Adv. Geom. **4** (2004), 319–340

Advances in Geometry

A family of flat Minkowski planes admitting 3-dimensional simple groups of automorphisms

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(Communicated by R. Löwen)

Abstract. In this paper we construct a new family of flat Minkowski planes of group dimension 3. These planes share the positive half with the classical real Minkowski plane and admit simple groups of automorphisms isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ acting diagonally on the torus. We further determine the full automorphism groups and the Klein–Kroll types of these flat Minkowski planes.

2000 Mathematics Subject Classification. MSC 2000: 51H15, 51B20

1 Introduction and result

A *flat Minkowski plane* $\mathscr{M}$ is an incidence structure of points, circles and two kinds of parallel classes whose point set is the torus $\mathbb{S}^1 \times \mathbb{S}^1$ (where the 1-sphere $\mathbb{S}^1$ usually is represented as $\mathbb{R} \cup \{\infty\}$), whose circles are graphs of homeomorphisms of $\mathbb{S}^1$ and whose parallel classes of points are the horizontals and verticals on the torus. We furthermore require that for every point p of $\mathscr{M}$ the associated incidence structure $\mathscr{A}_p$ whose point set A_p consists of all points of $\mathscr{M}$ that are not parallel to p and whose set of lines $\mathscr{L}_p$ consists of all restrictions to A_p of circles of $\mathscr{M}$ passing through p and of all parallel classes not passing through p is an affine plane. We call $\mathscr{A}_p$ the *derived affine plane* at p; compare [5] or [4], Chapter 4. This implies that three mutually non-parallel points can be joined by a unique circle and that for two non-parallel points p and q and a circle $K \ni p$ there is a unique circle which touches K at p and passes through q. The *classical flat Minkowski plane* is obtained in this way as the geometry of all graphs of fractional linear maps on $\mathbb{S}^1$. Each derived affine plane of the classical flat Minkowski plane is Desarguesian.

When the circle sets are topologized by the Hausdorff metric with respect to a metric that induces the topology of the torus, then the planes are *topological* in the sense that the operations of joining three mutually non-parallel points by a circle, intersecting of two circles, and touching are continuous with respect to the induced topologies on their respective domains of definition. For more information on topological Minkowski planes we refer to [5] and [4], Chapter 4. The flat Minkowski planes are precisely the 2-*dimensional topological Minkowski planes*.

The circle space $\mathscr{C}$ of a flat Minkowski plane has two connected components; one, $\mathscr{C}^+$, consists of all circles in $\mathscr{C}$ that are graphs of orientation-preserving homeomorphisms $\mathbb{S}^1 \to \mathbb{S}^1$ and the other, $\mathscr{C}^-$, consists of all circles in $\mathscr{C}$ that are graphs of orientation-reversing homeomorphisms. We call $\mathscr{C}^+$ and $\mathscr{C}^-$ the *positive* and *negative half of* $\mathscr{M}$, respectively. It turns out that these two halves are completely independent of each other, that is, we can interchange components from different flat Minkowski planes and obtain another flat Minkowski plane; see [4], 4.3.1.

An *automorphism* of a flat Minkowski plane is a homeomorphism of the torus such that parallel classes are mapped to parallel classes and circles are mapped to circles. The collection of all automorphisms of a flat Minkowski plane $\mathscr{M}$ forms a group with respect to composition, the automorphism group Γ of $\mathscr{M}$. This group is a Lie group of dimension at most 6 with respect to the compact-open topology; see [4], 4.4. We say that a flat Minkowski plane has *group dimension n* if its automorphism group is n-dimensional. All flat Minkowski planes of group dimension at least 4 have been classified by Schenkel [5], see also [4], 4.4.5. In particular, the classical flat Minkowski plane is the only flat Minkowski plane of group dimension at least 5 and every flat Minkowski plane of group dimension 4 fixes two parallel classes. Many flat Minkowski planes of group dimension 3 have also been constructed, see [4], 4.3 for a summary, but no complete classification flat Minkowski planes of group dimension 3 has yet been achieved.

In this paper we contribute to the eventual classification by constructing a new family of flat Minkowski planes of group dimension 3. These planes admit simple groups of automorphisms isomorphic to $\mathrm{PSL}_2(\mathbb{R})$. They are obtained from the classical flat Minkowski plane by replacing the circles in the negative half in such a way that $\mathrm{PSL}_2(\mathbb{R})$ acts diagonally. Thus these planes are not isomorphic to the well-known flat Minkowski planes that admit $\mathrm{PSL}_2(\mathbb{R})$ as a group of automorphisms in one of the kernels.

Main Theorem. *Each incidence structure $\mathscr{M}(k)$ for $k > 1$, see the beginning of Section 2, is a flat Minkowski plane. Furthermore, these planes are mutually non-isomorphic and the full automorphism group of each such plane is isomorphic to* $\mathrm{PGL}_2(\mathbb{R})$ *and acts diagonally on the torus. Each $\mathscr{M}(k)$ is of Klein–Kroll type* IV.A.1.

The author wishes to thank the referee for his suggestions and comments in the preparation of the final version of this paper.

2 The incidence structures $\mathscr{M}(k)$

We construct a flat Minkowski plane $\mathscr{M}(k)$ by replacing the negative half of the classical flat Minkowski plane by the images of the generating circle

$$C_k = \{(x, -x|x|^{k-1}) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\}$$

under the group

$$\Sigma = \{(x, y) \mapsto (\delta(x), \delta(y)) \mid \delta \in \mathrm{PSL}_2(\mathbb{R})\}.$$

More precisely, let $k > 1$. Then the incidence structure $\mathcal{M}(k)$ on the torus $\mathbb{S}^1 \times \mathbb{S}^1$ has circles of the following form.

- The graphs of elements in $\mathrm{PSL}_2(\mathbb{R})$, that is,

$$\left\{ \left(x, \frac{ax+b}{bx+d} \right) \Big| \, x \in \mathbb{S}^1 \right\}$$

  where $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$, with the obvious definitions for $x = \infty$ and when the denominator becomes 0. These circles are the same as the circles in the positive half of the classical flat Minkowski plane.

- The graphs of $\delta f_k \delta^{-1}$ for $\delta \in \mathrm{PSL}_2(\mathbb{R})$ where

$$f_k(x) = \begin{cases} -x|x|^{k-1}, & \text{if } x \in \mathbb{R}, \\ \infty, & \text{if } x = \infty. \end{cases}$$

We shall show in the following that $\mathcal{M}(k)$ is indeed a flat Minkowski plane. Note that the restriction g_k of $-f_k$ on $\mathbb{R}$, that is, the function given by

$$g_k(x) = x|x|^{k-1}$$

for $x \in \mathbb{R}$ is a multiplicative strictly increasing homeomorphism of $\mathbb{R}$. Moreover, g_k is continuously differentiable and its derivative is given by $g_k'(x) = k|x|^{k-1} \geqslant 0$. Hence $kg_k(x) = xg_k'(x)$ for all $x \in \mathbb{R}$.

The multiplicativity of g_k implies that f_k commutes with the transformation $\sigma \in \mathrm{PSL}_2(\mathbb{R})$ given by

$$\sigma(x) = -1/x.$$

Hence δ and $\delta\sigma$ define the same circle. In fact, this is the only instance that this happens, that is, if $\delta f_k \delta^{-1} = \gamma f_k \gamma^{-1}$ for $\gamma, \delta \in \mathrm{PSL}_2(\mathbb{R})$, then $\gamma = \delta$ or $\gamma = \delta\sigma$. More generally, we show the following.

Proposition 2.1. *Let $\alpha \in \mathrm{PSL}_2(\mathbb{R})$. Then the homeomorphism $\alpha^{-1} f_k^{-1} \alpha f_k$ of $\mathbb{S}^1$ fixes at least three points of $\mathbb{S}^1$ if and only if $\alpha = \mathrm{id}$ or $\alpha = \sigma$ as above.*

Proof. For $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ let $F_\alpha \subseteq \mathbb{S}^1$ be the set of all fixed points of $\alpha^{-1} f_k^{-1} \alpha f_k$ and let $G \subseteq \mathrm{PSL}_2(\mathbb{R})$ be the collection of all $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ such that F_α contains at least three points. By definition, $\alpha \in G$ if and only if the cardinality of $C_k \cap \alpha^{-1}(C_k)$ is at least 3. From $\sigma(C_k) = C_k$ we infer $\sigma \in G$. Applying α and σ to the intersection, we see that the set $\{\alpha, \alpha^{-1}, \alpha\sigma, \sigma\alpha = (\sigma\alpha^{-1})^{-1}, \sigma\alpha\sigma\}$ is contained in G if one of its elements is.

Given $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ one finds

α	$\alpha\sigma$	$\sigma\alpha$	$\sigma\alpha\sigma$
$x \mapsto \frac{ax+b}{cx+d}$	$x \mapsto \frac{bx-a}{dx-c}$	$x \mapsto \frac{-cx-d}{ax+b}$	$x \mapsto \frac{-dx+c}{bx-a}$

From the above table we see that if $\alpha : x \mapsto \frac{ax+b}{cx+d}$ belongs to G we may assume that either all coefficients $a, b, c, d \in \mathbb{R}$ are nonzero or that $c = 0$.

We first assume that $c = 0$. Then α can be written in the form $\alpha : x \mapsto r(x+s)$ where $r, s \in \mathbb{R}$, $r > 0$. Since $\alpha^{-1} \in G$ too, we may further assume that $s \geqslant 0$. Moreover, $\alpha^{-1} f_k^{-1} \alpha f_k$ fixes $\infty \in \mathbb{S}^1$ and has at least two fixed points $x_1 < x_2$ in $\mathbb{R}$. For these fixed points x_i one then finds $\alpha f_k(x) = f_k \alpha(x)$, that is, $|r|^{k-1} g_k(x_i + s) = g_k(x_i) - s$ for $i = 1, 2$. Eliminating r from these two equations, we obtain $h(s) = 0$ where

$$\begin{aligned} h(s) &= g_k(s+x_2)(s - g_k(x_1)) - g_k(s+x_1)(s - g_k(x_2)) \\ &= s(g_k(s+x_2) - g_k(s+x_1)) + g_k(x_2(s+x_1)) - g_k(x_1(s+x_2)). \end{aligned}$$

Since $x_1 < x_2$ and $s \geqslant 0$, the first term $s(g_k(s+x_2) - g_k(s+x_1))$ is nonnegative. The second term $g_k(x_2(s+x_1)) - g_k(x_1(s+x_2))$ is 0 if and only if $x_2(s+x_1) = x_1(s+x_2)$, that is, if and only if $s = 0$. Since $g_k(x_2(s+x_1)) - g_k(x_1(s+x_2)) = g_k(s)\big(g_k\big(x_2 + \frac{x_1 x_2}{s}\big) - g_k\big(x_1 + \frac{x_1 x_2}{s}\big)\big) > 0$ for large $s > 0$, it follows by continuity that $g_k(x_2(s+x_1)) - g_k(x_1(s+x_2)) > 0$ for all $s > 0$. Hence $h(s) \geqslant g_k(x_2(s+x_1)) - g_k(x_1(s+x_2)) > 0$ for $s > 0$. This shows that we must have $s = 0$. But then $r = 1$. Therefore $\alpha = \mathrm{id}$ in this case.

We now show that the second case where $a, b, c, d \neq 0$ is not possible. We write α in the form $\alpha(x) = r\frac{x+s}{x+t}$ where $r, s, t \in \mathbb{R}$, $r(t-s) > 0$. By passing over to $\alpha^{-1}, \alpha\sigma, \ldots,$ if necessary, we may further assume that $0 < s < t$ or $s < 0 < t$. This then implies that $r > 0$. Note that in this case neither ∞ nor $\alpha^{-1}(\infty)$ can be fixed by $\alpha^{-1} f_k^{-1} \alpha f_k$. Then $f_k \alpha(x) = \alpha f_k(x)$ is equivalent to $h_{r,s,t}(x) = 0$ where $x \in \mathbb{R}$ and

$$h_{r,s,t}(x) = |r|^{k-1} g_k(x+s)(g_k(x) - t) + g_k(x+t)(g_k(x) - s)$$

We show that $h_{r,s,t}$ has at most two real zeros. By looking at where the factors $g_k(x+s), g_k(x) - t, g_k(x+t), g_k(x) - s$ occurring in $h_{r,s,t}(x)$ are positive or negative we find that $h_{r,s,t}(x) > 0$ for $x > \max\{-s, g_k^{-1}(t), g_k^{-1}(s)\}$ or $x < \min\{-s, -t, g_k^{-1}(s)\}$ (note that $t > 0$).

Using $kg_k(x) = xg_k'(x)$ one finds for the derivative of $h_{r,s,t}$ that

$$xh_{r,s,t}'(x) - kh_{r,s,t}(x) = (|r|^{k-1} g_k'(x+s) + g_k'(x+t))(xg_k(x) + st). \tag{$*$}$$

The first factor $|r|^{k-1} g_k'(x+s) + g_k'(x+t)$ on the right-hand side is always positive. We now assume that $0 < s < t$. Then the second factor $xg_k(x) + st$ in ($*$) is also positive. This implies that $h_{r,s,t}'(x_0) > 0$ for every positive zero x_0 of $h_{r,s,t}$ and $h_{r,s,t}'(x_0) < 0$ for every negative zero x_0 of $h_{r,s,t}$. Hence there can be at most one positive and at most one negative zero of $h_{r,s,t}$. Since

$$h_{r,s,t}(0) = -t|r|^{k-1} g_k(s) - sg_k(t) = -st(|rs|^{k-1} + |t|^{k-1}) < 0,$$

we see that $h_{r,s,t}$ has precisely two zeros in case $0 < s < t$.

We finally assume that $s < 0 < t$. In this case one further finds that $h_{r,s,t}(x) > 0$ for $\max\{-t, g_k^{-1}(s)\} \leqslant x \leqslant \min\{-s, g_k^{-1}(t)\}$; see also Table 1 below.

x	$h_{r,s,t}(x)$	$xg_k(x) + st$
$-t$	$\lvert r\rvert^{k-1} t(\lvert t\rvert^{k-1} + 1) g_k(t-s) > 0$	$t(s + g_k(t))$
$g_k^{-1}(s)$	$-(t-s)\lvert r\rvert^{k-1} g_k(s) g_k(\lvert s\rvert^{1/k-1} + 1) > 0$	$s(t + g_k^{-1}(s))$
$g_k^{-1}(t)$	$(t-s) g_k(t) g_k(\lvert t\rvert^{1/k-1} + 1) > 0$	$t(s + g_k^{-1}(t))$
$-s$	$-s(\lvert s\rvert^{k-1} + 1) g_k(t-s) > 0$	$s(t + g_k(s))$

Table 1.

Thus every zero of $h_{r,s,t}$ must be in the open intervals

$$I_- = (\min\{-t, g_k^{-1}(s)\}, \max\{-t, g_k^{-1}(s)\}) \quad \text{and}$$
$$I_+ = (\min\{-s, g_k^{-1}(t)\}, \max\{-s, g_k^{-1}(t)\}).$$

Note that for $g_k^{-1}(s) = -t$ we have $I_- = \emptyset$ and $h_{r,s,t}(x) > 0$ for all $x < 0$. Likewise, $g_k^{-1}(t) = -s$ implies $I_+ = \emptyset$ and $h_{r,s,t}(x) > 0$ for all $x > 0$. We will see below that $h_{r,s,t}$ can have at most two zeros in $I_\pm$. Therefore in each of the above two cases where one of the intervals is empty we obtain the desired result. In order to avoid unnecessary special cases in the following, we now assume that $g_k^{-1}(s) \neq -t$ and $g_k^{-1}(t) \neq -s$, that is, that both intervals $I_\pm$ are nonempty.

The map $x \mapsto xg_k(x) + st$ has precisely two zeros $x_0 = \lvert st\rvert^{1/(k+1)}$ and $-x_0$. Since $s + g_k(t)$ and $t + g_k^{-1}(s)$ have the same sign, we see from Table 1 that $xg_k(x) + st$ takes on opposite signs at the boundary points of I_-. We similarly obtain that $xg_k(x) + st$ takes on opposite signs at the boundary points of I_+. This shows that $x_0 \in I_+$ and $-x_0 \in I_-$.

If $x \neq x_0$ is a zero of $h_{r,s,t}$, then we obtain from Equation $(*)$ that $h'_{r,s,t}(x) > 0$ for $x > x_0$ or $-x_0 < x < 0$ and $h'_{r,s,t}(x) < 0$ for $0 < x < x_0$ or $x < -x_0$. As before this implies that $h_{r,s,t}$ has at most one zero in each of the intervals $(\min\{-t, g_k^{-1}(s)\}, -x_0)$, $(-x_0, \max\{-t, g_k^{-1}(s)\})$, $(\min\{-s, g_k^{-1}(t)\}, x_0)$ and $(x_0, \max\{-s, g_k^{-1}(t)\})$. Thus $h_{r,s,t}$ has at most two zeros in $I_\pm$ unless perhaps $h_{r,s,t}(-x_0) = 0$ or $h_{r,s,t}(x_0) = 0$. Suppose that $h_{r,s,t}(x_0) = 0$. Then $h'_{r,s,t}(x_0) = 0$ too by $(*)$ and differentiating $(*)$ at x_0 we obtain

$$\begin{aligned} x_0 h''_{r,s,t}(x_0) &= (\lvert r\rvert^{k-1} g'_k(x_0 + s) + g'_k(x_0 + t))(g_k(x_0) + x_0 g'_k(x_0))) \\ &= (\lvert r\rvert^{k-1} g'_k(x_0 + s) + g'_k(x_0 + t))(k+1)) g_k(x_0). \end{aligned}$$

Hence $h''_{r,s,t}(x_0) > 0$ and it then follows that x_0 is the only zero of $h_{r,s,t}$ in I_+. The case $h_{r,s,t}(-x_0) = 0$ is dealt with similarly and results in only one zero of $h_{r,s,t}$ in I_-. So in any case $h_{r,s,t}$ has at most two zeros in $I_\pm$.

We still have to exclude the case that $h_{r,s,t}$ has more than two zeros in $I_+ \cup I_-$. Let

$$r_+ = \left(- \frac{g_k(x_0 + t)(g_k(x_0) - s)}{g_k(x_0 + s)(g_k(x_0) - t)} \right)^{1/(k-1)}$$

$$r_- = \left(- \frac{g_k(-x_0 + t)(g_k(-x_0) - s)}{g_k(-x_0 + s)(g_k(-x_0) - t)} \right)^{1/(k-1)}$$

that is, $r_\pm$ are such that $h_{r_+,s,t}(x_0) = h_{r_-,s,t}(-x_0) = 0$. Then

$$r_+^{k-1} - r_-^{k-1} = \frac{s}{t} \left(\left| \frac{x_0 - t}{x_0 - s} \right|^{k-1} - \left| \frac{x_0 + t}{x_0 + s} \right|^{k-1} \right) > 0.$$

Hence $r_+ > r_- > 0$. Since $g_k(x_0 + s)(g_k(x_0) - t) < 0$ on I_+, we obtain that $h_{r,s,t}(x_0) < 0, \ = 0, \ > 0$ for $r > r_+$, $r = r_+$, $r < r_+$, respectively. Similarly, $g_k(-x_0 + s)(g_k(-x_0) - t) > 0$ on I_- implies that $h_{r,s,t}(-x_0) < 0, = 0, > 0$ for $r < r_-$, $r = r_-$, $r > r_-$, respectively.

If $h_{r,s,t}$ has two zeros in I_+, then $h_{r,s,t}(x_0) < 0$ and thus $r > r_+$ from above. But then $r > r_-$ and $h_{r,s,t}(-x_0) > 0$. From what we have seen before, this then implies that $h_{r,s,t}$ has no zeros in I_-. The case that $h_{r,s,t}$ has two zeros in I_- is dealt with similarly. This concludes the proof that $h_{r,s,t}$ has at most two real zeros and the statement of the proposition is established. □

Corollary 2.2. *Two different circles of $\mathscr{M}(k)$ intersect in at most two points. Hence two points in a derived geometry at a point of $\mathscr{M}(k)$ are on at most one line.*

Proof. The circles of $\mathscr{M}(k)$ are the graphs of β and $\delta f_k \delta^{-1}$ for all $\beta, \delta \in \mathrm{PSL}_2(\mathbb{R})$. Since the first kind of homeomorphism is orientation-preserving and the latter kind is orientation-reversing, we obtain that any two such associated circles intersect in at most two points. The same is true for any two circles of the first kind because we are essentially in the classical flat Minkowski plane.

If the circles associated with $\gamma f_k \gamma^{-1}$ and $\delta f_k \delta^{-1}$ for $\gamma, \delta \in \mathrm{PSL}_2(\mathbb{R})$ have three distinct points in common, then $(\delta^{-1}\gamma)^{-1} f_k^{-1} (\delta^{-1}\gamma) f_k$ fixes three points so that $\delta^{-1}\gamma = \mathrm{id}$ or $\delta^{-1}\gamma = \sigma$ by Proposition 2.1 and the circles are the same. This shows that if $\gamma f_k \gamma^{-1}$ and $\delta f_k \delta^{-1}$ describe different circles in $\mathscr{M}(k)$, that is, $\gamma \neq \delta, \delta\sigma$, then these circles can have at most two points in common. □

From the definition of circles it is obvious that circles are described by homeomorphisms of $\mathbb{S}^1$. Hence, in order to verify that $\mathscr{M}(k)$ is a flat Minkowski plane, we only have to make sure that each derived incidence geometry is an affine plane. Since the group Σ is a group of automorphisms of the classical flat Minkowski plane, and, by construction, also acts on the negative half of $\mathscr{M}(k)$, we see that Σ is a group of automorphisms of $\mathscr{M}(k)$. Furthermore, Σ has two orbits on the torus, the circle

$$D = \{(x, x) \mid x \in \mathbb{S}^1\}$$

in the positive half and its complement $(\mathbb{S}^1 \times \mathbb{S}^1) \backslash D$. It therefore suffices to show that the derived incidence geometries at the points (∞, ∞) and $(\infty, 0)$ are affine planes.

Note moreover that Σ is even doubly transitive on the points of D and that D is the only circle fixed by Σ.

3 The derived geometry $\mathscr{A}$ at (∞, ∞)

The lines of the derived geometry $\mathscr{A}$ of $\mathscr{M}(k)$ at (∞, ∞) are the horizontal and vertical Euclidean lines (coming from parallel classes of $\mathscr{M}(k)$), all Euclidean lines of positive slope (coming from circles in the positive half that pass through (∞, ∞)), and the lines

$$\left\{ \left(x, rf_k\left(\frac{x-t}{r}\right) + t \right) \,\middle|\, x \in \mathbb{R} \right\}$$

for $r, t \in \mathbb{R}$, $r > 0$ (coming from circles in the negative half that pass through (∞, ∞)). The latter circles are the images of the generating circle C_k under the stabilizer $\Lambda = \Sigma_{(\infty, \infty)}$ of (∞, ∞), that is, the group

$$\Lambda = \{(x, y) \mapsto (rx + t, ry + t) \,|\, r, t \in \mathbb{R}, r > 0\} \cong L_2.$$

Note that the transformation $\hat{\sigma} : (x, y) \mapsto (-1/x, -1/y)$ in Σ leaves C_k invariant. Therefore the coset $\Lambda\hat{\sigma}$ gives rise to the same set of circles.

Using the restriction g_k of $-f_k$ on $\mathbb{R}$, the lines of the latter kind in $\mathscr{A}$ can then be rewritten as

$$y = sg_k(x - t) + t$$

for $s, t \in \mathbb{R}$, $s < 0$ $(s = -r^{1-k})$. Hence we obtain the following description of the the lines in $\mathscr{A}$.

The geometry $\mathscr{A}$. The lines of $\mathscr{A}$ are the verticals $\{c\} \times \mathbb{R}$ for $c \in \mathbb{R}$ and

$$L_{s,t} = \begin{cases} \{(x, sx + t)\} \,|\, x \in \mathbb{R}\}, & \text{for } s, t \in \mathbb{R}, s \geqslant 0, \\ \{(x, sg_k(x - t) + t)\} \,|\, x \in \mathbb{R}\}, & \text{for } s, t \in \mathbb{R}, s < 0. \end{cases}$$

Proposition 3.1. *The derived geometry $\mathscr{A}$ of $\mathscr{M}(k)$ at (∞, ∞) is an affine plane.*

Proof. We first show that two distinct points of $\mathbb{R}^2$ can be joined by a unique line in $\mathscr{A}$. Let (x_i, y_i), $i = 1, 2$, be two such points. If $(y_2 - y_1)(x_2 - x_1) \geqslant 0$, there is a unique Euclidean line of nonnegative slope or a vertical line through these points. Moreover, no line $L_{s,t}$ for $s < 0$ can pass through (x_1, y_1) and (x_2, y_2). If $(y_2 - y_1) \cdot (x_2 - x_1) < 0$, no such Euclidean line with $s > 0$ can exist and we have to find a unique line $L_{s,t}$ where $s < 0$ through both points. Without loss of generality we may assume that $x_1 < x_2$. From the system of equations

$$y_1 - t = sg_k(x_1 - t)$$
$$y_2 - t = sg_k(x_2 - t)$$

we obtain $(y_2 - t)/(y_1 - t) = g_k((x_2 - t)/(x_1 - t))$. Taking the inverse of the fractional linear map $t \mapsto (y_2 - t)/(y_1 - t)$ on both sides and using that g_k is multiplicative, we obtain

$$\frac{y_1 g_k(x_2 - t) - y_2 g_k(x_1 - t)}{g_k(x_2 - t) - g_k(x_1 - t)} = t.$$

The left-hand side defines a strictly decreasing homeomorphism h of $\mathbb{R}$. Hence h has a unique fixed point t_0. Since $x_1 \neq x_2$, we have $t_0 \neq x_i$ for at least one $i = 1, 2$. Then $s_0 = (y_i - t_0)/g_k(x_i - t_0)$ is well defined and L_{s_0, t_0} is the unique line in $\mathscr{A}$ through (x_1, y_1) and (x_2, y_2).

For the parallel axiom note that $L_{s,t}$ for $s > 0$ and $s < 0$ are graphs of orientation-preserving and orientation-reversing homeomorphisms of $\mathbb{R}$. We therefore see that the parallel axiom is clearly satisfied for horizontal or vertical lines and that the parameter s' of any parallel $L_{s',t'}$ of a line $L_{s,t}$ in $\mathscr{A}$ must have the same sign as s. Hence there is a unique parallel in $\mathscr{A}$ to a line $L_{s,t}$, $s > 0$ (that is, a Euclidean line of positive slope) through a given point. We thus only consider the case $s < 0$.

We first verify that two lines $L_{s,t}$ and $L_{s',t'}$ where $s, s' < 0$ are parallel if and only if $s = s'$. Straightforward computation shows that the automorphism $(x, y) \mapsto (ax + b, ay + b)$, where $a, b \in \mathbb{R}$, $a > 0$, takes $L_{s',t'}$ to the line $L_{a^{1-k}s', at'+b}$. Using the group Λ we may therefore assume that $s' = -1$ and $t' = 0$. Then

$$-g_k(x - t) + t \begin{cases} > -g_k(x) + t > -g_k(x), & \text{if } t > 0, \\ = -g_k(x), & \text{if } t = 0, \\ < -g_k(x) + t < -g_k(x), & \text{if } t < 0. \end{cases}$$

This shows that $L_{-1,t}$ is parallel to $L_{-1,0}$. If $s \neq -1$, then $x \mapsto g_k(x) + sg_k(x - t) + t$ is a continuous function on $\mathbb{R}$ that tends to $\pm\infty$ as x goes to $\pm\infty$ if $-1 < s < 0$ and to $\mp\infty$ for $x \to \pm\infty$ if $s < -1$. Therefore this function is surjective in any case and the value 0 is attained. This shows that $L_{s,t}$ intersects $L_{-1,0}$ in a point if $s \neq -1$.

Now given a point (x_0, y_0), a line parallel to $L_{-1,0}$ that passes through this point must be of the form $L_{-1,t}$. To find t just note that $g_k(t - x_0) = -g_k(x_0 - t)$ is strictly increasing in t and $y_0 - t$ is strictly decreasing in t. Furthermore, both functions are unbounded. Hence there is a unique $t_0 \in \mathbb{R}$ such that $-g_k(x_0 - t) = y_0 - t$, that is, L_{-1,t_0} is the unique line parallel to $L_{-1,0}$ that passes through (x_0, y_0).

Hence the axioms of an affine plane are satisfied. □

Note that Proposition 3.1 also follows from [3], Theorem 2.7. Rotation through 45° brings the geometry $\mathscr{A}$ in the form used in [3]. In the new coordinates the group Λ acts on $\mathbb{R}^2$ as $(x, y) \mapsto (rx, ry + t)$ and the distinguished line fixed under Λ is the y-axis. Straightforward computation shows that the triple $(\{F_1\}, \{F_2\}, \varphi)$, where F_1 and

F_2 are functions from $\mathbb{R}^+$ to $\mathbb{R}$ defined by $F_2(x) = -F_1(x)$, $F_1(x) = \sqrt{2}u - x$, where u is the unique solution of $u - f_k(u) = x$, and φ is given by $\varphi(x) = -x$, satisfies the conditions (F1)–(F3) of [3], p. 7, so that $\mathscr{A}$ is an affine plane by [3], Theorem 2.7.

The transitivity of Σ on D implies that Proposition 3.1 carries over to any point on D.

Corollary 3.2. *Each derived geometry of $\mathscr{M}(k)$ at a point of D is an affine plane.*

Note that $k = 1$ does not yield an affine plane because we do not get enough lines in $\mathscr{A}$. This of course means that we cannot extend the definition of $\mathscr{M}(k)$ to $k = 1$. Indeed, the orbit of the generating circle C_1 under Σ only yields a 2-dimensional family of circles so that we do not obtain enough circles in the negative half in this case. (However, $k = 1$ results in the Desarguesian affine plane for the derived geometry at $(\infty, 0)$, see the following section for this geometry.)

For later, when we determine isomorphism classes, we conclude this section by showing that $\mathscr{A}$ is not an affine plane that occurs as a derivation of the classical flat Minkowski plane.

Lemma 3.3. *$\mathscr{A}$ is not Desarguesian.*

Proof. We consider the triangles with vertices $p_1 = (0,0)$, $p_2 = (1,-1)$, $p_3 = (1,-3)$, and $q_1 = (-2,0)$, $q_2 = (-1,-1)$, $q_3 = (-1,-3)$, respectively. The lines through p_i and q_i are horizontals and thus are parallel for $i = 1,2,3$. Furthermore, corresponding lines through p_2 and q_2 are also parallel. (The lines p_2p_3 and q_2q_3 are verticals and the lines p_1p_2 and q_1q_2 are $L_{-1,0}$ and $L_{-1,-1}$, respectively.) Finally, the line p_1p_3 is $L_{-2,0}$ and the line through q_1 and q_3 is $L_{-3g_k(2)/2,-3/2}$. But $k > 1$ implies $g_k(2) > 2$ and thus $-\frac{3}{2}g_k(2) \neq -3$. Hence p_1p_3 and q_1q_3 are not parallel, compare the proof of Proposition 3.1, and Desargues' configuration does not close for the above six points. □

Note that the proof of Lemma 3.3 only uses horizontals, verticals and lines $L_{s,t}$ with $s < 0$, that is, parallel classes and circles in the negative half of $\mathscr{M}(k)$.

4 The derived geometry $\mathscr{B}$ at $(\infty, 0)$

For a description of the lines in the derived incidence geometry $\mathscr{B}$ of $\mathscr{M}(k)$ at $(\infty, 0)$ we use the coordinate transformation $(x, y) \mapsto (x, 1/y)$. A circle through $(\infty, 0)$ is the graph of a fractional linear map $x \mapsto b/(cx + d)$ where $bc = -1$ or the graph of $\delta f_k \delta^{-1}$ where $\delta \in \mathrm{PSL}_2(\mathbb{R})$, $\delta f_k \delta^{-1}(\infty) = 0$. Under the above coordinate transformation the former circles give rise to the lines $y = mx + t$ where $m, t \in \mathbb{R}$, $m < 0$. As for the latter circles, note that each such circle intersects the distinguished circle D in two points, say (u, u) and (v, v) where $u, v \in \mathbb{R}\backslash\{0\}$, $u \neq v$. Furthermore, because the derived geometry at (u, u) is an affine plane by Corollary 3.2, the points $(\infty, 0)$, (u, u) and (v, v) determine a unique circle. We must even have $uv < 0$. This follows from the fact that $\delta f_k \delta^{-1}$ is strictly decreasing on $\mathbb{R}\backslash\{w\}$ where $w = \delta f_k^{-1} \delta^{-1}(\infty) < 0$

so that the graph must intersect D in a point with negative coordinates and one with positive coordinates. Then $\delta^{-1}(u)$ and $\delta^{-1}(v)$ are both fixed points of f_k so that $\{\delta^{-1}(u), \delta^{-1}(v)\} = \{\infty, 0\}$, that is, $\{u, v\} = \{\delta(\infty), \delta(0)\}$. Let $\delta(x) = (ax+b)/(cx+d)$, $ad - bc = 1$. Then $u = \delta(\infty) = a/c$, $v = \delta(0) = b/d$, or $v = a/c$, $u = b/d$. In the first case we obtain $a = uc$, $b = vd$, and $1 = ad - bc = (u-v)cd$. Therefore $\delta(x) = (ucx + vd)/(cx+d)$ and $\delta^{-1}(x) = (dx - vd)/(-cx + uc) = -(d/c)(x-v)/(x-u)$. Furthermore, $-(vd)/(uc) = \delta^{-1}(0) = f_k\delta^{-1}(\infty) = f_k(-d/c) = g_k(d/c)$. Thus

$$\begin{aligned}
\delta f_k \delta^{-1}(x) &= \delta\left(-g_k\left(-\frac{d}{c}\frac{x-v}{x-u}\right)\right) \\
&= \delta\left(g_k\left(\frac{d}{c}\right)g_k\left(\frac{x-v}{x-u}\right)\right) \\
&= \delta\left(-\frac{vd}{uc}g_k\left(\frac{x-v}{x-u}\right)\right) \\
&= \delta\left(-\frac{vdg_k(x-v)}{ucg_k(x-u)}\right) \\
&= \frac{-vd\frac{g_k(x-v)}{g_k(x-u)} + vd}{-\frac{vdg_k(x-v)}{ug_k(x-u)} + d} \\
&= uv\frac{-g_k(x-v) + g_k(x-u)}{-vg_k(x-v) + ug_k(x-u)}
\end{aligned}$$

Under the above coordinate transformation we obtain the lines $L_{u,v}$ given by $y = F(u, v, x)$ where $u, v \in \mathbb{R}$, $uv < 0$, and

$$F(u, v, x) = \frac{1}{uv}\frac{ug_k(x-u) - vg_k(x-v)}{g_k(x-u) - g_k(x-v)}.$$

Note that the above denominator is never 0 so that the right-hand side is defined for all $x \in \mathbb{R}$.

In the second case the roles of u and v are interchanged and we obtain the same equation. Note that the above equation is symmetric in u and v. In particular, we can always assume that $u < 0 < v$.

Since g_k is multiplicative, it follows that $F(u, v, x) = \frac{1}{uv}\frac{ug_k(1-u/x) - vg_k(1-v/x)}{g_k(1-u/x) - g_k(1-v/x)}$ for $x \neq 0$ and thus $\lim_{x\to\pm\infty} F(u, v, x) = \pm\infty$. Furthermore, $\frac{\partial F}{\partial x}(u, v, x) = -\frac{k(v-u)^2|x-u|^{k-1}|x-v|^{k-1}}{uv(g_k(x-u) - g_k(x-v))^2} > 0$ for all $x \neq u, v$ so that $F(u, v, x)$ is strictly increasing in x. This verifies that $x \mapsto F(u, v, x)$ is indeed a homeomorphism of $\mathbb{R}$ for all admissible u and v.

In summary, we have found the following description of the lines of the derived geometry $\mathscr{B}$.

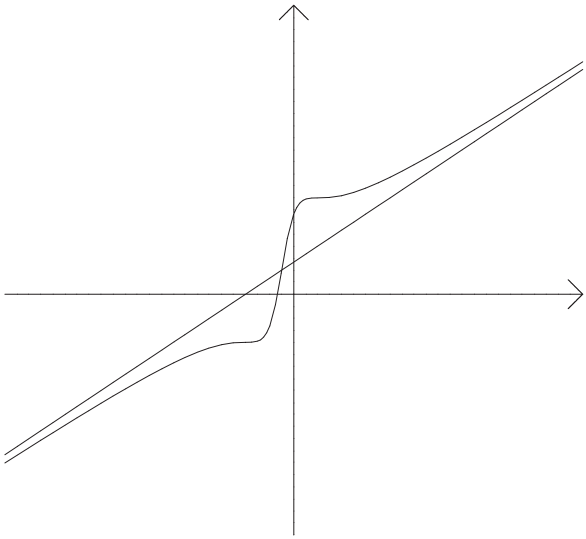

Figure 1. $E_{-1,1/2}$ and $L_{-1,1/2}$ for $k = 3$

The geometry $\mathscr{B}$. The lines of $\mathscr{B}$ are

- the verticals $\{c\} \times \mathbb{R}$ for $c \in \mathbb{R}$;
- the Euclidean lines $y = mx + t$ of nonpositive slope $m \leqslant 0$;
- the sets

$$L_{u,v} = \{(x, F(u,v,x))\} \mid x \in \mathbb{R}\}$$

for $u, v \in \mathbb{R}$, $u < 0 < v$.

Lemma 4.1. *The line $L_{u,v}$ has the Euclidean line $E_{u,v}$ given by $y = -\frac{1}{kuv} \cdot \left(x - \frac{k+1}{2}(u+v)\right)$ as an oblique asymptote. Furthermore, $L_{u,v}$ and $E_{u,v}$ have precisely the point $\left(\frac{u+v}{2}, \frac{u+v}{2uv}\right)$ in common and $E_{u,v}$ is below $L_{u,v}$ to the right of that point and above $L_{u,v}$ to the left.*

Proof. Since $k > 1$, the function g_k is continuously differentiable and even twice continuously differentiable for all $x \neq 0$. The respective derivatives are $g_k'(x) = k|x|^{k-1}$ and $g_k''(x) = k(k-1)x|x|^{k-3}$. If $z \in \mathbb{R}$ such that x and $x - z$ are in the same open interval $(-\infty, 0)$ or $(0, +\infty)$, Taylor's formula then yields

$$g_k(x-z) = g_k(x) - zg_k'(x) + \frac{z^2}{2}g_k''(\bar{z})$$

where $\bar{z}$ is between $x - z$ and x. Note that if z is fixed and x tends to $\pm\infty$ we obtain that $g_k''(\bar{z})|x|^{1-k}$ tends to 0 and that $g_k''(\bar{z})x|x|^{1-k} = g_k''(\bar{z})\frac{1}{x}|x|^{3-k}$ tends to $k(k-1)$.

Let $u < 0 < v$ so that $x - v < \bar{v} < x < \bar{u} < x - u$. Then

$$uvF(u,v,x) = \frac{(u-v)g_k(x) - (u^2-v^2)g_k'(x) + \frac{1}{2}(u^3 g_k''(\bar{u}) - v^3 g_k''(\bar{v}))}{-(u-v)g_k'(x) + \frac{1}{2}(u^2 g_k''(\bar{u}) - v^2 g_k''(\bar{v}))}$$
$$= \frac{(u-v)x - (u^2-v^2)k + \frac{1}{2}(u^3 g_k''(\bar{u}) - v^3 g_k''(\bar{v}))|x|^{1-k}}{-(u-v)k + \frac{1}{2}(u^2 g_k''(\bar{u}) - v^2 g_k''(\bar{v}))|x|^{1-k}}$$

and

$$uvF(u,v,x) + \frac{1}{k}x$$
$$= \frac{-(u^2-v^2)k + \frac{1}{2}(u^3 g_k''(\bar{u}) - v^3 g_k''(\bar{v}))|x|^{1-k} + \frac{1}{2k}(u^2 g_k''(\bar{u}) - v^2 g_k''(\bar{v}))x|x|^{1-k}}{-(u-v)k + \frac{1}{2}(u^2 g_k''(\bar{u}) - v^2 g_k''(\bar{v}))|x|^{1-k}}.$$

The numerator and denominator on the right-hand side tend to $-(u^2-v^2)k + \frac{1}{2k}(u^2-v^2)k(k-1) = -\frac{k+1}{2}(u^2-v^2)$ and $-(u-v)k$, respectively, as x goes to $\pm\infty$. Thus $\lim_{x\to\pm\infty} F(u,v,x) + \frac{1}{k}x = \frac{k+1}{2k}(u+v)$. This shows that the $E_{u,v}$ is an oblique asymptote of $L_{u,v}$.

Let

$$E(u,v,x) = -\frac{1}{kuv}\left(x - \frac{k+1}{2}(u+v)\right)$$

for $u, v, x \in \mathbb{R}$, $u < 0 < v$. Then $E\left(u,v,\frac{u+v}{2}\right) = F\left(u,v,\frac{u+v}{2}\right) = \frac{u+v}{2uv}$ so that $\left(\frac{u+v}{2}, \frac{u+v}{2uv}\right)$ is on $E_{u,v} \cap L_{u,v}$. Now consider the equation $E(u,v,x) = F(u,v,x)$ for fixed $u < 0 < v$. We write $x = \frac{v-u}{2}z + \frac{v+u}{2}$ for $z \in \mathbb{R}$. Then

$$\begin{aligned}
0 &= F(u,v,x) - E(u,v,x) \\
&= \frac{1}{uv}\left(\frac{ug_k(x-u) - vg_k(x-v)}{g_k(x-u) - g_k(x-v)} + \frac{1}{k}\left(x - (k+1)\frac{v+u}{2}\right)\right) \\
&= \frac{1}{uv}\left(\frac{ug_k(z+1) - vg_k(z-1)}{g_k(z+1) - g_k(z-1)} + \frac{1}{k}\left(\frac{v-u}{2}z - k\frac{v+u}{2}\right)\right) \\
&= \frac{(v-u)((z-k)g_k(z+1) - (z+k)g_k(z-1))}{2kuv(g_k(z+1) - g_k(z-1))}.
\end{aligned}$$

Hence $(z-k)g_k(z+1) - (z+k)g_k(z-1) = 0$ or $(z+k)/(z-k) = g_k(z+1)/g_k(z-1) = g_k((z+1)/(z-1))$. Let $t = (z+1)/(z-1)$ so that $z = (t+1)/(t-1)$. Then the above equation becomes

$$g_k(t) = \frac{(k+1)t - (k-1)}{(k+1) - (k-1)t}$$

and thus

$$h(t) = (k-1)|t|^{k+1} - (k+1)t|t|^{k-1} + (k+1)t - (k-1) = 0.$$

The function h is differentiable and even twice differentiable for $t \neq 0$. For the derivatives one finds

$$h'(t) = (k^2-1)t|t|^{k-1} - (k+1)k|t|^{k-1} + k + 1,$$
$$h''(t) = k(k^2-1)t|t|^{k-3}(t-1).$$

Hence $h''(t) > 0$ for $t < 0$ or $t > 1$ and $h''(t) < 0$ for $0 < t < 1$. Consequently, h' is strictly increasing on $(-\infty, 0)$ and $h'(1) = 0$ is a relative minimum of h' on $(0, +\infty)$. The latter implies that h is strictly increasing on $(0, +\infty)$ and thus 1 is the only positive zero of h. The former and the fact that $h'(0) = k+1 > 0$, $h'(-1) = -2(k^2-1) < 0$ imply that h' has precisely one negative zero t_- for which we have $-1 < t_- < 0$. Furthermore, h is strictly decreasing on the interval $(-\infty, t_-)$ and strictly increasing on $(t_-, 0)$. But $h(0) = -(k-1) < 0$ and $h(-1) = 0$. This shows that -1 is the only negative zero of h.

In summary we have found that h has precisely two zeros, namely $t = 1$ and $t = -1$. This in turn yields the only solution $z = 0$, that is, $x = (u+v)/2$, of our original equation. (Note that $t = -1$ corresponds to $z = 0$ and that $t = 1$ yields $z = \infty$ and thus does not contribute to a solution in $\mathbb{R}$.) This proves that $\left(\frac{u+v}{2}, \frac{u+v}{2uv}\right)$ is the only point of intersection of $E_{u,v}$ and $L_{u,v}$. The remaining statements on the relative positions of $E_{u,v}$ and $L_{u,v}$ readily follow from $E(u,v,u) - F(u,v,u) = -\frac{(k-1)(v-u)}{2kuv} > 0$ and $E(u,v,v) - F(u,v,v) = \frac{(k-1)(v-u)}{2kuv} < 0$. $\square$

Note that every Euclidean line of positive slope occurs precisely once as an asymptote $E_{u,v}$ for some $u < 0 < v$. Indeed, if $m, t \in \mathbb{R}$, $m > 0$, then $u = -\frac{t}{(k+1)m} - \sqrt{\frac{1}{km} + \frac{t^2}{(k+1)^2m^2}} < 0$ and $v = -\frac{t}{(k+1)m} + \sqrt{\frac{1}{km} + \frac{t^2}{(k+1)^2m^2}} > 0$ are such that $E_{u,v}$ is the Euclidean line given by $y = mx + t$.

In the coordinates of $\mathscr{B}$, the distinguished circle D induces the Euclidean hyperbola

$$H = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}.$$

The stabilizer $\Psi = \Sigma_{(\infty,0)}$ of $(\infty, 0)$ also fixes the points $(0, \infty)$, (∞, ∞) and $(0, 0)$. Hence

$$\Psi = \{(x, y) \mapsto (rx, ry) \mid r > 0\}.$$

This group induces a group Φ of collineations of $\mathscr{B}$. In the new coordinates of $\mathscr{B}$ one obtains

$$\Phi = \{(x, y) \mapsto (rx, y/r) \mid r > 0\}.$$

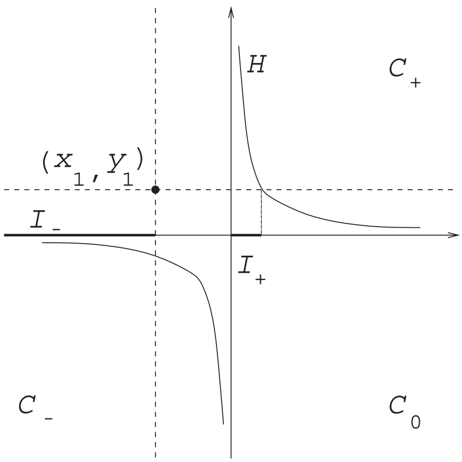

Figure 2.

Lemma 4.2. *The derived geometry* $\mathscr{B}$ *of* $\mathscr{M}(k)$ *at* $(\infty, 0)$ *is a linear space, that is, any two distinct points can be uniquely joined by a line.*

Proof. Given two distinct points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ we have to find a line of $\mathscr{B}$ that passes through them. Clearly, if $(x_2 - x_1)(y_2 - y_1) \leqslant 0$, then these points are on a vertical line or a Euclidean line of nonpositive slope. Furthermore, such a line is unique.

We now assume that $(x_2 - x_1)(y_2 - y_1) > 0$. Without loss of generality we may further assume that $x_1 < x_2$. Since the derived geometry at each point of D is an affine plane by Corollary 3.2, we may moreover assume that none of the points is on H, that is, $x_i y_i \neq 1$ for $i = 1, 2$. A line through the two points must then be of the form $L_{u,v}$ where $u < 0 < v$. By Corollary 2.2 a joining line will be unique and we only have to verify the existence of such a line.

Since each line $L_{u,v}$ is the graph of a strictly increasing homeomorphism of $\mathbb{R}$ and because $L_{u,v}$ passes through $(u, 1/u)$ and $(v, 1/v)$, we see that $(x_1 - u)\left(y_1 - \frac{1}{u}\right) > 0$ is a necessary condition, that is,

$$(x_1 - u)(uy_1 - 1) < 0.$$

We similarly find that

$$(x_1 - v)(vy_1 - 1) > 0;$$

compare Figure 2.

Depending on the position of (x_1, y_1) relative to the coordinate axes and to H one obtains from the above two inequalities certain restrictions for u and v where $I_- \subseteq$

$(-\infty, 0)$ and $I_+ \subseteq (0, +\infty)$ denote the maximal (open) intervals we can choose u and v from; see Table 2 below.

x_1	y_1	$x_1 y_1$	I_-	I_+	q_-	q_+	G_-	G_+	α
$\geqslant 0$	0		$(-\infty, 0)$	$(x_1, +\infty)$	> 0	< 0	$\uparrow$	$\downarrow$	$\downarrow$
< 0	0		$(-\infty, x_1)$	$(0, +\infty)$	> 0	< 0	$\uparrow$	$\downarrow$	$\downarrow$
$\leqslant 0$	> 0		$(-\infty, x_1)$	$(0, 1/y_1)$	> 0	< 0	$\uparrow$	$\downarrow$	$\downarrow$
> 0	> 0	> 1	$(-\infty, 0)$	$(1/y_1, x_1)$	> 0	> 0	$\uparrow$	$\uparrow$	$\uparrow$
> 0	> 0	< 1	$(-\infty, 0)$	$(x_1, 1/y_1)$	> 0	< 0	$\uparrow$	$\downarrow$	$\downarrow$
$\geqslant 0$	< 0		$(1/y_1, 0)$	$(x_1, +\infty)$	> 0	< 0	$\uparrow$	$\downarrow$	$\downarrow$
< 0	< 0	> 1	$(x_1, 1/y_1)$	$(0, +\infty)$	< 0	< 0	$\downarrow$	$\downarrow$	$\uparrow$
< 0	< 0	< 1	$(1/y_1, x_1)$	$(0, +\infty)$	> 0	< 0	$\uparrow$	$\downarrow$	$\downarrow$

Table 2.

For $L_{u,v}$ to pass through (x_1, y_1) we find the condition

$$y_1 uv(g_k(x_1 - u) - g_k(x_1 - v)) = ug_k(x_1 - u) - vg_k(x_1 - v)$$

which yields

$$\frac{ug_k(x_1 - u)}{y_1 u - 1} = \frac{vg_k(x_1 - v)}{y_1 v - 1}$$

that is, $G(u) = G(v)$ where

$$G(z) = \frac{zg_k(z - x_1)}{y_1 z - 1}$$

for $z \in \mathbb{R}$, $y_1 z \neq 1$. We denote by $G_\pm$ the restriction of G to the open interval $I_\pm$. G is differentiable and has derivative

$$G'(z) = \frac{|z - x_1|^{k-1}}{(y_1 z - 1)^2}(ky_1 z^2 - (k+1)z + x_1).$$

The first term $|z - x_1|^{k-1}/(y_1 z - 1)^2$ in $G'(z)$ above is always positive on $I_\pm$ and it readily follows that the last factor $q(z) = ky_1 z^2 - (k+1)z + x_1$ has no zero in $I_\pm$. (Note that $q(z) = kz(y_1 z - 1) + (x_1 - z)$ and that $z(y_1 z - 1)$ and $x_1 - z$ have the same sign on $I_\pm$.) In the above table the sign of $q_\pm$, that is, the restriction of q to $I_\pm$, is indicated in the columns labelled q_- and q_+. Hence $G_\pm$ is strictly increasing or strictly decreasing on $I_\pm$. In Table 2 this is indicated by an arrow up $\uparrow$ or an arrow down $\downarrow$, respectively.

Clearly, $G(0) = G(x_1) = 0$,

$$\lim_{z \to \pm\infty} G(z) = \begin{cases} \pm\infty, & \text{if } y_1 > 0, \\ -\infty, & \text{if } y_1 = 0, \\ \mp\infty, & \text{if } y_1 < 0, \end{cases}$$

and $\lim_{z\to 1/y_1} G(z) = \pm\infty$ depending on the relative position of $y_1 \neq 0$ to 0 and x_1, but the sign changes in any case when z approaches $1/y_1$ from opposite sides. It then follows that in any case $G_\pm$ takes $I_\pm$ onto the negative real numbers $(-\infty, 0)$. In particular, this shows that for each $u \in I_-$ there is a unique $v \in I_+$ such that $L_{u,v}$ passes through (x_1, y_1). In fact, there is a homeomorphism $\alpha_1 : I_- \to I_+$ such that $L_{u,\alpha_1(u)}$ passes through (x_1, y_1). Clearly, $\alpha_1 = G_+^{-1} G_-$, and α_1 is strictly decreasing if and only if $x_1 y_1 < 1$ (that is, (x_1, y_1) is between the two branches of the Euclidean hyperbola H) and strictly increasing if and only if $x_1 y_1 > 1$ (that is, (x_1, y_1) is above or below H); see Table 2. Moreover, α_1 is differentiable and its derivative is given by $\alpha_1'(u) = G_-'(u)/G_+'(\alpha_1(u))$.

(Note that there is no explicit formula for α_1 except for special cases. For example, in the case $x_1 = y_1 = 0$ one has $\alpha_1(u) = -u$. It then readily follows that there is a unique line $L_{u,-u}$ through a point (x_2, y_2) where $x_2 y_2 > 0$. Hence $(0,0)$ can be uniquely joined to any other point in $\mathscr{B}$.)

One similarly obtains a homeomorphism $\alpha_2 : \tilde{I}_- \to \tilde{I}_+$ such that $L_{u,\alpha_2(u)}$ passes through (x_2, y_2) where $\tilde{I}_- \subseteq (-\infty, 0)$ and $\tilde{I}_+ \subseteq (0, +\infty)$ are open intervals defined in a similar fashion as the intervals I_- and I_+ for α_1.

We consider the three connected components of $\mathbb{R}^2 \backslash H$; more precisely, let

$$C_+ = \{(x, y) \in \mathbb{R}^2 \mid xy > 1, x > 0\},$$
$$C_0 = \{(x, y) \in \mathbb{R}^2 \mid xy < 1\},$$
$$C_- = \{(x, y) \in \mathbb{R}^2 \mid xy > 1, x < 0\};$$

see Figure 2. The reflection ρ about the origin of $\mathbb{R}^2$ given by $\rho(x, y) = (-x, -y)$ is an automorphism of the incidence structure $\mathscr{B}$. (Note that $-F(u, v, -x) = F(-v, -u, x)$.) Furthermore, ρ interchanges C_+ and C_- and leaves C_0 invariant. Using ρ and perhaps relabelling the points, if necessary, we can assume that $x_1 < x_2$ and we can restrict ourselves to the four cases $(x_1, y_1) \in C_0$, $(x_2, y_2) \in C_+$ or $(x_1, y_1), (x_2, y_2) \in C_+$ or $(x_1, y_1), (x_2, y_2) \in C_0$ or $(x_1, y_1) \in C_-$, $(x_2, y_2) \in C_+$ for the relative positions of the two points (x_1, y_1) and (x_2, y_2). In each of theses cases we are looking at either $\alpha = \alpha_1 \alpha_2^{-1}$ or $\alpha = \alpha_2 \alpha_1^{-1}$ and verify that α fixes a point. Such a fixed point v leads to $u = \alpha_2^{-1}(v) = \alpha_1^{-1}(v)$ so that $L_{u,v}$ is a line through (x_1, y_1) and (x_2, y_2).

We encounter essentially two situations. In the first one $\alpha : I \to J$ is a strictly increasing homeomorphism and I and J are two open intervals in $\mathbb{R}$ such that J is finite and its closure $\bar{J}$ is contained in I. If $J = (c, d)$, we define v_n inductively by $v_0 = c$ and $v_{n+1} = \alpha(v_n)$ for $n \geqslant 0$, that is, $v_n = \alpha^n(c)$. Then the v_n's are increasing and bounded from above by d. Thus $v = \lim_{n\to\infty} v_n$ exists and by continuity of α it follows that α fixes v. In the other situation $\alpha : I \to J$ is a strictly decreasing homeomorphism and I and J are two open intervals in $\mathbb{R}$ such that $I \cap J$ is nonempty and J is finite. If $I = (a, b)$, $J = (c, d)$ and $w \in I \cap J$, we find that $\lim_{x\to a} \alpha(x) - x = d - a > w - w = 0$ and $\lim_{x\to b} \alpha(x) - x = c - b < w - w = 0$. By continuity of α it follows that there is a $v \in I$ such that $\alpha(v) - v = 0$, that is, α fixes v.

For example, if we assume that $(x_1, y_1) \in C_0$, $(x_2, y_2) \in C_+$, then $I_- \subseteq \tilde{I}_- = (-\infty, 0)$, $\tilde{I}_+ = (1/y_2, x_2)$ is finite and $I_+ \cap \tilde{I}_+ \neq \varnothing$ because this intersection contains the first coordinate of the point of intersection of the positive branch of H and the line segment from (x_1, y_1) to (x_2, y_2). Moreover, α_1 is strictly decreasing and α_2 is strictly increasing so that $\alpha = \alpha_2\alpha_1^{-1} : I_+ \to \alpha_2(I_-)$ is a strictly decreasing homeomorphism. Since $\alpha_2(I_-) \subseteq \tilde{I}_+$, we obtain that $\alpha_2(I_-)$ is finite. In order to show that $I_+ \cap \alpha_2(I_-)$ is nonempty we distinguish several cases.

If $x_1, y_1 \geqslant 0$, then $I_- = \tilde{I}_-$ and thus $\alpha_2(I_-) = \tilde{I}_+$. In case $x_1, y_1 \leqslant 0$ we have $I_+ = (0, +\infty)$ so that $\alpha_2(I_-) \subset I_+$. If $x_1 < 0 < y_1$, we have $I_- = (-\infty, x_1)$ and $I_+ = (0, 1/y_1)$; see Table 2. But then $\alpha_2(I_-) = (1/y_2, \alpha_2(x_1))$ and because $0 < y_1 < y_2$ we obtain that $I_+ \cap \alpha_2(I_-) = (1/y_2, \min\{1/y_1, \alpha_2(x_1)\})$. Finally, if $x_1 > 0 > y_1$, we have $I_- = (1/y_1, 0)$ and $I_+ = (x_1, +\infty)$; see Table 2. But then $\alpha_2(I_-) = (\alpha_2(1/y_1), x_2)$ and because $0 < x_1 < x_2$ we have $I_+ \cap \alpha_2(I_-) = (\max\{x_1, \alpha_2(1/y_1)\}, x_2)$.

The other cases are dealt with in a similar fashion. In any case one finds that α has a fixed point.

This finally shows that (x_1, y_1) and (x_2, y_2) can be joined by a line in $\mathscr{B}$. □

In order to show that $\mathscr{M}(k)$ is a flat Minkowski plane we still have to verify that the parallel axiom is satisfied in $\mathscr{B}$, that is, that $\mathscr{B}$ is an affine plane. As a first step in that direction we characterize parallelity in $\mathscr{B}$.

Lemma 4.3. *Two lines $L_{u,v}$ and $L_{u',v'}$ are parallel if and only if $uv = u'v'$.*

Proof. We first assume that $uv \neq u'v'$. Then the Euclidean lines given by

$$y = -\frac{1}{kuv}\left(x - \frac{k+1}{2}(u+v)\right) \quad \text{and} \quad y = -\frac{1}{ku'v'}\left(x - \frac{k+1}{2}(u'+v')\right)$$

have different slopes and intersect transversally in a point. Since these Euclidean lines are oblique asymptotes to the lines $L_{u,v}$ and $L_{u',v'}$ by Lemma 4.1 we see that $L_{u,v}$ and $L_{u',v'}$ must also intersect in a point.

Conversely assume that $uv = u'v'$. Assume that $L_{u,v}$ and $L_{u',v'}$ have a point (x_0, y_0) in common. Since we already have a linear space by Lemma 4.2 the two lines are either parallel (that is, $L_{u,v} = L_{u',v'}$) or (x_0, y_0) is the only common point of $L_{u,v}$ and $L_{u',v'}$. In the latter case the asymptotes of $L_{u,v}$ and $L_{u',v'}$ are different parallel Euclidean lines so that $F(u, v, x) - F(u', v', x)$ has the same sign for large $|x|$. This then implies that $L_{u,v}$ and $L_{u',v'}$ touch analytically at (x_0, y_0), that is, $\frac{\partial F}{\partial x}(u, v, x_0) = \frac{\partial F}{\partial x}(u', v', x_0)$.

We now show that $L_{u,v}$ is uniquely determined by the point (x_0, y_0) on it and the slope of the Euclidean tangent line at that point, that is, the partial derivative $\frac{\partial F}{\partial x}(u, v, x_0)$. Then the second case is not possible and the two lines must be parallel.

By using the group Φ we may assume that $uv = -1$. Let $x_0, y_0, y_0' \in \mathbb{R}$, $y_0' > 0$. We then have to find a unique $u < 0$ such that

$$\begin{aligned} F(u, v, x_0) &= y_0 \\ \frac{\partial F}{\partial x}(u, v, x_0) &= y_0', \end{aligned}$$

that is,

$$\begin{aligned}
y_0 &= \frac{1}{uv}\left(u + \frac{(u-v)(x_0-v)g_k'(x_0-v)}{k(g_k(x_0-u)-g_k(x_0-v))}\right) \\
&= \frac{1}{uv}\left(v + \frac{(u-v)(x_0-u)g_k'(x_0-u)}{k(g_k(x_0-u)-g_k(x_0-v))}\right) \\
y_0' &= \frac{(v-u)(g_k'(x_0-u)g_k(x_0-v)-g_k(u)g_k'(x_0-v))}{uv(g_k(x_0-u)-g_k(x_0-v))^2} \\
&= -\frac{(v-u)^2 g_k'(x_0-u)g_k'(x_0-v)}{kuv(g_k(x_0-u)-g_k(x_0-v))^2}.
\end{aligned}$$

(Note that $kg_k(x) = xg_k'(x)$ for all $x \in \mathbb{R}$.) Hence

$$y_0'(x_0-u)(x_0-v) + k(uy_0-1)(vy_0-1) = 0.$$

But $v = -1/u$ so that after multiplying through by u we obtain the quadratic equation

$$(ky_0 + x_0y_0')(u^2-1) + (k(y_0^2-1) - (x_0^2-1)y_0')u = 0$$

for u. If $ky_0 + x_0y_0' \neq 0$, then the above equation has precisely one positive and one negative zero (the coefficients of the quadratic and the constant terms have opposite signs). Thus there is at most one $u < 0$ that satisfies our two equations. If $ky_0 + x_0y_0' = 0$, then we must also have $k(y_0^2-1) - (x_0^2-1)y_0' = 0$. These two equations imply that $y_0 = -x_0$ and $y_0' = k$. The first of these identities yields $(x_0-u)g_k(x_0-u) = (x_0-v)g_k(x_0-v)$ and further $x_0 = (u+v)/2$. Thus $u^2 - 2x_0u - 1 = 0$ and $u = x_0 - \sqrt{x_0^2+1}$. Hence there is again a unique $u < 0$. □

Note that the proof of Lemma 4.3 further shows that there is at most one line $L_{u,v}$ with a given value uv through a given point, that is, we have the following.

Corollary 4.4. *Through each point there is at most one line $L_{u',v'}$ parallel to a line $L_{u,v}$.*

Proposition 4.5. *The derived geometry $\mathscr{B}$ of $\mathscr{M}(k)$ at $(\infty, 0)$ is an affine plane.*

Proof. It remains to show that the parallel axiom is satisfied in $\mathscr{B}$. Since Euclidean lines of negative slope and lines of the form $L_{u,v}$ are graphs of orientation-reversing and orientation-preserving homeomorphisms of $\mathbb{R}$ we see that a parallel of a Euclidean line or of $L_{u,v}$ in $\mathscr{B}$ must be of the same form. Hence there is a unique parallel in $\mathscr{B}$ to a horizontal line, a vertical line or a Euclidean line of negative slope through a given point. Given a point (x_0, y_0) and a line $L_{u,v}$ we know from Lemma 4.3 that any parallel through (x_0, y_0) must be of the form $L_{u',v'}$ where $u'v' = uv$. As in the proof of Lemma 4.3 we may assume that $uv = -1$. Then we have to find a unique $u' < 0$

such that $F(u', -1/u', x_0) = y_0$. But $h(z)$ defined by $h(z) = F(z, -1/z, x_0)$ for $z \in \mathbb{R}$, $z < 0$, is continuous and $\lim_{z \to -\infty} h(z) = +\infty$ and $\lim_{z \to 0^-} h(z) = -\infty$. This shows that h is onto $\mathbb{R}$ and there is at least one u' such that $h(u') = y_0$. But Corollary 4.4 shows that such a u' must be unique, that is, there is a unique parallel to $L_{u,v}$ through (x_0, y_0). □

The transitivity of Σ on points not on D implies that Proposition 4.5 carries over to any point not on D.

Corollary 4.6. *Each derived geometry of $\mathcal{M}(k)$ at a point not on D is an affine plane.*

Corollary 3.2 and Corollary 4.6 now imply the following.

Theorem 4.7. *Each incidence geometry $\mathcal{M}(k)$ for $k > 1$ as defined at the beginning of Section 2 is a flat Minkowski plane.*

5 Isomorphism classes and automorphisms

Since each derived affine plane of the classical flat Minkowski plane is Desarguesian, we immediately obtain the following from Lemma 3.3.

Theorem 5.1. *No flat Minkowski plane $\mathcal{M}(k)$ is classical.*

We now turn to isomorphisms between the planes $\mathcal{M}(k)$ and their automorphisms. We want to show that, in fact, these planes are mutually non-isomorphic. As a first step in this direction we prove that any isomorphism must respect the point orbits.

Lemma 5.2. *Let $\gamma : \mathcal{M}(k) \to \mathcal{M}(l)$ be an isomorphism between the flat Minkowski planes $\mathcal{M}(k)$ and $\mathcal{M}(l)$. Then γ takes the distinguished circle D in $\mathcal{M}(k)$ to the corresponding circle D in $\mathcal{M}(l)$.*

Proof. We assume that $\gamma(D) \neq D$. Then $\mathcal{M}(l)$ admits the 3-dimensional connected groups Σ and $\gamma\Sigma\gamma^{-1}$ as groups of automorphisms. Since D is the only circle fixed by Σ, it follows that $\Sigma \neq \gamma\Sigma\gamma^{-1}$ and hence that the automorphism group $\Gamma(l)$ of $\mathcal{M}(l)$ must be at least 4-dimensional. From the classification of flat Minkowski planes of group dimension at least 4 (see [5] or [4], 4.4.5) we see that $\mathcal{M}(l)$ must be classical or that $\Gamma(l)$ fixes two parallel classes. The former case is not possible by Theorem 5.1 and the latter cannot occur since Σ is already transitive on each set of all $(\pm)$-parallel classes. □

Theorem 5.3. *Two flat Minkowski planes $\mathcal{M}(k)$ and $\mathcal{M}(l)$ are isomorphic if and only if $k = l$.*

Proof. Let $\gamma : \mathcal{M}(k) \to \mathcal{M}(l)$ be an isomorphism between the flat Minkowski planes $\mathcal{M}(k)$ and $\mathcal{M}(l)$. Then γ takes the distinguished circle D in $\mathcal{M}(k)$ to the distinguished circle D in $\mathscr{M}(l)$ by Lemma 5.2. Since Σ is doubly transitive on D, we may assume that γ takes $p = (\infty, \infty)$ and $(0,0)$ in $\mathscr{M}(k)$ to the 'same' respective points (∞, ∞) and $(0,0)$ in $\mathscr{M}(l)$. Moreover, the stabilizer of these two points is transitive on the set of circles in the negative half through these two points. We therefore can further assume that γ takes the generating circle C_k in $\mathscr{M}(k)$ to the generating circle C_l in $\mathscr{M}(l)$.

The induced isomorphism $\bar{\gamma}$ from the derived affine plane $\mathscr{A}(k)$ of $\mathscr{M}(k)$ at p onto the derived affine plane $\mathscr{A}(l)$ of $\mathscr{M}(l)$ at p then takes $(0,0)$ to $(0,0)$, the lines $L_{1,0}$ induced from D and $L_{-1,0}$ in $\mathscr{A}(k)$ to $L_{1,0}$ and $L_{-1,0}$ in $\mathscr{A}(l)$, respectively. Furthermore, horizontal lines are mapped to horizontal lines and vertical lines to vertical lines, or these two sets of lines are interchanged. In the former case, $\bar{\gamma}$ is of the form $(x, y) \mapsto (\alpha(x), \beta(y))$ and $(x, y) \mapsto (\alpha(y), \beta(x))$ in the latter case, where α and β are homeomorphisms of $\mathbb{R}$. Since $L_{1,0}$ is taken to $L_{1,0}$ and $(0,0)$ to $(0,0)$, one finds $\alpha = \beta$ and $\alpha(0) = 0$ in both cases.

A line $L_{1,t}$, which is parallel to $L_{1,0}$, must be taken to a parallel to $L_{1,0}$ in $\mathscr{A}(l)$, that is, for every $t \in \mathbb{R}$ there is a $t' \in \mathbb{R}$ such that $\bar{\gamma}(L_{1,t}) = L_{1,t'}$. In the former case, this condition implies that $\alpha(x+t) = \alpha(x) + t'$. For $x = 0$ we obtain $t' = \alpha(t)$ so that

$$\alpha(x+t) = \alpha(x) + \alpha(t)$$

for all $x, t \in \mathbb{R}$. Hence $\alpha(x) = ax$ for some $a \in \mathbb{R}\backslash\{0\}$. We arrive at the same form for α in the second case where the horizontals and verticals are interchanged.

We finally look at $L_{-1,0}$. This line is taken by $\bar{\gamma}$ to the set

$$\{(x, -ag_k(x/a)) \mid x \in \mathbb{R}\} = \{(x, -ag_k(x)/g_k(a)) \mid x \in \mathbb{R}\}$$

which, of course, must be the line $L_{-1,0}$ in $\mathscr{A}(l)$. Thus

$$g_l(x) = \frac{ag_k(x)}{g_k(a)}$$

for all $x \in \mathbb{R}$. For $x = 1$ we find $a = g_k(a)$ and thus $g_l = g_k$. This shows that $k = l$.

In the second case we similarly obtain $g_l^{-1} = g_k$. But $g_l^{-1} = g_{1/l}$ so that $k = 1/l$. This clearly is not possible, because $k, l > 1$. □

Note that the transformation $\gamma : (x, y) \mapsto (y, x)$ is a homeomorphism of the torus that interchanges the horizontals with the verticals. The image $\gamma(\mathscr{M}(k))$ of $\mathscr{M}(k)$ is a again a flat Minkowski plane that, in addition, has a very similar description to our planes as $\mathscr{M}(1/k)$. Thus we could have extended the definition of our flat Minkowski planes as given in Section 2 to values for the parameter k between 0 and 1.

Since Σ is a group of automorphisms of the flat Minkowski plane $\mathscr{M}(k)$, each such plane has group dimension at least 3. In fact, all these planes have group dimension 3.

Theorem 5.4. *Each flat Minkowski plane $\mathscr{M}(k)$ has group dimension* 3. *The connected component Σ of the automorphism group of $\mathscr{M}(k)$ that contains the identity is isomorphic to the simple group* $\mathrm{PSL}_2(\mathbb{R})$. *Furthermore, $\mathscr{M}(k)$ also admits the automorphism*

$\alpha : (x, y) \mapsto (-x, -y)$. *The group generated by* α *and* Σ *is the full automorphism group of* $\mathcal{M}(k)$ *and is isomorphic to* $\mathrm{PGL}_2(\mathbb{R})$.

Proof. Let $\Gamma(k)$ be the full automorphism group of $\mathcal{M}(k)$. From Lemma 5.2 we know that every automorphism of $\mathcal{M}(k)$ must fix the circle D. Hence the orbit under $\Gamma(k)$ of any point on D is (at most) 1-dimensional. Choosing three distinct points on D the stabilizer of these points is 0-dimensional; see [4]. The dimension formula then implies that $\Gamma(k)$ is at most 3-dimensional. We thus conclude that a Minkowski plane $\mathcal{M}(k)$ has group dimension 3.

It is readily verified that α is indeed an automorphism of $\mathcal{M}(k)$. Let $\gamma \in \Gamma(k)$. Up to automorphisms in Σ, we may assume that γ fixes (∞, ∞), $(0, 0)$, D and the generating circle C_k. As in the proof of Theorem 5.3 we then see that γ must be of the form $(x, y) \mapsto (ax, ay)$ or $(x, y) \mapsto (ay, ax)$ where $a \in \mathbb{R}$ satisfies $a = g_k(a)$, that is, $a = \pm 1$. The former case gives us $\gamma = \mathrm{id}$ and $\gamma = \alpha$ for $a = 1$ and $a = -1$ respectively. However, the transformation $(x, y) \mapsto (y, x)$ does not define an automorphism of $\mathcal{M}(k)$, because the generating circle C_k is taken to $C_{1/k} = \{(x, -g_{1/k}(x)) \mid x \in \mathbb{S}^1\}$ and $C_{1/k} \neq C_k$ unless $k = 1$. This shows that $\Gamma(k)$ is generated by α and Σ and the remaining statements about $\Gamma(k)$ readily follow. □

Similar to the Lenz–Barlotti classification of projective planes with respect to central collineations, Minkowski planes have been classified by Klein and Kroll in [2] and [1] with respect to *central automorphisms*, that is, automorphisms that fix at least one point and induce central collineations in the projective extension of the derived affine plane at that fixed point; see [2] and [1] or [4] Section 4.5, for a definition of the so-called Klein–Kroll types.

Proposition 5.5. *Each flat Minkowski plane* $\mathcal{M}(k)$ *has Klein–Kroll type* IV.A.1.

Proof. The group Σ from Theorem 5.4 contains the translations $(x, y) \mapsto (x + t, y + t)$ for $t \in \mathbb{R}$. They form a $(p, B(p, D))$-transitive group of $(p, B(p, D))$-translations where $p = (\infty, \infty)$ and $B(p, D)$ is the tangent bundle of circles that touch the distinguished circle D at p. Since Σ is transitive on D, we see that $\mathcal{M}(k)$ is $(p, B(p, D))$-transitive for each point $p \in D$. Hence $\mathcal{M}(k)$ is of Klein–Kroll type at least IV. However, type V or higher implies classical; see [8], Corollary 4.2. But then $\mathcal{M}(k)$ must be of combined type IV.A.1 by [8], Theorem 6.1. □

There are flat Minkowski planes that admit the group $\mathrm{PSL}_2(\mathbb{R})$ as a group of automorphisms in one of the kernels, that is, the normal subgroups of all automorphisms that fix each $(+)$-parallel class or each $(-)$-parallel class. These planes are obtained from the classical flat Minkowski plane by replacing the circles in the negative half by the graphs of the composition of all fractional linear maps not in $\mathrm{PSL}_2(\mathbb{R})$ with a fixed orientation-preserving homeomorphism f of $\mathbb{S}^1$. The resulting plane $\mathcal{M}(f)$ has group dimension 3 or 4 or is classical, depending on the form of f; see [4], Theorem 4.3.3 or [7]. Clearly, such a plane $\mathcal{M}(f)$ cannot be isomorphic to any of our planes $\mathcal{M}(k)$.

There is however a looser connection between flat Minkowski planes via generalised quadrangles. From one half of a flat Minkowski plane one can construct an antiregular 3-dimensional compact generalised quadrangle that admits a Minkowski involution as the 'lifted Lie geoemtry', see [6] or [4] Chapter 6 for details. Vice versa such a generalised quadrangle gives rise to one half of a flat Minkowski plane by taking the set of fixed points S of the Minkowski involution τ as the point set, the fixed lines of τ as the parallel classes, and as circles the traces $S \cap p^{\perp}$ of points p not fixed by τ. By using different Minkowski involutions of the same antiregular 3-dimensional generalised quadrangle, one can establish a relationship between halves of different flat Minkowski planes. Following the notation in [6] we say halves of two flat Minkowski planes are sisters of each other if they can be obtained in this way.

Since the verification of the axioms of a Minkowski plane is straightforward for the planes $\mathscr{M}(f)$, the question arises whether or not the negative half $\mathscr{M}^-(k)$ of a flat Minkowski plane $\mathscr{M}(k)$ is a sister of the negative half of a plane $\mathscr{M}(f)$. Note that the positive half in both types of planes, $\mathscr{M}(f)$ and $\mathscr{M}(k)$, is the same as in the classical flat Minkowski plane. Furthermore, the negative half of $\mathscr{M}(f)$ is also isomorphic to the positive half of the classical flat Minkowski plane. Hence, in order for $\mathscr{M}^-(k)$ to be a sister of a half of some $\mathscr{M}(f)$ this half must be obtainable from the classical antiregular 3-dimensional compact generalised quadrangle. This implies that Desargues' configuration must close in the derived geometry of $\mathscr{M}^-(k)$ at each of its points where all occuring lines are horizontals, verticals or in the negative half of $\mathscr{M}(k)$. However, as seen in the proof of Lemma 3.3 this is not the case. Hence our planes $\mathscr{M}(k)$ are not related to the planes $\mathscr{M}(f)$ and there is no 'easy way' to verify the axioms of a Minkowski plane.

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Received 12 November, 2002; revised 29 January, 2003

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