with spread in* $\mathrm{PG}(3,q)$. *Let E denote a collineation group of the linear translation complement generated by affine elations. If E is solvable then E is elementary Abelian and is a group of elations all with the same axis or E is dihedral of order $2k$, k odd and there are exactly k elation axes.*

Theorem 16 (Johnson [16], (2.1)). *Let π be a translation plane of even order $q^2 \neq 64$ with spread in* $\mathrm{PG}(3,q)$ *admitting exactly $q+1$ elation axes. Then the net of elation axes is derivable if and only if the group generated by the elations is* $\mathrm{SL}(2,q)$ *or is a dihedral group of order $2(q+1)$ where the cyclic stem fixes at least two components of π.*

Theorem 17 (Johnson [18], (3.4)). *Let π be a translation plane of even order $q^2 \neq 64$ with spread in* $\mathrm{PG}(3,q)$. *Let π contain a derivable net N and for each component ℓ of N, assume there is an elation with axis ℓ leaving N invariant. Then, there is a Desarguesian plane Σ such that π is obtained by multiple derivation in Σ; π may be constructed by replacing an odd number of pairwise disjoint derivable nets in Σ.*

Theorem 18 (Johnson [18], (4.3)). *Let N be a net in a finite translation plane π. Let Σ be an affine Desarguesian plane such that each of the components of N is a Baer subplane in some derivable net of Σ that shares two fixed components. If N is not derivable, N is said to be 'twisted' through Σ.*

Let π be a translation plane of even order $q^2 \neq 64$ with spread in $\mathrm{PG}(3,K)$, *where K is isomorphic to* $\mathrm{GF}(q)$. *Assume that π admits $q+1$ elations with distinct axes and that the group D generated by the elations leaves invariant the net N defined by these axes. Then either N is a derivable net and π is multiply derived from a Desarguesian plane Σ or N is twisted through a Desarguesian plane.*

Note. The following is also true from the proof of the above theorem:

If the cyclic stem of the group generated by the elations does not fix at least two components of π, it does fix two Baer subplanes each of which lies across the set of $q+1$ elation axes and are kernel subplanes (i.e. K-subspaces).

The order $q^2 = 64$ is somewhat problematic since the case is not specifically considered in the work of Johnson [18], so is not included in Theorems 16, 17, or 18. Hence, assume that π is a translation plane of order 64 and kernel containing $\mathrm{GF}(8)$ that admits a group generated by elations. If the group is solvable then it is a dihedral group of order $2k$, where k is odd by Theorem 15. Assume that $k = 9$. Let C_9 denote the cyclic stem and let N denote the net defined by the $8+1$ elation axes. Since C_9 acts on $8(8-1)$ components of π, it follows that C_9 fixes two components, which we choose to be represented by $x = 0$ and $y = 0$. Let σ be an elation interchanging $x = 0$ and $y = 0$, where $\sigma : (x, y) \mapsto (y, x)$. Note that the axis of σ is $y = x$.

Then, $C_9 = \langle\chi\rangle$ such that $\chi : (x, y) \mapsto (xT, yT^{-1})$, where T is a 2×2 matrix over $\mathrm{GF}(8)$ of order 9. Note that the set N of images of $y = x$ under $\langle\chi\rangle$ is $\{y = xT^{-2i}; i = 0, 1, \ldots, 8\}$. Consider the $\mathrm{GF}(8)\langle T\rangle$-module generated over $\mathrm{GF}(8)$ by T. Since

$\langle T \rangle$ is clearly an irreducible group acting on a 2-dimensional GF(8)-space, it follows that the centralizer of T is a field $\mathscr{F}$ isomorphic to $GF(8^2)$. Hence, the set of images N of $y = x$ union $x = 0$ and $y = 0$ belong to a Desarguesian affine plane Σ_1 coordinatized by $\mathscr{F}$, and $\mathscr{F}$ extends the kernel subfield K isomorphic to GF(8). It follows that N is a regulus net in Σ_1. However, this implies that each Baer subplane of N incident with the zero vector is a K-subspace. That is, N is also a regulus net when considered in π.

We may consider N to have components

$$y = xm; \quad m^9 = k,$$

where $m \in \mathscr{F}$ and $k \in K$. In this context, N is an André regulus net of Σ_1 with opposite regulus $\{y = x^8 m; m^9 = k\}$. Let $T = t$ when considered in $\mathscr{F}$. Note that

$$(x, x^8 m) \mapsto (xt, x^8 m t^{-1})$$

and that $(xt)^8 m = x^8 m t^{-1}$ since $t^{8+1} = 1$. Hence, $\langle \chi \rangle$ fixes each Baer subplane $y = x^8 m$ of N incident with the zero vector.

Now χ fixes at least three K-subspaces of line size so we may apply Theorem 3 to construct a Desarguesian affine plane Σ consisting of χ-invariant 2-dimensional K-subspaces. Now the normalizer of $\langle \chi \rangle$ is a collineation group of Σ and $\langle \chi \rangle$ is a kernel homology group when acting on Σ. It follows exactly as in Johnson [18] Lemma (3.3), p. 333, that the orbits under C_9 are either of length 1 or 9. Since 9 is a prime power, Theorem (4.3) of Johnson [18] may now be easily extended to show that the component orbits of length 9 are derived regulus nets of Σ. All of this now proves the following theorem:

Theorem 19. *Let π be a translation plane of order* 64 *and kernel containing K isomorphic to* GF(8). *Assume that π admits a group generated by elation groups that is dihedral of order* $2(8 + 1)$.

(1) *Then the net defined by the elation axes is a regulus net.*

(2) *There is an associated Desarguesian affine plane Σ consisting of the* 2*-dimensional K-spaces fixed by the cyclic stem and π may be obtained from Σ by a multiple derivation of mutually disjoint regulus nets of Σ.*

Theorem 20 (Walker [28]). *Consider a finite translation plane π of even order q^2 that is multiply derived from a Desarguesian plane Σ by the replacement of a non-empty set of $\leqslant q/2$ mutually disjoint reguli. Then the full collineation group of the plane π is the group inherited from the associated Desarguesian plane Σ; the collineation group of π permutes the reguli that are replaced.*

Theorem 21 (Foulser, Johnson, Ostrom [4]). *Let π be a translation plane of order q^2 that admits a collineation group isomorphic to* $SL(2, q)$, *where $q = p^r$, p a prime, and the p-elements are elations. Then π is Desarguesian.*

Theorem 22 (André [1]). *If Π is a finite projective plane that has two homologies with the same axis ℓ and different centers then the group generated by the homologies contains an elation with axis ℓ.*

Theorem 23 (Hering [10]). *Let π be a translation plane of even order 2^t that admits a collineation group isomorphic to a Suzuki group $\mathrm{Sz}(2^{2s+1})$ that is generated by elations. Then there exists a Lüneburg–Tits subplane of order 2^{2s} invariant under $\mathrm{Sz}(2^{2s+1})$.*

2.4 Baer groups in translation planes.

Theorem 24 (Jha and Johnson [12]). *Let π be a translation plane of even order q^2 admitting at least two Baer groups of order $2\sqrt{q}$ in the translation complement. Then either π is Lorimer–Rahilly or Johnson–Walker of order 16 or π is Hall.*

Theorem 25 (Jha and Johnson [13]). *Let a translation net of degree $q+1$ and order q^2 contain at least three distinct Baer subplanes incident with the zero vector and coordinatized by the same field K isomorphic to $\mathrm{GF}(q)$. Then the net is a regulus net that corresponds to a regulus in $\mathrm{PG}(3, K)$. Such a regulus net is often called a 'K-regulus net.'*

Theorem 26 (Johnson and Ostrom [22], (3.1)). *Let π be a translation plane of order 2^{2r} and of dimension 2 over its kernel. Let G be a collineation group in the linear translation complement and assume that the involutions in G are Baer. Then the Sylow 2-subgroups of G are elementary Abelian.*

Theorem 27 (Johnson and Ostrom [22], part of (3.27)). *Let π be a translation plane of dimension 2 over $\mathrm{GF}(q)$, where $q = 2^r$. Let G be any subgroup of the translation complement. If the involutions of G are all Baer, let G_1 denote the subgroup of G generated by the Baer involutions in the linear translation complement. If G is nonsolvable then G_1 is isomorphic to $\mathrm{SL}(2, 2^s)$, for some s, and is normal in G.*

If G_1 is irreducible, π has an Ott–Schaeffer subplane of order 2^{2s} and s divides r.

If G_1 is reducible then π is derived from a plane π^* also admitting $\mathrm{SL}(2, 2^s)$ and the involutions in π^* are elations.

2.5 Flocks of quadratic cones.

Theorem 28 (Gevaert and Johnson [6]). *Let π be a translation plane of order q^2 with spread in $\mathrm{PG}(3, q)$ that admits an affine elation group E of order q such that there is at least one orbit of components union the axis of E that is a regulus in $\mathrm{PG}(3, q)$. Then π corresponds to a flock of a quadratic cone in $\mathrm{PG}(3, q)$.*

In the case above, the elation group E is said to be 'regulus-inducing' as each orbit of a 2-dimensional $\mathrm{GF}(q)$-vector space disjoint from the axis of E will produce a regulus.

Theorem 29 (Johnson [20]). *Let π be a translation plane of order q^2 with spread in $\mathrm{PG}(3, q)$ admitting a Baer group B of order q. Then the $q-1$ component orbits union*

$\mathrm{Fix}\,B$ are reguli in $\mathrm{PG}(3,q)$. *Furthermore, there is a corresponding partial flock of a quadratic cone with $q-1$ conics. The partial flock may be uniquely extended to a flock if and only if the net defined by* $\mathrm{Fix}\,B$ *is derivable.*

Theorem 30 (Payne and Thas [26]). *Every partial flock of a quadratic cone of $q-1$ conics in* $\mathrm{PG}(3,q)$ *may be uniquely extended to a flock.*

Theorem 31 (Gevaert, Johnson, Thas [7]). *Let π be a translation plane corresponding to a flock of a quadratic cone and let $\mathcal{B}$ denote the set of reguli of π that share a common line and define the associated spread. Then any two reguli of $\mathcal{B}$ may be embedded into a unique Desarguesian spread.*

Theorem 32 (Jha and Johnson [11]). *The full group of a derived conical flock plane of finite order by one of the base reguli is the group inherited from the conical flock plane or the conical plane is Desarguesian of order* 4 *or* 9.

3 Sketch of the proof

Notation 1. *In all of the sections, G is a collineation group of order $q(q+1)$ in the linear translation complement of a translation plane π of even order q^2, $q=2^r$, with spread in* $\mathrm{PG}(3,K)$, *where K is isomorphic to* $\mathrm{GF}(q)$.

In this section, we indicate the general nature of the proof that the linear group G of order $q(q+1)$, for $q=2^r$, must leave invariant some 2-dimensional K-subspace. Furthermore, we show that the invariant subspace is either a component or a Baer subplane. We recall that a prime 2-primitive divisor u of q^2-1, is a divisor of q^2-1 that does not divide 2^t-1 for $t<2r$. Let S_2 be a Sylow 2-subgroup of order q. A major step towards our goal is to show there is a unique Sylow 2-subgroup that is either a regulus-inducing elation group, a Baer group or the maximal elation or Baer group has order $q/2$. In either of the first two situations, we may use established results to show that our translation plane is either a conical flock plane or a derived conical flock plane.

By Theorem 12, if $q^2\neq 64$, there is always such a divisor u. In this case, let g_u be an element of G of prime order u. We first show that if g_u normalizes an elation subgroup E' then g_u centralizes E'. This will, in turn, imply the existence of an associated Desarguesian affine plane Σ, whose spread consists of the g_u-invariant linesize K-subspaces. In this setting, the normalizer of $\langle g_u\rangle$ in G becomes a subgroup in the translation complement $\Gamma\mathrm{L}(2,q^2)$ of Σ.

If there exist a non-trivial elation subgroup E' and g_u normalizes E' and we obtain an associated Desarguesian spread as a tool to use in the proof. Thus, when we establish that there is an invariant component, we ultimately will complete the proof of the main result by the consideration of an associated Desarguesian affine plane.

If g_u does not normalize E' then we show that there are at least two non-trivial elation subgroups and we may use Theorem 14 to determine the group generated by the elations within G. Since G has order $q(q+1)$, this will normally imply a contradiction when the group generated by the elations is non-solvable and of the form $SL(2, 2^s)$ or $Sz(2^e)$.

If a Sylow 2-subgroup S_2 is not an elation group then S_2 will fix a component ℓ and fix a unique 1-dimensional subspace pointwise on this component. Since we basically are trying to move the component ℓ, we assume that g_u leaves ℓ invariant and argue that g_u becomes an affine homology with axis ℓ. If S_2 contains a non-trivial elation subgroup E' then g_u must centralize E', implying that E' leaves invariant the coaxis (the 'coaxis' is the component whose projective extension contains the center of g_u), which, of course, cannot occur.

Hence, the use of 2-primitive collineations g_u and the existence of non-trivial elations will show that there is an associated Desarguesian affine plane whose spread consists of g_u-invariant linesize K-subspaces or the group generated by the elations is dihedral of order $2k$, where k is odd and equal to the number of elation axes.

If there are no affine elations then all involutions are Baer and we are able to show that the group generated by the Baer involutions is solvable. What this will mean is that there is an invariant Baer subplane, an invariant 2-dimensional K-subspace that is not a component. Again, we will later use the model of an associated Desarguesian affine plane Σ to complete our main result.

Of course, when $q^2 = 64$, we have noted in the background section that, even in this case, similar results apply that do not rely directly on 2-primitive collineations.

The pieces of the proof are as follows:

I. Establish that there is a G-invariant 2-dimensional K-subspace Ω that is either a component of π or a Baer subplane of π.

II. If g_u is a collineation whose order is a prime 2-primitive divisor u of $q^2 - 1$, use the invariance of Ω to show that there exists an associated Desarguesian affine plane Σ whose spread is the set of g_u-invariant 2-dimensional K-subspaces. The normalizer of $\langle g_u \rangle$ in G is a subgroup of the full translation complement $\Gamma L(2, q^2)$ of Σ.

III. When Ω is a component, show that there is an elation subgroup E of order q or $q/2$.
   (1) When the order of E is q, with the additional hypothesis that E leaves invariant some 2-dimensional K-subspace over which it induces a non-trivial group,
      (a) show the plane π is a conical flock plane,
      (b) then show that, in this case, π is Desarguesian. Finally,
      (c) Remove the hypothesis that E leaves invariant some 2-dimensional K-subspace.
   (2) When the order of E is $q/2$, show the plane π may be multiply derived from Σ by the replacement of a set of $q/2$ mutually disjoint regulus nets in Σ.

IV. When Ω is a Baer subplane, show that there is a Baer group B of order q or $q/2$.
   (1) When the order of B is q,
      (a) show that the plane π is a derived conical flock plane and
      (b) then show that, in this case, π is Hall.

(2) When B has order $q/2$, argue that there can be no such plane; this case does not occur.

V. General comments on the interaction of elations, Baer involutions and 2-primitive collineations:
   (1) Analyze, in general, 2-primitive collineations, where connections to the associated Desarguesian affine plane are obtained.
   (2) If non-trivial elations exist, this implies that either we have an invariant component and an associated Desarguesian affine plane or the group generated by elations is either $\mathrm{SL}(2, 2^s)$, $\mathrm{Sz}(2^e)$ or dihedral D_k, where k is odd.
   Furthermore, a general analysis of $\mathrm{SL}(2, 2^s)$ can also be used when there are Baer involutions:
   (a) study the situation when there is a subgroup isomorphic to $\mathrm{SL}(2, 2^s)$ generated by elations or Baer involutions.
   We then consider the remaining two groups by
   (b) analysis of $\mathrm{Sz}(2^e)$ and
   (c) consideration of D_k.
   (3) If the involutions are Baer, this ultimately implies that there is an invariant Baer subplane as a 2-dimensional K-subspace.

The analysis of the possible situations when there is not a 2-primitive divisor, when $q^2 = 64$, is interwoven in the general arguments.

Lemma 1. *In a translation plane π with spread in* $\mathrm{PG}(3, q)$, *q even, any linear involution is either an elation or a Baer* 2*-element* (*fixes an affine Baer subplane pointwise*).

Proof. This is almost obvious by results of Baer but we include a simple proof in the translation plane case. Since a linear 2-element must fix at least a 1-dimensional $\mathrm{GF}(q)$-subspace pointwise, either the element is quartic or fixes a 2-dimensional $\mathrm{GF}(q)$-subspace pointwise. If the fixed-point space is a line, the element is an elation. If the fixed-point-space is not a line, it is decomposed by the spread of π into a spread itself. Hence, there is an induced translation plane of order q on the fixed-point-space producing the Baer subplane.

The minimal polynomial of a linear involution is $x^2 - 1$ so it clearly follows that every involution cannot fix exactly a 1-dimensional $\mathrm{GF}(q)$-subspace pointwise. □

4 2-primitive collineations

Our first lemma is also given in [11], Lemma 2, as the proof is independent of order.

Lemma 2. *Let u be a* 2*-primitive divisor of $q^2 - 1$, for $q \neq 8$, and let g_u be a collineation of order u in G. If G admits a* 2*-collineation group E' such that g_u normalizes E' then g_u centralizes E'.*

Furthermore, the g_u-invariant subspaces of line size are K-subspaces and define the spread for a Desarguesian affine plane Σ (*with spread in* $\mathrm{PG}(3, K)$). *The normalizer of $\langle g_u \rangle$ in G is a collineation group of* Σ.

More generally, if g_u fixes at least three components of π, the same conclusion on the existence of a g_u-invariant Desarguesian plane still applies.

Lemma 3. *Assume that u is a prime 2-primitive divisor of $q^2 - 1$ and g_u is a collineation in G of order u.*

(1) *If g_u fixes a component ℓ and fixes a 1-dimensional K-space on ℓ then g_u is an affine homology.*

(2) *Let G_ℓ denote the subgroup of G fixing ℓ and let $S_{2,\ell}$ denote a Sylow 2-subgroup of G_ℓ. Assume that $S_{2,\ell}$ is not an elation group with axis ℓ. Let $S_{2,[\ell]}$ denote the subgroup of $S_{2,\ell}$ fixing ℓ pointwise. If g_u also fixes ℓ then either*
   (a) *g_u is an affine homology,*
   (b) *$q = 2$, or*
   (c) *$S_{2,\ell}/S_{2,[\ell]}$ has order 2 and there is a dihedral group of order $2z$ induced on ℓ by G_ℓ, for z odd > 1.*

Proof. Assume the conditions of (1). Since the number of points in a 1-dimensional K-subspace X is $q - 1$, it follows that g_u must fix X pointwise. However, there is a g_u-invariant complement of X by Maschke's theorem, which implies that g_u fixes ℓ pointwise. This proves (1).

Now assume that $S_{2,\ell}$ fixes ℓ and hence fixes a 1-dimensional K-subspace X pointwise. Since $S_{2,\ell}$ is not an elation group, the group induced on ℓ has order > 1.

If g_u also fixes ℓ and is not an affine homology then by (1), g_u must move X. Since ℓ may be considered an affine Desarguesian plane π_ℓ of order q, it follows that the involutions in G_ℓ acting on ℓ generate a normal subgroup W of G_ℓ that is in $GL(2, q)$. Since g_u moves X, there are at least two involutions in G_ℓ that act as elations with different axes of π_ℓ.

Hence, $W \mid \ell$ is either isomorphic to $SL(2, 2^t)$, or is dihedral of order $2z$ where z is odd. Note in any case, $g_u \mid \ell$ does not centralize $W \mid \ell$ since then g_u would fix X.

First assume that the generated group is $SL(2, 2^t)$, then t divides r and since g_u cannot fix a 1-dimensional K-subspace, it follows that u must divide the number of elation axes of π_ℓ. Hence, u divides $2^t + 1$ so that u divides $(2^{2t} - 1, 2^{2r} - 1) = 2^{2(t,r)} - 1$. Since u is a 2-primitive divisor of $2^{2r} - 1$, it follows that $(t, r) = r$ so that $SL(2, q)$ is generated on ℓ. But, the order of G is $q(q + 1)$, a contradiction unless $q - 1 = 1$ so that $q = 2$, possibility (b).

Now assume that $W \mid \ell$ is D_z, a dihedral group of order $2z$, where z is odd, possibility (c). Clearly, this proves all parts of the lemma. □

5 *g* cannot normalize any special linear group

In this section, we show that if $SL(2, 2^t) \lhd G$ then $q = 2$ or $t = 1$. This is established by first showing that in the non-degenerate case $SL(2, 2^t)$ must be centralized by an element g_u of order u, a 2-primitive divisor of $q^2 - 1$. Thus, g_u centralizes the elations or Baer involutions in $SL(2, 2^t)$. These cases are ruled out separately.

Lemma 4. *Assume that G contains a normal subgroup isomorphic to* $SL(2, 2^t)$, *where*

$q = 2^r$, and a 2-primitive element g_u for a prime 2-primitive divisor u of $q^2 - 1$. Then one of the two following situations occur:

(a) $q = 2$,

(b) *g_u centralizes* $\mathrm{SL}(2, 2^t)$.

Proof. There are $2^t + 1$ Sylow 2-subgroups permuted by g_u. If u divides $2^t + 1$, then the argument of the previous lemma that u divides $(2^{2t} - 1, 2^{2r} - 1) = 2^{2(t,r)} - 1$ shows that G must contain $\mathrm{SL}(2, q)$, which by order implies that $q = 2$. If g_u normalizes a Sylow 2-subgroup of $\mathrm{SL}(2, 2^t)$ but does not normalize all Sylow 2-subgroups then g_u fixes exactly two of them and hence divides $2^t - 1$, implying that $t = 2r$. However, this cannot occur by order.

Hence, it follows that g_u normalizes all Sylow 2-subgroups. There are $2^t - 1$ non-identity elements in each Sylow 2-subgroup and since each such group is elementary Abelian, it follows that either g_u centralizes the Sylow 2-groups and hence centralizes $\mathrm{SL}(2, 2^t)$ or again $\mathrm{SL}(2, q)$ is generated implying $q = 2$. □

Lemma 5. *Assume under the conditions of the previous lemma that g_u centralizes* $\mathrm{SL}(2, 2^t)$. *If* $\mathrm{SL}(2, 2^t)$ *is generated by elations then $t = 1$.*

Proof. Since g_u centralizes $\mathrm{SL}(2, 2^t)$, it must fix each of the $2^t + 1 \geqslant 3$ elation axes associated with the S_2-subgroups of $\mathrm{SL}(2, 2^t)$. So, by Lemma 2, g_u fixes a unique Desarguesian spread Σ, consisting of the rank two spaces invariant under g_u. Hence, by the centrality hypothesis, $\mathrm{SL}(2, 2^t) \times \langle g_u \rangle \leqslant \Gamma\mathrm{L}(2, q^2)$. So, in particular, $\mathrm{GF}(2^t)$ is a subfield of $\mathrm{GF}(2^{2r})$, recall that $q^2 = 2^{2r}$, so $t \mid 2r$. But, since $|\mathrm{SL}(2, 2^t)|$ divides $|G| = q(q + 1)$, we further have that $2^{2t} - 1 \mid 2^r + 1$. If $t \mid r$, we have a contradiction, unless $t = 1$, since this now implies $2^t - 1$ simultaneously divides $2^r - 1$ and $2^r + 1$ ($=q + 1$). Thus, $t \mid 2r$ and yet $t \nmid r$, forcing $t = 2z$, where z divides r.

But, $2^{2t} - 1 = (2^t - 1)(2^t + 1)$. Furthermore, $(2^t - 1) = (2^z - 1)(2^z + 1)$.

Since $2^z - 1$ divides $2^r - 1$, $(2^r - 1, 2^r + 1) = 1$ and $2^{2t} - 1$ must divide $|G| = q(q + 1)$, then $((2^z - 1), (q + 1)) = 1 = (2^z - 1)$, implying that $z = 1$. Hence, $t = 2$.

But, if $t = 2$ and t does not divide r then r is odd so that

$$2^{2 \cdot 2} - 1 \text{ divides } 2^r + 1, \text{ which divides } 2^{2r} - 1.$$

However, this implies that 4 divides $2r$, a contradiction. Hence, the only possibilitity is that $t = 1$.

This proves the lemma. □

We have seen that the group $\mathrm{SL}(2, 2^t) \lhd G$, for $t > 1$, is not possible if its involutions are elations. It remains to rule out the case when its involutions are all Baer. Part of the argument involves reducing to the elation case by derivation.

Lemma 6. *Assume the conditions of the previous lemma and assume that g_u centralizes* $\mathrm{SL}(2, 2^t)$. *If* $\mathrm{SL}(2, 2^t)$ *is generated by Baer 2-elements then $t = 1$.*

Proof. Assume that $t > 1$. We now apply Theorem 27, using the notation introduced there. Thus, either G_1 is irreducible and π has an Ott–Schaeffer subplane of order 2^{2s} or G_1 is reducible and π is derived from a plane also admitting $SL(2, 2^s)$ and the involutions in the derived plane are elations. In the latter case, the Sylow 2-subgroups fix Baer subplanes pointwise and since the Baer groups are linear, it follows that there are $2^s + 1 > 3$ Desarguesian Baer subplanes in the same derivable net. It follows from Theorem 25 that the net is a GF(q)-regulus net. Since G_1 is normal and fixes pointwise the set of $q + 1$ infinite points of this net and fixes no other points, it follows that G leaves invariant this regulus net. Then we may derive and apply the previous lemma to complete the proof when G_1 is reducible.

Hence, assume that G_1 is irreducible and we have an Ott–Schaeffer subplane π_o of order 2^{2s}. We know that s must divide r when $q = 2^r$ by Theorem 27. The kernel of π_o is isomorphic to GF(2^s) and may be considered a subfield of GF(q). Thus, there are $(2^r - 1)/(2^s - 1)$ Ott–Schaeffer subplanes on the same set of $2^{2s} + 1$ components and these subplanes are in an orbit under the kernel homology group of order $q - 1$. Each Sylow 2-subgroup fixes exactly one component of a set of $2^s + 1$ components of π_o and, since the group is GF(q)-linear, fixes a 1-dimensional GF(q)-subspace pointwise on the unique fixed component. Since G_1 is normal, it follows that this set S of $2^s + 1$ 1-dimensional GF(q)-subspaces must be permuted by the full group G. Thus, there are exactly $(2^r - 1)/(2^s - 1)$ Ott–Schaeffer subplanes left invariant by G_1. Since we have a 2-primitive collineation g_u in G, g_u must fix each 1-dimensional subspace of S, implying a contradiction.

Hence, $t = 1$ as above. □

Corollary 1. $SL(2, 2^t)$ *cannot be normal in* G *unless* $t = 1$.

6 Non-solvable elation groups and 2-primitive collineations

We note from Theorem 9 that any translation plane of order 16 with kernel GF(4) is either Desarguesian or Hall or there is a component that is invariant under the full translation complement of the plane. Since our goal is to establish that there is an invariant component when the plane is not Desarguesian or Hall, we may assume that $q > 4$.

Theorem 33. *If* $2^6 \neq q^2$ *and there are elation groups of order at least* 4 *then we obtain a G-invariant component.*

Proof. If there is not a G-invariant component then the group generated by elations is isomorphic to $SL(2, 2^s)$ or $Sz(2^{2t+1})$. By Section 5, if we have $SL(2, 2^s)$ then $s = 1$, contrary to our assumptions. Hence, we may assume that we have the latter case.

Thus, by Theorem 23 there is a Lüneburg–Tits subplane of order $2^{2(2r+1)}$ left invariant by $Sz(2^{2t+1})$. We have an element g_u of order a prime 2-primitive divisor of $q^2 - 1$ permuting the elation axes.

If g_u acts semi-regularly on the elation axes then u divides

$$(2^{2r} - 1, 2^{4(2t+1)} - 1);$$

this implies that r divides $(4(2t + 1), 2r) = 2r$. Hence, $2t + 1 = r$ or $r/2$. So, either the plane is a Lüneburg–Tits plane or the group is $Sz(\sqrt{q})$. In the first case, the plane does not admit a 2-primitive collineation by Theorem 10.

Hence, the group is $Sz(\sqrt{q})$.

So, there are exactly $q+1$ elation axes. Therefore, there is an $Sz(\sqrt{q})$-invariant Lüneburg–Tits plane of order q. However, we are in a translation plane of order q^2 and kernel containing K isomorphic to $GF(q)$, so this subplane cannot be left invariant under the kernel homology group. Since the kernel of the subplane is exactly $GF(\sqrt{q})$, this implies that we have at least $\sqrt{q}+1$ Lüneburg–Tits subplanes of order q on the same set of $q+1$ components.

But, also by order, $q(q+1)(\sqrt{q}-1)$ must divide $q(q+1)$, a contradiction.

So, g_u does not act semi-regularly on the set of elation axes and hence must fix an elation axis, implying that it fixes at least two. If u divides $2^{2(2t+1)}-1$, it can only be that $r=(2t+1)$, which has been considered previously above.

Hence, g_u fixes at least three elation axes. But this means that g_u normalizes at least three Sylow 2-subgroups of $Sz(2^{2t+1})$. Since $q>2$, it follows from Lemma 2 that g_u commutes with at least three Sylow 2-subgroups. A given Sylow 2-subgroup of $Sz(2^{2t+1})$ acts transitively on the remaining Sylow 2-subgroups so that the Suzuki group is generated by two of its Sylow 2-subgroups. This implies that g_u centralizes $Sz(2^{2t+1})$. But, then by Lemma 2, $Sz(2^{2t+1})$ is a collineation group of a Desarguesian affine plane, a contradiction. This completes the proof of the theorem. □

7 The elations generate a solvable group, $q^2 \neq 64$

Theorem 34. *Assume that* $q^2 \neq 16$ *or* 64. *If the elations generate a solvable group then there is a G-invariant component or* $q=2$.

We shall give the proof as a series of lemmas. We assume throughout this section that the elations generate a solvable group. Note that some arguments used in proving the theorem will used in the next section for the case $q^2=64$.

Assume that there is not an invariant component.

Lemma 7. *The subgroup generated by the elations is dihedral of order* $2(q+1)$.

Proof. Since the elations generate a solvable group by hypothesis and there is not an invariant component then, by Theorem 15, the elations generate D_k, a dihedral group of order $2k$, where k is odd and >1. Then this group contains a characteristic cyclic group C_k of order k, which is then normalized by G. Furthermore, any Sylow 2-subgroup of order 2^r must permute the k elation axes and fix at least one such axis. Since $q^2 \neq 64$, there is a 2-primitive divisor u and a corresponding group element g_u.

Let ℓ be an elation axis of an elation σ. Then the Sylow 2-subgroup S_2 of G containing σ must leave ℓ invariant. Hence, the group induced on ℓ is elementary Abelian of order $q/2$.

Assume that g_u fixes an elation axis. We now apply Lemma 3, and the reader is directed back to this lemma for the notation.

Either g_u is an affine homology, $q=2$, or the group induced on ℓ by $S_{2,\ell}$ has order 2.

In the latter case, $q/2=2$ so $q=4$, which has been excluded from consideration.

Thus, either $g=2$ or g_u is an affine homology. But, then the affine homologies must centralize the unique non-identity elation σ with axis ℓ, implying that σ must leave invariant the co-axis of g_u, a contradiction.

Since u cannot fix an elation axis without g_u centralizing the elation, we may assume that g_u does not centralize D_k. But, we have otherwise that u divides k and is semi-regular on the elation axes.

We claim that the stabilizer of a component must fix the 1-dimensional K-subspace fixed pointwise by a Sylow 2-subgroup S_2, since if not then, as we have a group of order $q/2$ induced on an elation axis, it would follow that $\mathrm{SL}(2,2^{r-1})$ is generated on any axis. But, then

$$(2^{2(r-1)}-1) \text{ divides } (q+1).$$

Also,

$$(2^{2(r-1)}-1)=(2^{2(r-1)}-1,2^{2r}-1)=(2^{2(r-1,2r)}-1)=(2^2-1).$$

Therefore, $r=2$ so that $q=2^r=4$, contrary to assumption.

Hence, the stabilizer of a component must fix a 1-dimensional K-subspace X.

The full group in $\mathrm{GL}(2,q)$ acting on a component and fixing X has order dividing $q(q-1)^2$. Thus, the stabilizer of a elation axis in G can have order exactly q since $q+1$ is odd. So, we have at least $q+1$ elation axes and, furthermore, $q+1$ must divide the number k of elation axes. Let $k=k'(q+1)$. Since G has order $q(q+1)$, the normalizer of a Sylow 2-subgroup leaves invariant an elation axis. Hence, there are then exactly $q+1$ Sylow 2-subgroups and there is an unique non-identity elation in each such group, implying that the number k of elation axes is $q+1$. Hence, $k'=1$. This completes the proof of the lemma. □

Lemma 8. *If a Sylow* 2*-subgroup* S_2 *acts transitively on the remaining* q *elations axes not fixed by* S_2 *then* $q=2$ *or* 4.

Proof. If we assume that each Sylow 2-subgroup acts transitively on the remaining q elation axes other than the axis fixed by the contained elation, we have sharp 2-transitivity, which implies by Theorem 5 that $q+1$ is an odd prime power t^b. When $q+1=t^b$, by Theorem 13, it follows that $b=1$, since $q\neq 8$. Now, we have a sharply 2-transitive group of degree $q+1$, implying there is an associated nearfield plane of order $q+1=t$. But, nearfield planes of odd prime order are Desarguesian. The group may be then be identified with the group

$$x\mapsto xa+b \quad \text{for } a\neq 0, b\in \mathrm{GF}(t).$$

This means that the group of order $t-1=q$ is cyclic. However, this group fixes a 1-dimensional K-space pointwise and induces a faithful group of order $q/2$ on a component ℓ. The group induced is an elation group on the associated Desarguesian affine plane of order q defined on the component. Hence, the group of order $q/2$ must be elementary Abelian. But, since it is also cyclic, it follows that $q/2 = 1$ or 2, so that $q = 2$ or 4. □

Assume the background hypothesis for this section: there is not a G-invariant component. So, Lemmas 7 and 8 hold.

Lemma 9. *G leaves invariant a component, unless $q = 2$ or 4.*

Proof. Suppose $q > 4$. Then Lemma 8 implies that some Sylow 2-subgroup S fixes two distinct elation axes, associated with, say elations α_1 and α_2. Let α be the unique elation in S. Then α clearly normalizes α_1 and α_2, hence centralizes them. Thus, each α_i, $i = 1, 2$, must leave invariant the components $\mathrm{Fix}(\alpha_i)$, so α_1 and α_2 have the same axis, viz. α, contrary to hypothesis. □

8 $q^2 = 64$

We extend Theorem 34 when $q^2 = 64$.

Theorem 35. *If $q = 8$ and there is an elation in G then there is an invariant component.*

Proof. If there are elation groups of order 4, the previous arguments apply to show there is an invariant component. Now assume that there is a elation but no invariant component. Then we have a dihedral group D_k generated by the elations of order $2k$, k is odd. The group G has order $8 \cdot 9$, so $k = 3$ or 9. We claim that $k = 9$. So, assume that $k = 3$ and let g_3 be an element of D_3 of the cyclic stem C_3 of order 3. Note that a Sylow 2-group of order 8 must fix an elation axis L. Hence, g_3 cannot fix an elation axis. Let S_3 be a Sylow 3-subgroup. Then S_3 also acts on the 3 elation axes, implying that there is an element of order 3 that fixes an elation axis. Since there is a unique elation per elation axis, it follows that there is an element h_3 of order 3 that commutes with D_3. But, the group of order 8 acting on a component L fixes a 1-dimensional K-subspace X pointwise, and induces a faithful group on L of order 4. Hence, h_3 must leave invariant X and permutes 7 non-zero points. This implies that h_3 fixes X pointwise and is therefore a planar group. But, it can only be that h_3 is Baer but then a Baer group has order dividing 7, a contradiction.

Hence, we have a dihedral group of order $2(8 + 1)$.

If we now apply Theorem 19, we see that the proofs of Lemmas 7, 8 and 9 apply when $q = 8$ to obtain a contradiction. □

9 Baer 2-groups

Theorem 36. *If a translation plane of even order q^2 admits a linear collineation group of order $q(q + 1)$ such that all involutions are Baer then there is an invariant Baer subplane or $q = 2$, 4 or 8.*

We give the proof as a series of lemmas.

Lemma 10. *G is solvable.*

Proof. If the involutions of G are all Baer, let G_1 denote the subgroup of G generated by the Baer involutions in the linear translation complement. If G is nonsolvable then G_1 is isomorphic to $SL(2, 2^s)$ for some s and is normal in G by Theorem 27. However, Corollary 1 implies that $s = 1$, a contradiction. Hence, G_1 is solvable, implying that G is solvable. □

Lemma 11. *Assume that $q > 4$. If there is not a G-invariant Baer subplane then there are exactly $q + 1$ Sylow 2-subgroups, each Sylow 2-subgroup fixes exactly one component and these components are in an orbit $O(\ell)$ under G.*

Proof. A Sylow 2-group S_2 is elementary Abelian by Theorem 26. Hence each element σ of S_2 fixes a Baer subplane π_σ pointwise that is also left invariant by S_2. Thus, $S_2 \mid \pi_\sigma$ is an elation group acting on π_σ, a Desarguesian subplane of order q. If S_2 is a Baer group and π_σ is not G-invariant then the plane is Hall or order 16 by Theorem 24. Similarly, we may assume that if π_σ is not G-invariant then S_2 cannot contain a Baer group of order $\geqslant 2\sqrt{q}$. Hence, S_2 induces a faithful group on π_σ of order at least $q/\sqrt{q} = \sqrt{q}$. Moreover, S_2 fixes a unique component, since otherwise S_2 would be, in fact, a Baer 2-group.

We have $\sqrt{q} > 2$, by assumption.

Note that every element of order dividing $q + 1$ is q-primitive. If any such element g of prime power order leaves invariant the component ℓ containing $X = \text{Fix}\, S_2$, then either a non-solvable group is generated on ℓ, as this group is generated by elation groups on ℓ of order >2, or X is invariant and g is an affine homology with axis ℓ. Then S_2 must fix the coaxis or there is a generated elation. Hence, S_2 fixes two components implying that S_2 is Baer.

Therefore, the orbit $O(\ell)$ containing ℓ has length $q + 1$. Let S_2 and S_2' be distinct Sylow 2-subgroups. Assume that they both fix a common component in $O(\ell)$. Since the order of the group is $q(q + 1)$, this is a contradiction. Now each Sylow 2-subgroup fixes at least one of the $q + 1$ components in $O(\ell)$, implying that there are exactly $q + 1$ Sylow 2-subgroups and each such group fixes a unique component in $O(\ell)$. □

Lemma 12. *Let $q > 4$ and assume the hypothesis of the previous lemma. Let $\sigma \in S_2 - \{1\}$ and let π_σ denote the Baer subplane fixed pointwise by σ. Then the stabilizer of π_σ in G is S_2.*

Proof. We know that the elation group S_2 induces on a fixed-point subplane π_σ, for $\sigma \in S_2$, a group of order at least $\sqrt{q}$; an elation group acting on π_σ. If some element h not in S_2 fixes π_σ then the elation axis cannot be left invariant and hence there are two elation axes in π_σ. Since S_2 and $S_2^h \neq S_2$ now fix π_σ, the group generated by S_2 and S_2^h induced on π_σ contains the group generated by elations. If $\sqrt{q} > 2$, we have a non-solvable group. Hence, $\sqrt{q} = 2$, so that $q = 4$. Thus, the stabilizer of a Baer subplane π_σ is S_2 for $\sigma \in S_2$. □

Lemma 13. *Assume the conditions of the previous lemma. The subplane π_σ of the previous lemma shares exactly one component with $O(\ell)$. Hence, S_2 is transitive on $O(\ell) - \{\ell\}$, assuming that S_2 fixes ℓ.*

Proof. Assume that π_σ shares two of its components with $O(\ell)$, say ℓ and m. Assume that S_2 fixes ℓ. There exists an element h that maps ℓ into m so that π_σ and $\pi_\sigma h$ share m so that σ and σ^h fix m. Since $\pi_\sigma \neq \pi_\sigma h$ then $\sigma \neq \sigma^h$. Hence, the group $\langle \sigma, \sigma^h \rangle$ properly contains S_2, implying that the stabilizer of m has order strictly larger than q. Thus, any Baer subplane has exactly one component in $O(\ell)$. This means that S_2 is transitive on $O(\ell) - \{\ell\}$. □

Lemma 14. $q + 1$ *is a prime number u or* $q = 8$.

Proof. Our group is of order $q(q+1)$ and acts transitively on the orbit $O(\ell)$. Since S_2 acts transitively on $O(\ell) - \{\ell\}$, the group is sharply doubly transitive on $O(\ell)$. Therefore, $q + 1$ is a prime power and since q is even, $q + 1$ must be a prime u by an application of Theorem 13, or $q = 8$. □

Lemma 15. *If there is not an invariant Baer subplane then* $q = 2, 4$.

Proof. Assume that $q > 8$. Then we may apply the previous lemmas. By Theorem 5, we now have that the group corresponds to a nearfield plane of order u. Since q is even, the nearfield group must arise from a Desarguesian subgroup of $A\Gamma L(1, u)$. Since u is prime, it follows that the group corresponds to the group generated by the following elements:

$$x \mapsto xa + b \quad \text{for all } a, b \in GF(u), \text{ for } a \neq 0.$$

Hence, a Sylow 2-subgroup is cyclic. Since we know the group is elementary Abelian, it follows that $q = 2$.

If $q = 8$ and we do not obtain a cyclic Sylow 2-subgroup, as above, then we have a proper nearfield group in $A\Gamma L(1, 9)$. However, here the Sylow 2-subgroups are quaternion. Since we have elementary 2-subgroups, we have a contradiction. □

10 There is an invariant 2-space

Theorem 37. *Let π be a translation plane of even order q^2 with spread in* $PG(3, q)$ *that admits a linear collineation group G of order $q(q+1)$. Then either*

(1) *G leaves invariant a 2-dimensional* $GF(q)$*-subspace, or*

(2) *π is Desarguesian of order* 4 *or* 16.

Proof. Apply Theorems 33, 34, 35 and 36. □

11 There is an invariant line and there exists an elation group of order q acting non-trivially on a Baer subplane

Theorem 38. *If there is an elation group E of order q fixing a 2-dimensional K-subspace and inducing a non-trivial group on this subspace then the plane is a conical flock plane.*

Proof. If there is an invariant line, assume that there is an elation. If the Sylow 2-subgroup is an elation group E then this group must fix a K-invariant Baer subplane π_o. Also assume that $q^2 \neq 64$ so there is a 2-primitive divisor u and corresponding element g_u of order u. Since the order of E is q, it follows that g_u must centralize E. Since g_u fixes at least two components, it is forced to fix at least three since the element g_u commutes with E. Hence, there is a Desarguesian plane Σ whose spread is the set of g_u-invariant 2-dimensional K-subspaces including the axis of E. If g_u leaves π_o invariant then it fixes π_o pointwise, which cannot be the case since any Baer group has order dividing $q(q-1)$.

We have the following conditions: (1) g_u cannot leave π_o invariant, (2) π_o is a Baer subplane of Σ and (3) E and g_u act as collineations of Σ, (4) g_u is a kernel homology of Σ.

Furthermore, E acts transitively on the nonaxis components of a Desarguesian subplane of order q and leaves invariant a Baer subplane π_o of Σ.

This Baer subplane defines a regulus $\mathcal{N}$ of Σ containing the axis of an elation in E. Since E leaves invariant $\mathcal{N}$ and is transitive on the non-axis components, it follows that E is 'regulus inducing.' So, we must have a conical flock plane by Theorem 28.

Now assume that the order is 64. We are assuming the situation when there is an E elation group of order q. Moreover, our hypothesis requires that E also fixes a Baer subplane π_o. We have a group of order 9 that acts on the G-invariant component ℓ. Note that our group is linear and 3 is a q-primitive divisor. Hence, elements of order 3 centralize E. An element σ of order 3 fixes another component that cannot be fixed by E so it fixes at least three components. By Theorem 19, there is a Desarguesian spread admitting G since σ is normal in the full collineation group. But E fixing π_o implies that E is regulus-inducing and hence, the plane is a conical flock plane. So, in all cases, if there is a normal elation group, we have a conical flock plane. □

12 There is an invariant line and the elation group is non-trivial of order $<q$

Hence, assume that we do not have an elation group of order q. However, assume that there are elations.

Theorem 39. *If there are elations in G but the Sylow 2-group is not an elation group then the elation group with fixed axis has order $q/2$.*

Proof. In this case, $q = 8$ or there is a 2-primitive divisor. Even when $q = 8$, there is a q-primitive divisor 3. Let ℓ denote an elation axis and G_ℓ the subgroup of G that leaves ℓ invariant. Note that G_ℓ contains a Sylow 2-subgroup S_2 of G. Hence, there must exist elements of G_ℓ that fix exactly a 1-dimensional K-subspace X of ℓ pointwise. Assume that the group S_2 induces a group of order 2^s on the component ℓ, where $q = 2^r$. Assume that X is not left invariant by g_u, where g_u is a element of order u, a p-primitive divisor of $q^2 - 1$ or $u = 3$. Then either $\mathrm{SL}(2, 2^s)$ must be generated on ℓ or $s = 1$ and a dihedral group D_k is generated on ℓ. In this first case, we still see that g_u must normalize $\mathrm{SL}(2, 2^s)$. If g_u normalizes $\mathrm{SL}(2, 2^s)$ suppose that u divides the number of Sylow 2-subgroups $2^s + 1$. Then u divides $2^{2s} - 1$, implying that $2s$ is divisible by $2r$ so that $q = 2^s$. This implies that $q = 2$. Thus, we must have the second alternative so that D_k is generated on ℓ.

Hence, either X is invariant or there is an elation group of order $q/2$ (since we have assumed that the full Sylow 2-subgroup is not an elation group in this setting).

Assume that X is G-invariant. In this setting, since there are elations, it follows that the g_u-element (or the 3-element if $q = 8$) centralizes the elation group or the elation group has order q^2, and the latter is contrary to order. On the other hand, the g_u-element must fix X and, by Maschke's theorem, is forced to be an affine homology, which is a contradiction since an elation is then forced to fix the coaxis, which cannot occur. □

Theorem 40. *If the elation group has order $q/2$ then either π is Desarguesian or π is a translation plane obtained from a Desarguesian plane by multiple derivation of a set of $q/2$ mutually disjoint regulus nets that are in an orbit under an elation group of order $q/2$.*

Proof. We use the notation developed in the previous arguments.

We then consider the situation when there is an elation group E of order $q/2$ so that g_u does not fix X (S_2 now induces a group of order 2 on ℓ). We see that g_u centralizes E, fixes the axis of E and fixes at least one additional component which cannot be fixed by E so that g_u fixes at least three components for $q/2 > 1$. Hence, there is a corresponding Desarguesian affine plane Σ of g_u-invariant components. We claim that $\langle g_u \rangle$ is normal in G. We note that the group acting on the invariant component ℓ is a dihedral group of order $2k$ that acts as a transitive group on k 1-dimensional K-subspaces. Let h be any element of G of order dividing $q + 1$. Suppose that h^j has prime power order. Since h normalizes E, it follows that, since $|h^j|$ cannot divide $|E - \{1\}|$, there are fixed points under h^j. Since E is elementary Abelian, it also follows that acting on the elements ('points') of E by conjugation, h^j fixes all points of E; h^j centralizes E. It then follows that any element of order dividing $q + 1$ centralizes E. Therefore, we have that there can be no affine homologies in G with axis ℓ. So G must act transitively on the 1-dimensional K-subspaces of ℓ. The kernel homology group of order $q - 1$ and the group of order $2(q + 1)$ normalize $\langle g_u \rangle$ in the quotient group $G/G_{[\ell]}$ ($G_{[\ell]}$ is the group fixing ℓ pointwise), is E, as noted above. Hence, $h g_u h^{-1} \in \langle g_u \rangle E$, say equal to $g_u^j b$, where $b \in E$. Then, this implies that $1 = (g_u^j b)^u = b^u$, since g_u commutes with E, implying $b = 1$, so that $\langle g_u \rangle$ is normal in G. We have that the group induced on ℓ is dihedral of order $2(q + 1)$. Since E and G/E are solvable, then G is solvable, and $(q, q + 1) = 1$. So, G contains a subgroup H of order $q + 1$. Now H acting on Σ is a subgroup of $\Gamma\mathrm{L}(2, q^2)$ and H is naturally in $\mathrm{GL}(4, q)$. Let $h \in H$, such that h is semi-linear with automorphism τ so that $h(\alpha x)$, $\alpha \in \mathrm{GF}(q^2)$, is $\alpha^\tau h(x)$. Then $h(\alpha^\tau h(x)) = \alpha^{2\tau} h^2(x)$, so that $\alpha^{|h|\tau} = 1$. Since H is K-linear, it follows that if $\alpha \in \mathrm{GF}(q)$, $\alpha^\tau = 1$. Hence, τ is given by either $z \mapsto z^q$ or $z \mapsto z^{q^2} = z$. In the former case, as $\alpha^{|h|q} = 1$ for all $\alpha \in \mathrm{GF}(q^2)$, it must be that $|h|$ is even. Therefore, it follows that H is in $\mathrm{GL}(2, q^2)$ acting on Σ and commutes with E of order $q/2$. Thus, it follows that H fixes at least two components on Σ, since it fixes one, implying that H fixes at least three components. We note that G is a subgroup of $\Gamma\mathrm{L}(2, q^2)$ acting on Σ and H commutes with an elation group E of order $q/2$. Thus, H fixes at least $1 + q/2$ components of Σ. Therefore, H is, in fact, the kernel group of Σ of order $q + 1$. Now let π and Σ share the set of lines S each of which is invariant under H. Since q is even, it follows that the lines of $\pi - \Sigma$ are Baer subplanes of Σ that are in orbits of length $q + 1$ under H. That is, these orbits constitute sets of opposite lines to reguli in the associated spread for Σ. Hence, the components of $\pi - \Sigma$ consist of opposite reguli to reguli in Σ. Furthermore, these reguli are permuted semi-regularly by the elation group of order $q/2$, for if an elation fixes a regulus of $q + 1$ components distinct from the axis of ℓ, it must fix a component, which cannot be the case. We have then that either $\pi = \Sigma$ or there are at least $q/2$ reguli that are disjoint from ℓ in an orbit under E. □

Remark 1. *In the above situation, define*

$$A = \{a; (x, y) \mapsto (x, xa + y) \text{ is in } E\}.$$

Then $A^q = \{a^q; a \in A\} = A$.

Proof. Now we have $q/2$ reguli in an orbit under E and there is a group of order q acting on this set which is in $\Gamma\mathrm{L}(2, q^2)$ acting on Σ. The stabilizer of one of these reguli has order 2 and hence there is a group of order 2 in the Sylow 2-subgroup that fixes at least two reguli in the E-orbit of reguli. This implies that there is a Baer involution τ that fixes at least two of these reguli. Choose $y = 0$ to belong to one of the fixed reguli. Since $x = 0, y = 0, y = x$ define a unique regulus, it follows that we may choose $y = x$ to be in a second disjoint regulus and $x = 0, y = 0, y = x$ fixed by τ, implying that $\tau : (x, y) \mapsto (x^q c, y^q c)$. Now with an appropriate choice of basis on $x = 0$, we may assume that the fixed point space of τ on $x = 0$ is such that c is forced to be 1. This implies that if $A = \{a; (x, y) \mapsto (x, xa + y)$ is in $E\}$ then $A^q = A = \{a^q; a \in A\}$. Note also that there is not an invariant Baer subplane in this case. □

Remark 2. *To summarize the situation, when there is an invariant line, either there are no elations, the Sylow* 2*-group is an elation group or the elation subgroup has order* $q/2$.

13 There is an invariant line but no elations

Theorem 41. *If there are no elations but there is an invariant line, the plane is a derived conical flock plane.*

Proof. The same argument as above shows that there exists an affine homology group of order u (or 3 when $q = 8$), which is either moved by the 2-group, thus implying an elation by Theorem 22, or the axis and coaxis are fixed by the 2-group, forcing the Sylow 2-group to be a Baer group of order q, centralized by g_u. However, again by Theorems 29 and 30, we have a derived flock of a quadratic cone. □

14 There is an invariant component; summary

Theorem 42. *Let π be a translation plane of even order q^2 with spread in* $\mathrm{PG}(3, q)$ *such that π admits a linear collineation group G of order $q(q+1)$. Assume that a Sylow 2-subgroup fixes a 2-dimensional K-subspace and acts non-trivially on it. If G leaves invariant a line of the spread then one of the following three situations occur*:

(1) *π is a conical flock plane*,

(2) *π is a derived conical flock plane*, *or*

(3) *π is a translation plane obtained from a Desarguesian plane by multiple derivation of a set of $q/2$ mutually disjoint regulus nets that are in an orbit under an elation group of order $q/2$.*

Proof. Apply the previous sections when there is an invariant line of the spread. □

15 There is a G-invariant Baer subplane but no G-invariant component and the Sylow 2-subgroups are not Baer

Theorem 43. *If $q > 2$ and if there is neither an invariant line nor a Baer group of order q then there is a Baer group of order $q/2$ and the group induced on the Baer subplane is dihedral of order* $2(q+1)$.

Proof. Assume that π_o is a G-invariant 2-dimensional K-subspace. The component intersections with π_o form a spread for π_o so that π_o is a G-invariant Desarguesian Baer subplane. We may assume by the previous sections that there is no elation in G. Let S_2 be a Sylow 2-subgroup of G and note that all involutions are Baer. Hence, S_2 is elementary Abelian by Theorem 26. Then S_2 fixes a component ℓ on which S_2 fixes a unique 1-dimensional subspace X_ℓ pointwise. Since S_2 has order q, it cannot act fixed point free on π_o. Moreover, S_2 is in $\mathrm{GL}(2, q)$ acting on π_o. Hence, $S_2 \mid \pi_o$ is an elation group with axis $\ell \cap \pi_o$, since we have assumed that S_2 is not a Baer group of order q. Assume that $S_2 \mid \pi_o$ has order > 2. Assume that $q \neq 8$. Then, for a prime 2-primitive divisor u, we have an element g_u which cannot leave invariant any component of π_o, since otherwise g_u would fix a component pointwise and u must then divide q or $q-1$ components of π_o, a contradiction. Similarly, if $u = 3$, then g_u acting on π_o is in $\Gamma\mathrm{L}(2, 8)$. However, G is in $\mathrm{GL}(4, q)$ and if G fixes a 2-dimensional K-subspace then G restricted, G_{π_o} to that subspace is naturally in $\mathrm{GL}(2, q)$. Hence, this

$GL(2,q)$ permutes the 1-dimensional K-subspaces of π_o, implying that, as a Desarguesian affine plane, π_o admits a collineation group isomorphic to $\Gamma L(2,q)$. It follows that G_{π_o} is a subgroup of $GL(2,q)$ acting on π_o. Thus, g_3 cannot fix a component of π_o.

Therefore, the elations generate $SL(2,2^s)$ on π_o and this group is normalized by g_u acting on π_o (as it acts faithfully on π_o). Note that the action of g_u on π_o cannot centralize $SL(2,2^s)$ as then it would fix each of the elation axes on π_o, which it cannot do. This means that $SL(2,q)$ is induced on π_o implying that $q(q^2-1)$ divides the order of G, so that $q-1=1$, or $q=2$, contrary to our assumptions. Hence, $S_2 \mid \pi_o$ has order 2.

In $PG(2,q)$ the corresponding group is a subgroup of a dihedral group of order dividing $2(q+1)$ or is A_4, S_4 or A_5. Since no element of order dividing $q+1$ can fix a Baer subplane pointwise, if the group induced on π_o contains a factor A_4, S_4 or A_5 then $q+1$ is 3 or 5, so that $q+1=3$ and the plane is of order 16, previously considered. There is no group of order 4(5) in such a setting.

It now follows that the group induced is isomorphic to a dihedral group of order $2(q+1)$, implying that there is a Baer group of order $q/2$. □

Theorem 44. *Assume that $q>2$. If there is a Baer group of order $q/2$, which is G-invariant, there is an associated Desarguesian spread Σ admitting G as a collineation group in $\Gamma L(2,q^2)$. The Baer subplane fixed pointwise by the Baer group is a component in Σ.*

Proof. Assume there is a Baer group B of order $q/2$ fixing a Baer subplane π_o pointwise. Furthermore, there is a Baer element σ in a Sylow 2-subgroup S_2 such that $\mathrm{Fix}\,\sigma$ and π_o share exactly one component, since S_2 induces a non-trivial elation on π_o. We have seen in the previous theorem that the group induced on π_o is transitive on the components of π_o. Furthermore, we know that the group induced on π_o is dihedral of order $2(q+1)$. If q is not 8 then there is a prime 2-primitive divisor u of q^2-1 and an element g_u of order u in G. If $q=8$, we take $u=3$ and g_3 a q-primitive element. Since g_u acts on the remaining q^2-q components not in the net containing π_o, it follows that g_u must fix at least two of these components. Together with π_o, g_u fixes at least three 2-dimensional K-subspaces that are mutually disjoint, implying that there is an associated Desarguesian spread Σ by Lemma 2.

We claim that $\langle g_u\rangle$ is normal in G. To see this, note that we may assume that K^*, the kernel group of order $q-1$, acts on π_o also as a kernel group and GK^* has order $2(q^2-1)$, acting on π_o. We note that the Baer group B of order $q/2$ is normal in G and g_u acts on this group. Since the Baer group is elementary Abelian and u is a 2-primitive divisor (or q-primitive divisor, of $q=8$ and $u=3$), it must be that g_u centralizes B. The kernel group and the group of order $2(q+1)$ must normalize $\langle g_u\rangle$ in the quotient group $G/G_{[\pi_o]}$ and $G_{[\pi_o]}$ (the group fixing π_o pointwise), can only be B. Hence, $hg_uh^{-1}\in\langle g_u\rangle B$, say equal to g_u^jb, where $b\in B$. This implies that $1=(g_u^jb)^u=b^u$, since g_u commutes with B, implying $b=1$, so that $\langle g_u\rangle$ is normal in G.

This means that G acts on the Desarguesian spread Σ as a collineation group and it follows that B is an elation group of Σ (as π_o is a component of Σ). Therefore, we have a group of order $q(q+1)$ acting on Σ, with an invariant component π_o, containing an elation group of order $q/2$ and a kernel homology g_u. Hence, G is a subgroup of $\Gamma L(2, q^2)$ acting on Σ. However, G leaves invariant a component π_o of Σ and is in $GL(4, q)$ and acting in Σ is a subgroup of $\Gamma L(2, q^2)$. □

Theorem 45. *Under the assumptions of the previous theorem, there is a subgroup H of order $q+1$ acting on Σ as a subgroup of* $GL(2, q^2)$ *that acts as a kernel homology group of Σ.*

Proof. By order and our previous analysis, we see that there is a subgroup H of order $q+1$. Furthermore, H must normalize B and induces on π_o a subgroup of $GL(2, q)$. Also, H acting on Σ is a subgroup of $GL(2, q^2)$. Since every element of H has order dividing $q+1$ and not $q-1$, then certainly every element of prime power order must commute with B as B is elementary Abelian of order $q/2$. But this implies that every element of H, and hence H, must commute with B. Furthermore, either H fixes exactly two components or H is a kernel homology group. Assume the former; then B would be forced to fix a second component as B and H commute. Hence, H is a kernel homology group of order $q+1$ acting on Σ. □

Theorem 46. *Assume the hypothesis of the previous theorem. Then there is a Baer group B of order exactly $q/2$ and the plane π is obtained from a Desarguesian plane by multiple derivation of a set of $q/2$ mutually disjoint reguli in an orbit under B together with the derivation of a regulus containing $\pi_o =$ Fix B as an elation axis of B and two orbits of B of both length $q/2$ forming a regulus containing π_o.*

Proof. Now π and Σ share exactly those components that are fixed by H, and, since q is even, the other H-orbits of Baer subplanes within Σ, as lines of π, are subplanes of a regulus in the spread of Σ. That is, the net of degree $q+1$ defined by π_o is a regulus net. Moreover, the Baer group of order $q/2$ acts as an elation group of Σ and the components of $\pi - \Sigma$ lie in opposite reguli to reguli in Σ defined by a given Baer subplane and its images under H. Furthermore, no element of B can fix any regulus net in Σ defined by a component of $\pi - \Sigma$ (note that π_o is a component of Σ). This means that the plane is obtained from a Desarguesian plane by multiple derivation of a set of $q/2$ mutually disjoint reguli in an orbit under B together with the derivation of a regulus containing π_o as an elation axis of B and two orbits of B of length $q/2$ forming a regulus containing π_o. □

16 The Sylow 2-subgroups are Baer

Theorem 47. *If there is a Baer group of order q, the plane is a derived conical flock plane.*

Proof. Assume that there is a Baer group of order q. By Theorems 29 and 30, the plane is a derived conical flock plane. □

17 When there is an invariant Baer subplane; summary

We note that, in all cases, we obtain from our original hypothesis either a conical flock plane or a derived conical flock plane with a group G of order $q(q+1)$ acting on it or there is an elation group or a Baer group of order $q/2$ and where the plane is constructed from a Desarguesian plane by multiple derivation.

Theorem 48. *Let π be a translation plane of even order q^2 with spread in $PG(3, q)$ that admits a linear collineation group G of order $q(q+1)$. If G leaves invariant a 2-dimensional K-space that is not a line of the spread of π then one of the following occur:*

(1) *π is a derived conical flock plane,*

(2) *π is a conical flock plane, or*

(3) *there is a Baer group of order exactly $q/2$ and the plane is obtained from a Desarguesian plane by multiple derivation of a set of $q/2$ mutually disjoint reguli in an orbit under B together with the derivation of a regulus containing π_o as an elation axis of B and two orbits of B both of length $q/2$ forming a regulus containing π_o.*

18 Multiply-derived planes; $q/2$-case

In this section, we consider the multiply derived situation mentioned previously in Theorems 42 and 46. In one case, there is always an elation group E of order $q/2$ and the regulus nets to be multiply derived are in an E-orbit. The Baer $q/2$-case, as in Theorem 46, arises in this situation when there is a regulus net R containing the axis of E and disjoint from the $q/2$ regulus nets used in the multiple derivation. When R is also derived, E becomes a Baer group of order $q/2$.

Thus, we further investigate the possibility that we could have an elation group of order $q/2$ within a Sylow 2-subgroup of order q and a translation plane admitting a collineation group of order $q(q+1)$, where there is a group of order $q+1$ acting as a kernel homology group H of an associated Desarguesian plane Σ of order q^2. Furthermore, there is a Baer involution σ which generates with H a dihedral group of order $2(q+1)$.

Lemma 16. *The orbit of $\mathrm{Fix}\,\sigma$ under H defines a regulus net in Σ that shares the axis $x=0$ of the elation group E of order $q/2$.*

Proof. The fixed point space of σ is a Baer subplane of order q. No non-identity element of H can fix σ, as otherwise the element would induce a kernel homology on $\mathrm{Fix}\,\sigma$, forcing the element to have order dividing $q-1$. Hence, the orbit of $\mathrm{Fix}\,\sigma$ under H defines a set of $q+1$ Baer subplanes that form the opposite regulus of a regulus of Σ. □

Lemma 17. (1) *The group G permutes the set of $q/2$ reguli.*

(2) *The Baer involution σ fixes at least two of the reguli and fixes exactly one component in each fixed regulus.*

Proof. Suppose that there are two sets of mutually disjoint reguli of cardinality $q/2$, this implies that we have at least $q(q+1)+1$ components, a contradiction. Hence, the full group G permutes this set of $q/2$ reguli. Since E is transitive on the set of $q/2$ reguli, there is a Baer involution, which we may take as σ, that fixes at least two of these reguli. These reguli are disjoint from $x=0$, so σ fixes at least one component of each fixed regulus. Moreover, since the orbit of $\operatorname{Fix}\sigma$ under H is also a regulus, σ can fix exactly one component of each fixed regulus. □

We now may fix our representation. Choose two fixed components to be $y=0$ and $y=x$ in distinct reguli. Hence, representing σ as $(x,y)\mapsto(x^q,y^q)$, any such fixed Baer subplane has the form $y=x^q m+xn$ such that $m^q=m$ and $n^q=n$. Moreover, the action of the kernel homology group H has elements of the form $(x,y)\mapsto(ax,ay)$, where a has order dividing $q+1$ in the associated field isomorphic to $GF(q^2)$ coordinatizing the Desarguesian plane. The elation group E of order $q/2$ has the form $(x,y)\mapsto(x,xb+y)$ where $b\in A$, where A is an additive subgroup of $GF(q^2)$ of order $q/2$ and $b\in A$ implies that $b^q\in A$.

Note that we have also considered the Baer group $q/2$-situation and in this case, the Sylow 2-subgroups are elementary Abelian. If this would be the case here, this says that A would be an additive subgroup of $GF(q)$, since we must have that $b^q=b$ for all b in A in this case.

Note that if σ fixes a regulus of Σ then σ fixes both a unique line and a unique Baer subplane of that regulus.

If $y=x^q m+xn$ intersects $y=0$ nontrivially, and is fixed by σ then $m,n\in GF(q)$. Furthermore, $x^{q-1}=(nm^{-1})$ for some non-zero x, implying that

$$(nm^{-1})^{q+1}=1=(nm^{-1})^2,$$

but this implies that $n=m$. Note that $y=x^q m+xn$ maps to $y=x^q ma^{1-q}+xn$ under kernel homologies. If the order of a is $1+q$, then the order of a^{1-q} is also $1+q$.

Hence, we may assume that

$$\{y=x^q ma+xm;\ a \text{ of order dividing } q+1\},$$

is a regulus of $\pi-\Sigma$, for some $m\in GF(q)-\{0\}$.

Similarly, if a component $y=x^q m^*+xn^*$ intersects $y=x$ nontrivially and is fixed by σ then $m^*+1=n^*$.

Since E is transitive on the $q/2$ reguli, there is an elation that maps

$$\{y=x^q ma+xm;\ a \text{ of order dividing } q+1\},$$

onto

$$\{y = x^q m^* a + x(m^* + 1);\ a \text{ of order dividing } q + 1\},$$

Now map $y = x^q m + xm$ onto $y = x^q m^* a + x(m^* + 1)$ by an elation

$$(x, y) \mapsto (x, xb + y).$$

Since $y = x^q m + xm$ maps onto $y = x^q m + x(m + b)$, it follows that

$$m^* a = m \quad \text{and} \quad m + b = m^* + 1.$$

Since m and m^* are in GF(q), it then follows that $a = 1$ so that $m^* = m$, implying that $b = 1$.

In general, the image Baer subplanes under EH have the form $y = x^q ma + x(m + b)$ for all $b \in A$ and all elements a of order dividing $q + 1$. Hence, σ fixes all such image reguli such that $b^q = b$, because it fixes the Baer subplane when $a = 1$. We have noted that whenever b corresponds to an element of E then so does b^q. Thus, $b + b^q$ also corresponds to an element of E and this element defines an regulus net which is invariant under σ.

Thus, we have shown:

Theorem 49. *Assume that there is a spread of even order in* PG$(3, q)$ *that is constructed by multiple derivation in a Desarguesian affine plane* Σ *by replacement of* $q/2$ *mutually disjoint reguli in an orbit under an elation group of* Σ.

Furthermore, assume that this set of reguli is left invariant by a group of order $q(q + 1)$ *consisting of a kernel homology group* H *of order* $q + 1$, *an elation group* E *of order* $q/2$ *and a Baer involution* σ.

Then the spread $\Sigma_{m,E}$ *has the following form*:

$$x = 0, \quad EH(y = x^q m + xm) \cup (y = xn),$$

for all components $y = xn$ *of* Σ *that are disjoint from the* $q/2$ *reguli defined by the images of* $y = x^q m + xm$, *for some* $m \in$ GF(q).

In this case, the Baer involution $\sigma : (x, y) \mapsto (x^q, y^q)$ *acts as a collineation of the plane*.

Theorem 50. *Referring to the above theorem, let* A *denote the set of elements* b *such that* $(x, y) \mapsto (x, xb + y)$ *is an elation. Hence, the set* A *is an additive group of order* $q/2$. *The above construction provides a spread if and only if the set of* $q/2$ *elements* $\alpha \in$ GF(q), *including* 0 *such that*

$$a^2 \neq am^{-2}\alpha + 1$$

for all non-identity elements a *of order dividing* $q + 1$ *contains* $\{b^{q+1}; b \in A\}$.

Proof. Consider the image set $\{y = x^q ma + x(m + b) \mid a$ of order dividing $q + 1,$

$b \in A\}$. If $y = x^q ma + x(m+b)$ nontrivially intersects $y = x^q m + xm$, then this implies that

$$(m(1+a))^{q+1} = b^{q+1}.$$

Since m is in $\mathrm{GF}(q)$, then we have

$$m^2(1 + a^{q+1} + a^q + a) = b^{q+1}.$$

Since $a^{q+1} = 1$, this previous equation is equivalent to

$$a^q + a = m^{-2}b^{q+1},$$

which is, in turn, equivalent to

$$a^{2q} + m^{-2}b^{q+1}a^q + 1 = 0.$$

Consider the analogous equation:

$$a^2 + m^{-2}\alpha a + 1 = 0.$$

Since there are exactly q elements not equal to 1 of order dividing $q+1$ and none of these are in $\mathrm{GF}(q)$ then there are exactly $q/2$ distinct elements α that provide solutions. So, if the set of non-solutions to the above equation including 0 contains $\{b^{q+1}; b \in A\}$, a spread is obtained. □

Corollary 2. (1) *If* $q = 4$, *there is an associated spread of order* 16 *of the* $q/2$*-elation type. This plane is either Hall or Desarguesian.*

(2) *If* $q = 8$, *there are* $q/2 - 1$ *associated spreads of order* 64 *of the* $q/2$*-elation type.*

Proof. (1) If $q = 4$ then $q/2 = 2$ and, as any non-solution generates an additive group of order 2, we obtain a spread. To complete the proof of Part (1), we note that the only other translation plane with kernel containing GF(4) is a semifield plane and this plane is not obtained in the manner indicated.

(2) Let t in $\mathrm{GF}(8^2)$ be such that $t^2 = t + 1$. This produces a subfield GF(4) of $\mathrm{GF}(8^2)$. Note that GF(4) must be fixed by the Frobenius automorphism. So, we let $A = \mathrm{GF}(4)$, which has cardinality $q/2$. Note that for $b \in A$, we must have $b^{q+1} = b^9 = 1$ or 0. Hence, to avoid a solution to $a^{2q} + m^{-2}b^{q+1}a^q + 1 = 0$, we may choose m in any of $q/2 - 1$ ways. □

19 The collineation group of the multiply derived planes

Let π be a translation plane of even order q^2 that may be constructed from a Desarguesian affine plane Σ by the multiple derivation of a set of $q/2$ regulus nets that are in an orbit under an elation group E of Σ of order $q/2$. Using Theorem 20, the full collineation group of π is the inherited group.

Then E acts as an elation group or Baer group of π and π also admits the kernel homology group F^* of order $q^2 - 1$ as a collineation group. In this setting, there is an orbit of components of length $q(q+1)/2$ in π under EF^*. If E is a Baer group, there is an associated regulus net that may be derived to produce an associated plane admitting an elation group. Hence, assume without loss of generality that E is an elation group of π.

Assume that there is a collineation τ of π that moves the axis $x = 0$ of E. Also, assume that $q/2 > 2$. Then the group $\mathscr{E}$ generated by elations in π is $\mathrm{SL}(2, 2^a)$ or $\mathrm{Sz}(2^b)$ where b is odd. Since $q > 4$, it follows that $q/2 > \sqrt{q}$.

If the group $\mathscr{E}$ is isomorphic to $\mathrm{SL}(2, 2^a)$, it follows from Theorem 6 that $\mathrm{SL}(2, q)$ is generated by elations, implying by Theorem 21 that π is Desarguesian. This is certainly a contradiction since the full group is inherited.

We also offer a slightly different proof to see that this situation cannot occur. If the set of $q/2$ regulus nets in Σ used in the construction process is denoted by $\mathscr{N}$ then π and Σ share $\Sigma - \mathscr{N}$, a net of degree $q^2 + 1 - q(q+1)/2 > q + 1$. However, two Desarguesian affine planes of order q^2 that share a net of degree strictly larger than $q + 1$ are necessarily equal, implying that the multiply derived net $\mathscr{N}^*$ of Baer subplanes of Σ is actually a net of components of Σ, a contradiction.

If the group $\mathscr{E}$ is isomorphic to $\mathrm{Sz}(2^b)$, there is a Lüneburg–Tits subplane of order 2^{2b} by Theorem 23. However, $2^b \geqslant q/2$, implying that $2^{2b} \geqslant q^2/4$. Since the maximal order of a proper subplane is q and $q^2/4 > q$ as $q > 4$, it follows that π is a Lüneburg–Tits plane of order q^2. However, then π cannot admit a collineation group of order $q^2 - 1$ by Theorem 10.

Hence, the axis $x = 0$ of E is invariant under the full collineation group $\mathscr{G}$ of π.

We now claim that E is normal in $\mathscr{G}$. If not then there is an elation group E^+ of order divisible by q acting on π. Since there is an orbit Λ of length $q/2$ of regulus nets in π under E and E^+ permutes the orbits of E but cannot leave Λ invariant, it follows that there is another set of $q/2$ mutually disjoint regulus. However, this implies that there are $q(q+1) > q^2 + 1$ components, a contradiction. Thus, E is normal in $\mathscr{G}$. Furthermore, we see that $\mathscr{G}$ must leave invariant the orbit Λ and is thus a collineation group of Σ. We state all of this formally within a theorem.

Theorem 51. *Let π be a translation plane of even order $q^2 > 16$ and kernel containing* $\mathrm{GF}(q)$ *that is constructed from a Desarguesian affine plane Σ by the multiple derivation of a set of $q/2$ mutually disjoint regulus nets that are in an orbit under an elation group E of order $q/2$.*

(1) *Then the full translation complement of π leaves invariant the fixed point space of E, normalizes E and is a subgroup of the translation complement of the Desarguesian plane Σ. In particular, E is an elation group or a Baer group of π.*

In particular,

(2) (a) *If E is an elation group of π then π is not Desarguesian, Hall, André or generalized André, and*

(b) *if E is a Baer group of π then π is not Hall, Desarguesian, derived André or derived generalized André.*

Proof. The proof that π is not Desarguesian is contained within the previous remarks. Also, if E is an elation group of π then π admits an elation group of order >2, so π cannot be a Hall plane. Since none of the proper André or generalized André planes admits non-trivial elations by Theorem 11, we have the proof of the theorem for the case when E is an elation group. If E is a Baer group then the derived plane π^* is not Desarguesian, Hall, André or generalized André so π is not a derived Desarguesian (which is Hall) derived Hall (which is Desarguesian), derived André or a derived generalized André plane. □

Definition 1. *A translation plane π constructed as in the previous theorem is called an 'elation $q/2$-type plane' or a 'Baer $q/2$-type plane' if and only if E is an elation group or Baer group of π, respectively.*

We now determine exactly when two elation $q/2$-type planes are isomorphic provided they exist.

Theorem 52. *Using the notation established in Theorem 49, assume the elation $q/2$-type planes $\Sigma_{m,E}$ and $\Sigma_{n,T}$ exist and are isomorphic. Define $A(Z) = \{d \in \mathrm{GF}(q^2); (x, y) \mapsto (x, xd + y) \in Z\}$, for Z an elation group of order $q/2$ in $\{E, T\}$. Then there exists an element a of order dividing $q + 1$ and an element c such that $ca^{-1} \in \mathrm{GF}(q)$ and the following conditions hold:*

$$n = m^{\sigma} ca^{-1} \quad \textit{and} \quad A(E)^{\sigma} c = A(T), \quad \sigma \in \mathrm{GalGF}(q^2).$$

Proof. We may assume as previously that an isomorphism has the following form:

$$g : (x, y) \mapsto (x^{\sigma}, y^{\sigma}) \begin{bmatrix} 1 & d \\ 0 & c \end{bmatrix}.$$

Then it follows that

$$g : y = x^q m + xm \mapsto x^q m^{\sigma} c + x(m^{\sigma} c + d).$$

This image component must have the form $y = x^q na + x(n + b)$, where $b \in A(T)$ and a has order dividing $q + 1$. The first condition is clear and the second condition follows from noting that $g^{-1}Eg = T$. □

The converse of the previous theorem is similar and is given as follows:

Theorem 53. (1) *If $\Sigma_{m,E}$ is an elation $q/2$-type plane and if $c \in \mathrm{GF}(q)$ then $\Sigma_{m^{\sigma}c, E_c^{\sigma}}$, where $\tau_b \in E$ if and only if $bc \in E_c$, $c \in \mathrm{GF}(q) - \{0\}$, is also an elation $q/2$-type plane. The two planes are isomorphic by the mapping $(x, y) \mapsto (x^{\sigma}, y^{\sigma}) \left[\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right]$ that takes $\Sigma_{m,E}$ onto $\Sigma_{m^{\sigma}c, E_c^{\sigma}}$.*

(2) *If there is an elation type $q/2$-type plane of order q^2 then it is isomorphic to $\Sigma_{1,E}$ for some elation group E.*

Proof. $\Sigma_{m,E}$ is a plane if and only if

$$\mathrm{trace}_{\mathrm{GF}(q)}(m^{-2}b^{q+1})^{-1} = 0,$$

for all $b \in A$, where $A = \{b; (x,y) \mapsto (x, xb + y) \in E\}$. It then follows that Σ_{mc,E_c} is also a plane since $c^{q+1} = c^2$, and

$$\mathrm{trace}_{\mathrm{GF}(q)}((mc)^{-2}(bc)^{q+1})^{-1} = \mathrm{trace}_{\mathrm{GF}(q)}(m^{-2}b^{q+1})^{-1}.$$

It then follows easily that the two planes are isomorphic by the indicated mapping. This proves (1). Given a plane $\Sigma_{m,E}$, let $c = m^{-1}$, so that $\Sigma_{m,E}$ is isomorphic to $\Sigma_{1,E_{m^{-1}}}$. This proves (2). □

Corollary 3. *Any elation $q/2$-type plane may be taken as a $\Sigma_{1,E}$, for some group of elations E of order $q/2$. Let $A = \{b \in \mathrm{GF}(q^2); (x,y) \mapsto (x, xb + y) \in E\}$.*

Thus, an elation $q/2$-type plane exists if and only if

$$\mathrm{trace}_{\mathrm{GF}(q)}(b^{q+1})^{-1} = 0,$$

for all $b \in A$, where $b \in A$ implies $b^q \in A$.

Proof. See Theorem 55. □

20 Analysis of Baer $q/2$-type planes

We may use the study of elation $q/2$-type planes in the Baer $q/2$-type planes situation with the additional hypothesis that there is a regulus net R containing the axis of the elation group E of order $q/2$ that is disjoint from the reguli used in the multiple derivation construction procedure.

Theorem 54. *Assume that $q > 2$.*

(1) *There is a Baer $q/2$-type plane if and only if there is an additive subgroup $\mathscr{A}$ of* $\mathrm{GF}(q)$ *of order $q/2$ such that $a^2 + a\beta + 1 \neq 0$ for all $\beta \in \mathscr{A}$ and for all a of order dividing $q+1$ and $a \neq 1$.*

(2) *If a has order $q+1$ and* $\mathrm{trace}_{\mathrm{GF}(q)}(a + a^q) = 1$ *then the additive subgroup of trace* 0 *elements in* $\mathrm{GF}(q)$ *satisfies Part* (1) *and produces a Baer $q/2$-type plane.*

Proof. A Baer $q/2$-type plane is equivalent to an elation type $q/2$ plane that fixes a regulus containing the axis $x = 0$ of E and disjoint from an orbit of $q/2$ mutually disjoint reguli. Moreover, there is a Baer involution that fixes exactly one component of the regulus net R containing $x = 0$.

We claim that any regulus net containing $x = 0$ has a partial spread of the form:

$$x = 0,\ y = x(t\alpha + s); \quad \alpha \in \mathrm{GF}(q),$$

where $t \neq 0$ and s are fixed elements in $\mathrm{GF}(q^2)$. To see this, simply map $y = 0$ onto $y = xz$ by $(x, y) \mapsto (x, xz + y)$. Hence, we have a regulus of the form

$$x = 0, y = x(\alpha + z); \quad \alpha \in \mathrm{GF}(q).$$

For $w \neq 0$, map by $(x, y) \mapsto (x, yw)$ so that

$$y = x(\alpha + z) \mapsto y = x(\alpha + z)w = x(\alpha w + zw).$$

Then let $w = t$ and $zw = s$. Note that the group

$$\langle (x, y) \mapsto (x, xz + y), (x, y) \mapsto (x, yw); z, t \neq 0 \in \mathrm{GF}(q^2) \rangle,$$

is sharply doubly transitive on the components of Σ not equal to $x = 0$.

We still have that there is an orbit of $q/2$ mutually disjoint reguli and there is an elementary Abelian 2-group of order q that acts and hence there is an involution σ which fixes at least two of these mutually disjoint reguli.

Consider the type of plane (Baer $q/2$-type) that is obtained from the derivation of R and the set of $q/2$ regulus nets in an orbit under the elation group E. Since σ is a Baer involution of Σ that fixes at least two of the regulus nets disjoint from $x = 0$, it follows that σ remains a Baer involution in the new plane π_1. Hence, we have that all involutions in G, acting on π_1, are Baer and the group G is in $\mathrm{GL}(4, q)$. By Theorem 26, the Sylow 2-subgroups of G are elementary Abelian, implying that σ commutes with E.

Another way to see that σ commutes with E is to note that $\langle \sigma, E \rangle$ fixes a 1-dimensional $\mathrm{GF}(q)$-space of $x = 0$ pointwise. If there is an invariant regulus net R then σ must fix at least one of the $q + 1$ Baer subplanes of R. But, E would then fix all of the Baer subplanes of R incident with the zero vector, implying that $\langle \sigma, E \rangle$ induces an elation group on a Desarguesian subplane π_o and acts faithfully on π_o. Hence, $\langle \sigma, E \rangle$ is elementary Abelian.

Since the group is elementary Abelian in this case, it follows that σ fixes every one of the $q/2$ disjoint reguli.

Choose components so that $y = 0$ and $y = x$ are in distinct reguli and fixed by σ. It follows that σ has the form: $(x, y) \mapsto (x^q c, y^q c)$ where $c^{q+1} = 1$. If $b^{q-1} = c$, conjugate by $(x, y) \mapsto (xb, yb)$ to transform σ into the form $(x, y) \mapsto (x^q, y^q)$.

So, if $y = x^q m + xm$ non-trivially intersects $y = 0$, we have $m^q = m$, as previously. The images under EH are of the form

$$y = x^q ma + x(m + b)$$

for all a of order dividing $q + 1$ and for all $b \in A$. Since there must be a fixed component in each regulus (for each fixed b), this can only imply that $b \in \mathrm{GF}(q)$ (also, this follows since σ commutes with E).

Now we have a regulus R with the following components:

$$x = 0, y = x(t\alpha + s); \quad \alpha \in \mathrm{GF}(q),$$

which is fixed by E and by σ. Since $A \subseteq \mathrm{GF}(q)$, it follows that

$$t\alpha + s + b = tf(\alpha, b) + s$$

for all α in GF(q) and for all $b \in A$, where f is a function from $\mathrm{GF}(q) \times \mathrm{GF}(q)$ into GF(q); $f(\alpha, b) \in \mathrm{GF}(q)$. Hence, $t\alpha + b = tf(\alpha, b)$. If $t \notin \mathrm{GF}(q)$, we have a contradiction, since then $t(\alpha + f(\alpha, b)) = b \in \mathrm{GF}(q)$, implying that $\alpha = f(\alpha, b)$ and $b = 0$. Thus, t is in GF(q) and we may incorporate t in the representation and take $t = 1$ without loss of generality. Hence,

$$x = 0, y = x(\alpha + s); \quad \alpha \in \mathrm{GF}(q),$$

are the components of the fixed regulus. However, then the fact that σ fixes this regulus implies

$$\alpha + s^q = g(\alpha) + s, \quad \text{for all } \alpha \in \mathrm{GF}(q),$$

where g is a function from GF(q) to GF(q). Note that σ fixes only the component $x = 0$.

Hence, $s^q = g(0) + s$, so

$$\alpha + s^q = \alpha + g(0) + s = \alpha + s \Leftrightarrow g(0) = 0.$$

Thus, $g(0) = \alpha_o \neq 0$.

We require that the images

$$y = x^q ma + x(m + b)$$

are all disjoint from

$$x = 0, y = x(\alpha + s); \quad \alpha \in \mathrm{GF}(q).$$

Note that $y = x(\alpha + s)$ is fixed by the kernel homology group so is disjoint from component $y = x^q m + x(m + b)$ if and only if it is also disjoint from $y = x^q ma + x(m + b)$.

There is an intersection between $y = x(\alpha + s)$ and $y = x^q m + x(m + b)$ if and only if for $\alpha = \beta + b$, $\beta \in \mathrm{GF}(q)$,

$$\begin{aligned} m^{q+1} = m^2 = (m + \beta + s)^{q+1} &= (m + \beta)^{q+1} + s^{q+1} + (m + \beta)(s + s^q) \\ &= m^2 + \beta^2 + m^2\beta + \beta^2 m + s^{q+1} + (m + \beta)(s + s^q). \end{aligned}$$

Using the fact that $s + s^q = \alpha_o$, we have the equivalent following equation:

$$\beta^2(1 + m) + (m^2 + \alpha_o)\beta + (s^{q+1} + m\alpha_o) = 0.$$

Now choose $m = 1 = \alpha_o$, $s^q + s = 1$, $s^{q+1} = \psi \neq 1$. Let $s^2 = s\alpha + \beta$ for $\alpha \neq 0$. Then,

$$(s/\alpha)^2 = (s/\alpha) + \beta/\alpha^2.$$

So, if there exist non-zero elements where $\beta \neq \alpha^2$, the above is possible. If $t^2 = t + 1$ then there is a subfield of $\mathrm{GF}(q^2)$ isomorphic to GF(4). Hence, if $q^2 > 4$, there exist s such that $s^2 = s + \psi$ for $\psi \neq 1$. The set

$$\{y = x^q ma + x(m + b); |a| \text{ divides } q + 1, b \in A\}$$

forms a partial spread if and only if

$$a^2 + am^{-2}b^2 + 1 \neq 0$$

for all $b \in A$. If $\mathscr{A}$ is an additive subgroup of order $q/2$ provided A is an additive subgroup of order $q/2$, define $A = \{\sqrt{\alpha}m; \alpha \in \mathscr{A}\}$. Then, we obtain a partial spread and by above a spread of the appropriate type. This proves (1).

Now assume that $q + 1$ is prime and a has order $q + 1$. If the $\mathrm{trace}_{\mathrm{GF}(q)}(a + a^q) = 1$ then we may choose $m = 1$ and, since $\mathrm{trace}_{\mathrm{GF}(q)}\, b = \mathrm{trace}_{\mathrm{GF}(q)}\, b^2$, the additive subgroup of order $q/2$ of trace 0 elements shows that

$$a^2 + ab^2 + 1 \neq 0 \Leftrightarrow a + b^2 + 1/a = a + a^q + b^2 \neq 0.$$

That is,

$$a + a^q + b^2 = 0 \Rightarrow \mathrm{trace}_{\mathrm{GF}(q)}(a + a^q + b^2) = 0$$

and if $\mathrm{trace}_{\mathrm{GF}(q)}(a + a^q) = 1$ and $\mathrm{trace}_{\mathrm{GF}(q)}\, b^2 = 0$, we have a contradiction. □

Example 1. In situation (2) above, assume that $q = 4$, $q + 1 = 5$. If the order of a is 5, $\mathrm{trace}_{\mathrm{GF}(4)}(a + a^4) = a + a^4 + a^2 + a^3$. Since $1 + a + a^2 + a^3 + a^4 = 0$, we have $\mathrm{trace}_{\mathrm{GF}(4)}(a + a^4) = 1$.

Theorem 55. *Let* $\{1, t\}$ *be a* $\mathrm{GF}(q)$*-basis for* $\mathrm{GF}(q^2)$ *where* $t^2 = t + \theta$. *Note that then* $t^{q+1} = \theta$ *and* $t^q + t = 1$.

(1) $\mathrm{trace}_{\mathrm{GF}(q)}\, \theta = 1$.

(2) $(t\alpha + \beta)^{q+1} = 1$ *if and only if* $\theta\alpha^2 + \beta^2 + \alpha\beta = 1$.

(3) $(t\alpha + \beta)^q + (t\alpha + \beta) = \alpha$.

(4) $\theta + \alpha^{-2}$ *has trace* 0 *if and only if* $\alpha = a + a^q$ *for some element* a *not* 1 *of order dividing* $q + 1$. *Hence, the set of non-solutions to* $a + a^q$ *for* a *not* 1 *is*

$$\{(\rho + \rho^2)^{-1}, 0; \rho \in \mathrm{GF}(q) - \{0, 1\}\}.$$

(5) *A spread of the Baer* $q/2$*-type exists if and only if*

$$\{(\rho + \rho^2)^{-1}, 0; \rho \in \mathrm{GF}(q) - \{0, 1\}\}$$

is an additive group of order $q/2$.

Proof. (1) Assume that trace $\theta = 0$. By Hilbert's theorem 90, $\theta = \rho + \rho^2$, for $\rho \in \mathrm{GF}(q)$. Then, $t + t^2 = \rho + \rho^2$, implies that $(t + \rho) + (t + \rho)^2 = 0$, and this implies that $(t + \rho) = 1$ so that $t \in \mathrm{GF}(q)$. Hence, trace $\theta = 1$. Clearly, $t + t^q = 1$ so that (2) and (3) are immediate.

To prove (4), assume that

$$\theta + \alpha^{-2} = f + f^2 \neq 0.$$

Let $f\alpha = \beta$. Then,

$$\theta + \alpha^{-2} = (\beta/\alpha) + (\beta/\alpha)^2$$

if and only if

$$\theta\alpha^2 + \beta^2 + \alpha\beta = 1.$$

This holds if and only if there is an element $a = t\alpha + \beta$ for $\alpha \neq 0$ of order dividing $q + 1$ such that $a + a^q = \alpha$.

Hence, α^{-2} has trace 1, for all solutions to $a + a^q = \alpha$. Since the trace of δ and the trace of δ^2 are equal, it must be that α^{-1} has trace 1 for all solutions.

Assume that a and a^* have order dividing $q + 1$ and neither a nor a^* is 1. Then,

$$a + a^q = \alpha = a^* + a^{*q}$$

if and only if

$$(a + a^*) = (a + a^*)^q.$$

That latter equality holds if and only if

$$a^2 + a\alpha + 1 = a^{*2} + a^*\alpha + 1,$$

which is equivalent to

$$\alpha(a + a^*) = (a + a^*)^2.$$

We then obtain the equivalent equality

$$a + a^* = \alpha.$$

Now, if $a^{q+1} = 1$, we claim that $(a + \alpha)^{q+1} = 1$. Note that

$$(a + \alpha)^{q+1} = a^{q+1} + \alpha^2 + \alpha(a + a^q) = 1 + \alpha^2 + \alpha^2 = 1.$$

Thus, there are exactly $q/2$ non-zero elements α in $\mathrm{GF}(q)$ that are solutions for

$a + a^q = \alpha$, where a is not 1. Since the trace of α^{-1} is 1 for all solutions, the set of non-solutions is

$$\{(\rho + \rho^2)^{-1}, 0; \rho \in \mathrm{GF}(q) - \{0, 1\}\}.$$

Now, in order that we have a Baer $q/2$ type spread, this fact implies that the set of non-solutions is an additive group. Our previous result implies that this condition is necessary and sufficient. □

21 Baer $q/2$-type planes do not exist when $q > 4$

Theorem 56. *If $q = 2^r$ and r is odd and larger than 1 then there does not exist a Baer $q/2$-type plane.*

Proof. If such a plane exists, then $A = \{(\rho + \rho^2)^{-1}, 0; \rho \in \mathrm{GF}(q) - \{0, 1\}\}$ is additive. Since this means that $(\rho^2 + \rho^4)^{-1} = (\rho + \rho^2)^{-2}$ is an element of A, then, for any ρ,

$$\sum_{i=0}^{r-1} (\rho + \rho^2)^{-2^i}$$

is an element of the set in question and thus has the form $(\gamma + \gamma^2)^{-1}$ or 0. But, this implies that

$$\mathrm{trace}(\rho + \rho^2)^{-1} = (\gamma + \gamma^2)^{-1}, \text{ or } 0$$

for some γ. Assume that $(\gamma + \gamma^2)^{-1} = 1$, then we have $(\gamma + \gamma^2) = 1$, a contradiction, since the left hand side has trace 0 and the right hand side, that is '1', has trace 1.

Hence, $\{(\rho + \rho^2)^{-1}, 0; \rho \in \mathrm{GF}(q) - \{0, 1\}\}$ is the set of trace 0 elements. There are $q/2 - 1$ trace 0 elements that are non-zero and each of these has a non-zero inverse in the set. Assume that for each non-zero $x \in A$, $x^{-1} \neq x$. Note that

$$\{x, x^{-1}\} \cap \{y, y^{-1}\} \neq \phi \Leftrightarrow x = y^{-1}$$

for $x \neq y$, $x, y \in A$, if and only if $\{x, x^{-1}\} = \{y, y^{-1}\}$. Thus, $A - \{0\} = q/2 - 1$ is even, a contradiction for $q > 2$. Hence, there exists an element $\rho \neq 0$ or 1 such that

$$(\rho + \rho^2)^{-1} = (\rho + \rho^2)$$

for some nonzero term $\rho + \rho^2$, but this implies that

$$(\rho + \rho^2)^2 = (\rho^2 + \rho^4) = 1,$$

a contradiction, as $\rho^2 + \rho^4$ has trace 0 but 1 has trace 1. Hence, there are no Baer $q/2$ type planes when $q = 2^r$ and r is odd, $r > 1$. □

Theorem 57. *There are no Baer $q/2$-type planes when $q = 2^{2e}$, for $e > 1$.*

Proof. If there is such a plane then

$$\{(\rho + \rho^2)^{-1}, 0; \rho \in \mathrm{GF}(q) - \{0, 1\}\}$$

corresponds to the additive group $\mathscr{A}$ of order $q/2$. We note that when $q = 2^{2e}$, then $\mathrm{GF}(2^e)$ has trace 0 over $\mathrm{GF}(q)$ and is contained in the set of non-solutions since inverses exist within non-zero elements of $\mathrm{GF}(2^e)$. Hence, there are 2^{e-1} cosets of $\mathrm{GF}(2^e)$ in $\mathscr{A}$, $z_i + \mathrm{GF}(2^e)$ for $i = 0, 1, \ldots, 2^{e-1} - 1$ and $z_o = 0$. Furthermore, elements in the coset $z_i + \mathrm{GF}(2^e)$ have trace 1 if and only if z_i has trace 1. The subset of trace 0 elements forms an additive subgroup which has index 1 or 2 and hence has order $q/2$ or $q/4$.

Let $\{1, z\}$ be a $\mathrm{GF}(2^e)$ basis of $\mathrm{GF}(2^{2e})$ such that $z^2 = z + \mu$ where μ is in $\mathrm{GF}(2^e)$. If

$$z\rho + \delta + \mathrm{GF}(\sqrt{q}) = z\rho + \mathrm{GF}(\sqrt{q})$$

is a coset of the non-solution set, where $\rho \neq 0$, then the inverse of each of these elements has trace 0.

The inverse of $z\rho + \gamma$ for $\rho \neq 0$, $\gamma \in \mathrm{GF}(2^z)$ is

$$z(\rho/((\rho + \gamma)\gamma + \mu\rho^2)) + (\rho + \gamma)/((\rho + \gamma)\gamma + \mu\rho^2).$$

We note that

$$\mathrm{trace}_{\mathrm{GF}(q)}(z\rho + \gamma) = \mathrm{trace}_{\mathrm{GF}(q)}\, z\rho.$$

Hence, the cardinality of the set $\{\rho; \mathrm{trace}_{\mathrm{GF}(q)}\, z\rho = 0\}$ is $(q/2)/\sqrt{q}$. We claim that

$$\{\rho/((\rho + \gamma)\gamma + \mu\rho^2); \rho \text{ is fixed and non-zero}, \gamma \in \mathrm{GF}(2^e)\}$$

has cardinality $\sqrt{q}/2 = 2^{e-1}$.

To see this, we note that

$$\rho/((\rho + \gamma)\gamma + \mu\rho^2) = \rho/((\rho + \gamma^*)\gamma^* + \mu\rho^2), \quad \gamma^* \neq \gamma$$

if and only if

$$\gamma + \gamma^* = \rho.$$

So, it follows that

$$z(\rho/((\rho + \gamma)\gamma + \mu\rho^2)) + (\rho + \gamma)/((\rho + \gamma)\gamma + \mu\rho^2)$$

has trace 0 for ρ fixed and non-zero, for all elements $\gamma \in \mathrm{GF}(\sqrt{q})$.

We claim that

$$\mathrm{trace}_{\mathrm{GF}(q)}\, z\delta = (\mathrm{trace}_{\mathrm{GF}(\sqrt{q})}\, \mu)(\mathrm{trace}_{\mathrm{GF}(\sqrt{q})}\, \delta). \qquad (*)$$

To prove this, we note that $z^2 = z + \mu$ so that $(z\delta)^2 = z\delta^2 + \mu\delta^2$. It thus follows that

$$(z\delta)^{2^j} = z\delta^{2^j} + (\mu + \mu^2 + \cdots + \mu^{2^{j-1}})\delta^{2^j}.$$

Furthermore,

$$(z\delta)^{2^e} = z\delta + (\mu + \mu + \cdots + \mu^{2^{e-1}})\delta = z\delta + ((\mathrm{trace}_{\mathrm{GF}(\sqrt{q})}\,\mu)\delta).$$

Thus, every term $z\delta^{2^j}$ for $j = 0, 1, \ldots, e-1$ is doubled and since

$$\mathrm{trace}_{\mathrm{GF}(\sqrt{q})}((\mathrm{trace}_{\mathrm{GF}(\sqrt{q})}\,\mu)\delta) = (\mathrm{trace}_{\mathrm{GF}(\sqrt{q})}\,\mu)(\mathrm{trace}_{\mathrm{GF}(\sqrt{q})}\,\delta),$$

we have the proof to $(*)$.

Since $z^2 = z + \mu$ then

$$\mathrm{trace}_{\mathrm{GF}(\sqrt{q})}\,\mu = z^{2^e} + z.$$

Hence, $\mathrm{trace}_{\mathrm{GF}(\sqrt{q})}\,\mu = 1$. Thus,

$$\mathrm{trace}_{\mathrm{GF}(q)}\,z\delta = 0 \Leftrightarrow \mathrm{trace}_{\mathrm{GF}(\sqrt{q})}\,\delta = 0. \qquad (*)$$

Therefore, we have for ρ fixed and non-zero,

$$\mathrm{trace}_{\mathrm{GF}(\sqrt{q})}\,\rho/((\rho + \gamma)\gamma + \mu\rho^2) = 0, \quad \text{for all } \gamma \in \mathrm{GF}(\sqrt{q}).$$

Note that the set of trace 1 elements in $\mathrm{GF}(\sqrt{q})$ is

$$\{\gamma + \gamma^2 + \mu; \gamma \in \mathrm{GF}(\sqrt{q})\}.$$

However, this means that there are $\sqrt{q}/2$ non-zero elements in $\mathrm{GF}(\sqrt{q})$ whose trace is 0 over $\mathrm{GF}(\sqrt{q})$, a contradiction, since there are exactly $\sqrt{q}/2 - 1$ nonzero elements of trace 0. Hence, $\rho = 0$ and $q/2 = \sqrt{q}$ so that $q = 4$. □

Theorem 58. *A Baer $q/2$-type plane of order q^2, for $q > 2$ exists if and only if $q = 4$ and the plane is Desarguesian.*

Proof. The above results show that the only possibility is when $q = 4$. Let $q = 4$ and consider an elation $q/2$-plane. There are 2 disjoint regulus nets. Since any two disjoint regulus nets may be embedded into a linear set of $q - 1$ regulus nets and since $q - 1 = 3$, replacement of the two nets is equivalent to derivation; an elation $q/2$-plane must be Hall. A Baer $q/2$-plane involves the replacement of $q/2 + 1$ mutually disjoint regulus nets. Multiple derivation then produces a Desarguesian affine plane. □

22 The associated Desarguesian spread

In our previous sections, we have shown that translation planes with spreads in $PG(3, q)$, for q even, with collineation groups in $GL(4, q)$ of order $q(q+1)$ correspond to conical planes or derived conical planes when the Sylow 2-subgroup fixes a 2-dimensional K-subspace and induces a non-trivial group on that subspace. Our intent is to now prove our results without resorting to the assumption that a Sylow 2-subgroup fixes a 2-dimensional K-subspace and acts non-trivially on it. That is, we shall eventually prove that this property must hold. In order to obtain this result, we establish connections with a Desarguesian spread when the Sylow 2-subgroups do have this property.

Theorem 59. *If π is a conical flock or derived conical flock plane of order q^2 admitting a linear group G of order $q(q+1)$ then there is an associated Desarguesian affine plane Σ such that G acts on Σ as a collineation group. Furthermore, if π is not Hall or Desarguesian, G is solvable and contains a group H of order $q+1$ that contains a non-identity subgroup acting as a kernel homology group of Σ.*

Proof. First assume that π is a conical flock plane.

Initially, assume that there is a prime 2-primitive divisor u of $q^2 - 1$ or $q = 8$ and we take $u = 3$ (a q-primitive divisor) and there is an element g_u of G of order u. We may assume that π is not Desarguesian and there is a G-invariant component ℓ. It follows that g_u centralizes the elation group E. Furthermore, the quotient on ℓ in $PG(2, q)$ either is isomorphic to A_4, S_4 or A_5, or the group is a subgroup of a dihedral group of order divisible by $2(q+1)$. In this case, there is a normal subgroup $\langle g_u Z \rangle$, where Z is the intersection with G and the kernel homology group of order $q-1$ acting on ℓ. Since g_u commutes with E, it follows that there is a normal subgroup $\langle g_u \rangle$ in G. But, g_u fixes at least three components of π and hence there is an associated Desarguesian affine plane admitting G by Lemma 2.

Now assume that the group induced on ℓ is A_4, S_4 or A_5. The only possible affine homology groups normalize E so they have order dividing $q-1$ which is impossible since $(q+1, q-1) = 1$. Then this forces $q+1$ to be even.

Now assume that the plane is a derived conical flock plane of order q^2. Since we have a Baer group of even order q by Theorem 32, this implies that the full collineation of a derived conical flock plane leaves invariant the derived net, provided the order is not 4. Hence, we may derive back to a conical flock plane that admits such a group, implying the existence of a Desarguesian affine plane Σ admitting G as a collineation group.

Thus, in all case, there is a Desarguesian affine plane Σ admitting G, G is solvable and has a normal subgroup H of order $q+1$ containing a non-trivial kernel homology group acting on Σ. □

23 When the plane is a conical plane

Now assume the plane π is a conical flock plane. Note that the group G is a subgroup of $GL(4, q)$ and acts on the Desarguesian plane Σ as a subgroup of $\Gamma L(2, q^2)$. Then

G actually acts on Σ as a subgroup of $\mathrm{GL}(2, q^2)$, since the elation group of order q is in $\mathrm{GL}(2, q^2)$ and $q+1$ is odd. Since the order of G is $q(q+1)$ and there is an elation group of order q, the group is solvable and there is a subgroup H of order $q+1$ that necessarily is in $\mathrm{GL}(2, q^2)$ by order.

Now assume that there is a subgroup of order >2 of H which is not a kernel homology group. Then, by order H must fix exactly one component or fixes all components. Assume that H is not a kernel homology group. Then the group leaves invariant a regulus net R of Σ and if g is an element of H then the order of g modulo the kernel subgroup H^- has order dividing $(q-1, q+1) = 1$. Thus, H^- has order $(q+1)$, but this is contrary to our assumption.

Hence, we do have that there is a kernel homology subgroup H of order $(q+1)$, which, since q is even, is transitive on the 1-dimensional K-subspaces on any component of the Desarguesian plane Σ. Suppose that there is a component π_o of $\pi - \Sigma$. Then π_o is a Baer subplane of Σ and $\pi_o H$ defines the lines of the opposite regulus to a regulus of Σ. Furthermore, $\pi_o HE$ defines a set of q reguli in π since if an elation τ fixes a regulus disjoint from the axis of E then τ would fix a component of the regulus, a contradiction. However, this means that $\pi_o HE$ is a set of at least $q(q+1)$, components. Thus, $\Sigma = \pi$.

We have shown:

Theorem 60. *If π is a conical flock plane then π is Desarguesian.*

24 When the plane is a derived conical plane

Now assume that the plane π is a derived conical flock plane. Since we are replacing a base regulus net by assumption, by Theorem 32 the full collineation group leaves invariant the net containing the Baer 2-group of order q or the order is 4. Hence, we may derive the plane back to a conical flock plane and retain the group. This gives the following result.

Theorem 61. *If π is a derived conical flock plane then π is Hall.*

25 The main theorem

Our main theorem is as follows:

Theorem 62. *Let π be a translation plane with even order q^2 with spread in* $\mathrm{PG}(3, K)$, *K isomorphic to* $\mathrm{GF}(q)$. *Assume that π admits a linear collineation group G of order $q(q+1)$. Then π is one of the following types of planes*:

(1) *Desarguesian*,

(2) *Hall*,

(3) *a translation plane obtained from a Desarguesian plane by multiple derivation of a set of $q/2$ mutually disjoint regulus nets that are in an orbit under an elation group of order $q/2$.*

Proof. If a Sylow 2-subgroup fixes a 2-dimensional K-subspace and acts non-trivially on it then the plane π is either a conical flock plane or a derived conical flock plane and must be Desarguesian or Hall, respectively, or there is an elation or Baer group of order $q/2$ and the previous arguments show that we only obtain a type (3) situation.

Now assume that if a Sylow 2-subgroup S_2 fixes a 2-dimensional K-subspace, then S_2 fixes it pointwise. Since S_2 must fix a 1-dimensional subspace pointwise, there is a unique component containing this subspace. Thus, this component must be fixed pointwise. That is, the group is an elation group of order q. Since $q+1$ is odd, it follows immediately that $q = 2$ and π is Desarguesian of order 4 or we have an invariant component (otherwise, the group $\mathrm{SL}(2, q)$ is contained in the group generated by elations).

We then may assume that we have an element g_u or g_3 that normalizes and hence centralizes the elation group E of order q. Since g_u fixes two components, it must then fix three. Furthermore, the group of order $q+1$ acts faithfully on the axis ℓ of E and must act transitively on the 1-dimensional K-subspaces. Indeed, there is a group H of order $q+1$ that acts faithfully as a subgroup of $\mathrm{GL}(2, q)$ and hence is cyclic. Thus, there is a normal u-group or 3-element group, implying that G acts on a Desarguesian affine plane Σ just as before. In this setting, we cannot be certain that E is regulus inducing. However, by order, E is normal in G. Thus, G permutes the E-orbits and since there is a cyclic subgroup C_{q+1}, this again implies that C_{q+1} fixes an E-orbit and is in $\mathrm{GL}(2, q^2)$, so that C_{q+1} is a kernel homology group of Σ. Now it again follows that if π is not Σ, we have at least $q(q+1)$ components of $\pi - \Sigma$, a contradiction. This completes the proof. □

26 Open problem

We have shown that there are elation $q/2$-type planes when $q = 4$ or 8 and there are Baer $q/2$ planes when $q = 4$. However, the Baer $q/2$-type planes of order 4^2 are Desarguesian. Furthermore, we have shown that Baer $q/2$ planes cannot exist for larger orders. However, the elation case is essentially open.

Problem 1. *If $q > 4$, show that an elation $q/2$ plane has order 8^2, or find a class of examples.*

Equivalently, we state the above problem using the trace function.

Problem 2. *Let A be an additive subgroup of order $q/2$ of* $\mathrm{GF}(q^2)$. *If*

$$\mathrm{trace}_{\mathrm{GF}(q)}(b^{q+1})^{-1} = 0, \quad \text{for all } b \in A - \{0\}, \text{ and } A^q = A,$$

show that $q = 4$ or 8 (note that we have shown that if $A \subseteq \mathrm{GF}(q)$ then $q = 4$).

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