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Symmetrization of starlike domains in Riemannian manifolds and a qualitative generalization of Bishop's volume comparison theorem

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Abstract. We introduce a new type of symmetrization in starlike domains in Riemannian manifolds that maintains the Ricci curvature in the radial direction. We prove that this symmetrization is volume increasing. We get, as its direct consequence, a generalization of Bishop's volume comparison theorem. Moreover, this generalization shows that this kind volume comparison theorem is qualitative in nature, instead of being quantitative. Using this symmetrization, we get some volume upper bounds in terms of some integrals of the Ricci curvature. Finally, we introduce a new type of symmetrization in geodesic balls within the injectivity radius, which is volume decreasing.

Key words. Volume comparison, Symmetrization, Ricci curvature.

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1 Introduction

Symmetrization is a very useful tool in mathematics. In particular, symmetrization of Riemannian manifolds is a powerful tool in geometrical analysis. Frequently when we consider a class of objects that shares some features, the maximum or the minimum of a given property is attained at the most symmetrical object in this class. In addition a symmetric object is usually simpler to study, what makes the symmetrization a very interesting tool to consider.

For instance let M_κ^n be the n-dimensional space form with constant sectional curvature $\kappa \in \mathbb{R}$. Let $\mathscr{M}_{\mathscr{V}}^n$ be the set of compact n-dimensional manifolds with smooth boundary in M_κ^n, with fixed volume $\mathscr{V}$. Consider the isoperimetric quotient

$$\Im_{n,\mathscr{V}}(\Omega) = \frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Vol}(\Omega)^{(n-1)/n}}$$

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in $\mathcal{M}^n_{\mathcal{V}}$, where Area$(\cdot)$ denotes the $(n-1)$-dimensional volume and Vol$(\cdot)$ denotes the n-dimensional volume. It is a well known fact that the minimum of $\Im_{n,\mathcal{V}}$ is attained at the geodesic disc with volume $\mathcal{V}$ (see [5] and references therein).

Another well known example is given by the Faber–Krahn inequality. It says that the lowest fundamental tone of a Riemannian manifold in $\mathcal{M}^n_{\mathcal{V}}$ is given by the geodesic disc with volume $\mathcal{V}$ (see also [5] and references therein).

There are several other examples that illustrate this kind of situation: The maximum or the minimum of "interesting" functionals is either attained at the most symmetrical object, or it is not attained (for instance, substitute "minimum" in the examples given above by "maximum").

Let us begin to present this work. Let M^n be an n-dimensional Riemannian manifold. Denote the tangent space at $p \in M$ by T_pM and its unit vectors by $\mathcal{S}_pM$. We can define a polar coordinate system with the origin at p in a neighborhood of p. Denote it by (t,θ), where $\theta \in \mathcal{S}_pM$ and t is the radial component. The canonical metric on $\mathcal{S}^{n-1}$ is denoted by $d\theta^2$ and its volume element by dA. The curve parametrized by arclength $\gamma(\cdot,\theta)$ represents the geodesic such that $\gamma(0,\theta) = p$ and $\gamma'(0,\theta) = \theta$, where the superscript $'$ stands for the derivative in the radial variable. Denote by $\tilde{c}(\theta)$ the cut point of p along $\gamma(\cdot,\theta)$.

We say that a domain $D \subset M^n$ is starlike with respect to $p \in D$ if given $x \in D$ then there exists a unique minimizing geodesic $\gamma : [0, c_x] \to M$ connecting p and x such that $\gamma([0,c_x]) \subset D$. It is not difficult to see that we can define a global polar coordinate system in starlike domains. Moreover, it can be defined as $\{\exp_p(t,\theta) \in D; \theta \in \mathcal{S}_pM; 0 \leqslant t < c(\theta)\}$, where $c(\theta) \leqslant \tilde{c}(\theta)$. In order to be more explicit, we denote a starlike domain by $D(p,c)$. Notice that geodesic balls, not necessarily within the injectivity radius, are starlike domains or the union of starlike domains with some of its closure points. We denote the geodesic ball with center p and radius r by $B(p,r)$.

Remark 1.1. Whenever we mention Bishop's volume comparison theorem, we are referring to the version where the Ricci curvature is bounded from below (see Theorem 2.1).

The main purpose of this paper is to introduce two new types of symmetrizations in starlike domains and to study their influence on the volume element in a polar coordinate system. One of these symmetrization is volume increasing and one of its consequences is a generalization of Bishop's volume comparison theorem. Furthermore this generalization shows that this kind of volume comparison theorem is qualitative in essence instead of being quantitative. We also get some volume upper bounds in terms of some integrals of the Ricci curvature thanks to this symmetrization. The other symmetrization that we introduce in this work is volume decreasing, and further details about it will be given afterwards.

This paper is divided as follows: In Section 2, we introduce some notation and basic facts. In Section 3, we summarize our work defining the symmetrizations and presenting the main theorems without proofs. In addition we justify their importance in Riemannian geometry. In Section 4, we complete all the details about symmetrizations and we prove the qualitative generalization of Bishop's volume comparison theorem.

Finally, in Section 5, we prove another volume comparison theorem and some volume upper bounds.

2 Notation and basic facts

Let us introduce some notation and present some basic facts. Calligraphic mathematical letters will indicate an object related to the tangent space.

Let (M^n, g) be an n-dimensional Riemannian manifold with metric g, ∇ the Levi–Civita connection, $R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X, Y]}$ the curvature tensor and $\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} \langle R(e_i, X)Y, e_i\rangle$ the Ricci tensor, where $\{e_1, \dots, e_n\}$ is an orthonormal frame and $\langle\cdot,\cdot\rangle$ denotes the metric g. If $|Z| = 1$, then $\mathrm{Ric}(Z, Z)$ is the Ricci curvature in the direction Z. An important object for the study of the geometry along geodesics in polar coordinates is the Ricci curvature in the radial direction $\frac{\partial}{\partial t}$. It will be called *radial Ricci curvature* and it will be denoted by $\mathrm{Ricr}(t, \theta)$.

The exponential map $\exp_p$ restricted to $\mathscr{D}(0, c) = \{X \in T_pM : \theta \in \mathscr{S}^{n-1}; 0 \leqslant \|X\| < c(\theta)\}$ induces spherical coordinates $(t, \xi_1, \xi_2, \dots, \xi_{n-1})$ on $D(p, c)$, where t is the radial coordinate and ξ_i, for $i = 1, \dots, n-1$, denotes the angular coordinates. Its relationship with Cartesian coordinates $(x_1, \dots, x_n)$ is given by

$$\begin{aligned}
x_1 &= r\sin\xi_1,\\
x_2 &= r\cos\xi_1\sin\xi_2,\\
x_3 &= r\cos\xi_1\cos\xi_2\sin\xi_3,\\
&\ \vdots\\
x_{n-1} &= r\cos\xi_1\dots\cos\xi_{n-2}\sin\xi_{n-1},\\
x_n &= r\cos\xi_1\dots\cos\xi_{n-2}\cos\xi_{n-1}.
\end{aligned}$$

The spherical coordinates will be useful in some calculations.

Let $D(p, c)$ be a starlike domain. For each $t \in [0, c(\theta))$, let $T^{\perp}_{\gamma(t,\theta)}M$ be the orthogonal complement of $\gamma'(t, \theta)$ in the tangent space $T_{\gamma(t,\theta)}M$. Define the *radial curvature operator* $\mathbf{R}(t, \theta) : T^{\perp}_{\gamma(t,\theta)}M \to T^{\perp}_{\gamma(t,\theta)}M$ by $\mathbf{R}(t, \theta)X := R(X, \gamma'(t, \theta))\gamma'(t, \theta)$. Observe that $\mathrm{Ricr}(t, \theta)$ is the trace of $\mathbf{R}(t, \theta)$.

Fix θ. Define the path of linear operators $\mathscr{A}(t, \theta) : T^{\perp}_{\gamma(0,\theta)}M \to T^{\perp}_{\gamma(0,\theta)}M$ by $\mathscr{A}(t, \theta)X = (\tau_t)^{-1}J(t)$, where $J(t)$ is the Jacobi field along $\gamma(\cdot, \theta)$ satisfying $J(0) = 0$ and $\nabla_{\gamma'(0,\theta)}J(0) = X$, and τ_t is the parallel transport from $T^{\perp}_{\gamma(0,\theta)}M$ to $T^{\perp}_{\gamma(t,\theta)}M$ along $\gamma(\cdot, \theta)$. Define also the path of linear operators $\mathscr{R}(t, \theta) : T^{\perp}_{\gamma(0,\theta)}M \to T^{\perp}_{\gamma(0,\theta)}M$ by $\mathscr{R}(t, \theta)X = (\tau_t)^{-1}\mathbf{R}(t, \theta)(\tau_t X)$. It is well known that $\mathscr{A}(t, \theta)$ is the solution of the equation $\mathscr{A}''(t, \theta) + \mathscr{R}(t, \theta)\mathscr{A}(t, \theta) = 0$ with initial conditions $\mathscr{A}(0, \theta) = 0$ and $\mathscr{A}'(0, \theta) = I$, where I denotes the identity operator. Moreover, the volume element in polar coordinate system is given by $\sqrt{\det\mathbf{g}(t, \theta)}.dt.dA = \det\mathscr{A}(t, \theta).dt.dA$.

Now we recall Bishop's celebrated volume comparison theorem (see [3] and compare with [5]). Denote by $\mathbf{S}_\kappa$ the function

$$\mathbf{S}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} t) & \kappa > 0 \\ t & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} t) & \kappa < 0. \end{cases}$$

Theorem 2.1 (Bishop). *Let M be a Riemannian manifold and fix $p \in M$. Consider a geodesic $\gamma(\cdot,\theta)$ parametrized by arclength in M such that $\gamma(0,\theta) = p$. Suppose that the radial Ricci curvature along $\gamma(\cdot,\theta)$ is greater than or equal to $(n-1)\kappa$ for every $t \in (0, c(\theta)]$. Then*

$$\left(\frac{\det \mathscr{A}(t,\theta)}{\mathbf{S}_\kappa^{n-1}(t)}\right)' \leqslant 0 \tag{1}$$

on $(0, c(\theta))$ and

$$\det \mathscr{A}(t,\theta) \leqslant \mathbf{S}_\kappa^{n-1}(t)$$

on $[0, c(\theta)]$. We have equality in (1) at $t = t_0 \in (0, c(\theta))$ if and only if

$$\mathscr{A}(t,\theta) = \mathbf{S}_\kappa(t) I, \quad \mathscr{R}(t,\theta) = \kappa I$$

for every $t \in (0, t_0]$.

Denote the geodesic ball of radius r in the space form of constant curvature κ by $B_\kappa(r)$. A global version of Theorem 2.1 is given below.

Theorem 2.2 (Bishop). *Let M be a Riemannian manifold and fix $p \in M$. Suppose that the radial Ricci curvature is greater than or equal to $(n-1)\kappa$ in $B(p,r)$. Then*

$$\mathrm{Vol}(B(p,r)) \leqslant \mathrm{Vol}(B_\kappa(r)) \tag{2}$$

with equality if and only if $B(p,r)$ is isometric to $B_\kappa(r)$.

We end this section remarking that we will usually simplify the notation if there is no possibility of misunderstandings (for example, $\mathscr{A}(t)$ instead of $\mathscr{A}(t,\theta)$).

3 Symmetrizations and the main theorems: a summary

In order to define the symmetrizations, we present some types of starlike domains.

Definition 3.1. We have the following types of metrics on starlike domains, from the more general to the more specific (All metrics are written in a polar coordinate system).

1. *General starlike domains*: Metrics of type $ds_g^2 = dt^2 + g_{ij}(t,\theta)\, d\theta^i\, d\theta^j$.

2. *Starlike domains with scalar radial curvature operator*: Metrics of type $ds_f^2 = dt^2 + f^2(t,\theta)\,d\theta^2$.

3. *Radially symmetric starlike domains*: Metrics of type $ds_h^2 = dt^2 + h^2(t)\,d\theta^2$.

4. *Starlike domains with constant curvature* κ: Metrics of type $ds_\kappa^2 = dt^2 + \mathbf{S}_\kappa^2(t)\,d\theta^2$.

For the sake of brevity, we call them starlike domains of type i $(i = 1,2,3,4)$. The definition of radially symmetric starlike domain may seem a little bit artificial. In fact, it is more suitable to geodesic balls within the injectivity radius. But we conserve Definition 3.1 as it is, in order to emphasize the hierarchy that exists on the metrics in starlike domains.

The reason why the second type of starlike domains is called *starlike domains with scalar radial curvature operator* will be explained in Proposition 4.1.

It is a natural idea to create a symmetrization process such that the first step is to transform a general starlike domain into a starlike domain with scalar radial curvature operator, the second step is to transform a starlike domain with scalar radial curvature operator into a radially symmetric starlike domain and so on. In order to define these symmetrizations, we have to determine what properties we want to keep.

Let us define these symmetrizations. One of them transforms a starlike domain of type 1 into a starlike domain of type 2, and it is called *symmetrization* (*of a starlike domain*) *along the radial geodesics*. For the sake of brevity, we call it 1-2 *symmetrization*. It is characterized by transforming a general starlike domain $D_g(p,c)$ into a starlike domain with scalar radial curvature operator $D_f(p,c)$ with the same radial Ricci curvature. The other symmetrization transforms a geodesic ball $B_f(p,r)$ of type 2 within the injectivity radius into a geodesic ball $B_h(p,r_h)$ of type 3, with $r_h \leqslant r$. It is called *symmetrization* (*of a geodesic ball of type* 2 *within the injectivity radius*) *along spheres that are equidistant to the origin*, and for the sake of brevity we call it 2-3 *symmetrization*. The 2-3 symmetrization is characterized by the property that the average of the radial Ricci curvature on $\partial B_f(p,t)$, $t \in (0,r_h)$, is equal to the correspondent average on $\partial B_h(p,t)$. We could create a "3-4 symmetrization", but these two symmetrizations are enough for our purposes. The formalization and all details about these symmetrizations will be done in Section 4.

A remark must be made: The metric of these symmetrizations can loose its smoothness at $(t = 0)$ and become only continuous at this point. But this loss of regularity will not harm the volume calculations.

The following theorem generalizes Bishop's volume comparison theorem.

Theorem 3.2. *Let* (M^n, g) *be a Riemannian manifold,* $D_g(p,c) \subset M$ *a starlike domain, and let* $D_f(p,c)$ *be the* 1-2 *symmetrization of* $D_g(p,c)$. *Fix* θ. *If* $\sqrt{\det \mathbf{g}(t,\theta)}.dt.dA$ *and* $\sqrt{\det \mathbf{f}(t,\theta)}.dt.dA$ *are respectively the volume element of* $D_g(p,c)$ *and* $D_f(p,c)$ *in a polar coordinate system, then*

$$\left(\frac{\sqrt{\det \mathbf{g}(t,\theta)}}{\sqrt{\det \mathbf{f}(t,\theta)}}\right)' \leqslant 0 \tag{3}$$

on $(0, c(\theta))$ *and*

$$\sqrt{\det \mathbf{g}(t,\theta)}.dt.dA \leqslant \sqrt{\det \mathbf{f}(t,\theta)}.dt.dA. \tag{4}$$

on $[0, c(\theta))$. *In particular,* $\mathrm{Vol}(D_g(p,c)) \leqslant \mathrm{Vol}(D_f(p,c))$.

Equality is achieved in (3) (*as well as in* (4)) *at* $t_0 \in (0, c(\theta))$, *if and only if*

$$\mathscr{R}_g(t,\theta) = \mathscr{R}_f(t,\theta)$$

for every $t \in [0, t_0]$, *where* $\mathscr{R}_g$ *and* $\mathscr{R}_f$ *are identified in the natural way.*

The radial curvature operator has a leading rule to determine the volume element behavior through the equation

$$\begin{cases} \mathscr{A}''(t,\theta) + \mathscr{R}(t,\theta).\mathscr{A}(t,\theta) = 0 \\ \mathscr{A}(0,\theta) = 0 \\ \mathscr{A}'(0,\theta) = I. \end{cases} \tag{5}$$

The difference between Theorem 2.1 and Theorem 3.2 is that in the former, the solution of (5) is compared to the solution of the equation

$$\begin{cases} \mathscr{A}''(t,\theta) + \left[\frac{\inf_{s\in[0,c(\theta))} \mathrm{Ricr}(s,\theta)}{(n-1)}.I\right].\mathscr{A}(t,\theta) = 0 \\ \mathscr{A}(0,\theta) = 0 \\ \mathscr{A}'(0,\theta) = I, \end{cases} \tag{6}$$

and in Theorem 3.2, the solution of (5) is compared to the solution of the equation

$$\begin{cases} \mathscr{A}''(t,\theta) + \left[\frac{\mathrm{Ricr}(t,\theta)}{(n-1)}.I\right].\mathscr{A}(t,\theta) = 0 \\ \mathscr{A}(0,\theta) = 0 \\ \mathscr{A}'(0,\theta) = I. \end{cases} \tag{7}$$

Therefore Theorem 3.2 is a qualitative generalization of Theorem 2.1.

The symmetrization along radial geodesics allow us to get some upper bounds for the volume element in a polar coordinate system. These estimates are given in terms of some integrals of $\mathrm{Ricr}(t,\theta)$. This is possible for three reasons: First of all, the symmetrization along radial geodesics is volume increasing. Secondly, we do not loose any information about the radial Ricci curvature along $\gamma(\cdot,\theta)$, as it happens in Equation (6). Finally, Equation (7) is simple enough to get the desired upper bounds.

Let us be more explicit. Consider $D(p,c)$ a starlike domain. Fix θ. Set $\mathrm{Ricr}^-(t,\theta) := \max(0, -\mathrm{Ricr}(t,\theta))$. Bishop's volume comparison theorem implies that

$$\sqrt{\det \mathbf{g}(s,\theta)} \leqslant \left(\frac{\sinh(\sqrt{\kappa}.s)}{\sqrt{\kappa}}\right)^{n-1} \tag{8}$$

where $\kappa = \sup_{t\in[0,c(\theta))}(\mathrm{Ricr}^-(t,\theta))$ (If $\kappa = 0$, then the inequality above has the obvious meaning). Therefore we can estimate the volume element in polar coordinate system in terms of a L^∞ norm of $\sqrt{(\mathrm{Ricr}^-)}$ along $\gamma(\cdot,\theta)$. In Theorem 3.3, we use Theorem 3.2 in order to get an $(L^2)^2$ version of (8).

Theorem 3.3. *Let (M^n, g) be a Riemannian manifold, $D_g(p,c) \subset M$ a starlike domain, and $\sqrt{\det \mathbf{g}(t,\theta)}.dt.dA$ its volume element in a polar coordinate system. Fix $\theta \in \mathscr{S}^{n-1}$ and define $\|\mathrm{Ricr}^-(s,\theta)\|_{L^1} := \int_0^s \mathrm{Ricr}^-(t,\theta)\,dt$. Then there exist constants A_1, A_2, B_1 and B_2 such that the following (equivalent) estimates hold*:

$$\sqrt{\det \mathbf{g}(s,\theta)} \leqslant \left[A_1.s.\frac{\sinh\left(\frac{B_1.\|\mathrm{Ricr}^-(s,\theta)\|_{L^1}.s}{(n-1)}\right)}{\left(\frac{B_1.\|\mathrm{Ricr}^-(s,\theta)\|_{L^1}.s}{(n-1)}\right)}\right]^{n-1} \tag{9}$$

and

$$\sqrt{\det \mathbf{g}(s,\theta)} \leqslant A_2^{n-1} s^{n-1} e^{\|\mathrm{Ricr}^-(s,\theta)\|_{L^1}.B_2.s}. \tag{10}$$

If $\|\mathrm{Ricr}^-(s,\theta)\|_{L^1} = 0$, then (9) has the obvious meaning.

We have the following estimate for geodesic balls as a consequence of Theorem 3.3. The geodesic ball is not necessarily within the injectivity radius.

Corollary 3.4. *Let M^n be a complete Riemannian manifold and $B(p,r) \subset M$ a geodesic ball. Then there exist constants $A_3, B_3 > 0$ such that*

$$\mathrm{Vol}[B(p,r)] \leqslant A_3^{n-1} r^{n-1} \int_{\mathscr{S}^{n-1}} \int_0^r e^{B_3.r^2.\mathrm{Ricr}^-(t,\theta)}\,dt.dA. \tag{11}$$

Finally we have the volume comparison theorem that is related to the 2-3 symmetrization.

Theorem 3.5. *Let $(B_f(r), dt^2 + f^2(t,\theta)\,d\theta^2) \subset M$ be a geodesic ball within the injectivity radius. If $B_h(r_h)$ is the* 2-3 *symmetrization of $B_f(r)$, then we have that $\mathrm{Vol}[\partial B_h(t)] \leqslant \mathrm{Vol}[\partial B_f(t)]$ for every $t \in [0, r_h)$. In particular, $\mathrm{Vol}[B_h(r_h)] \leqslant \mathrm{Vol}[B_f(r_h)]$. If $n = 2$, then we have that $r_h = r$, $\mathrm{Vol}[\partial B_h(t)] = \mathrm{Vol}[\partial B_f(t)]$ for every $t \in [0, r)$, and $\mathrm{Vol}[B_h(r)] = \mathrm{Vol}[B_f(r)]$. This theorem is still valid if $B_f(r)$ is the* 1-2 *symmetrization of another geodesic ball.*

Let us make some remarks about these results. Theorems 3.2 and 3.3 deal with the geometry along geodesics that emanate from some point. We control the curvature in order to get upper bounds for the volume element in polar coordinates. Notice that the radial curvature operator and the radial Ricci curvature are very important to this kind of theory.

Myers' classical theorem says that if $\mathrm{Ric}(\cdot) \geqslant (n-1)$ along a geodesic γ with arc-length greater than π, then γ is not minimizing (See [10]). In particular, if the Riemannian manifold M is complete, then M is compact and its diameter is less than or equal to π. Afterwards Ambrose, Avez, Calabi, Galloway and Markvorsen among others generalized Myers' theorem imposing weaker conditions on the Ricci curvature (See [1], [2], [4], [8], [9]). These works show that if we have some "positiveness" on the Ricci curvature along a geodesic, then the solution of Equation (5) becomes singular after some time, and the geodesic is no longer minimizing after that. Theorem 3.3 is similar to these works because we can get a upper bound for the volume element in terms of some lower bounds of the Ricci curvature. The difference is that the upper bound for the volume element does not go to zero as in these former works.

In order to generalize Myers' theorem, Ambrose, Calabi and Markvorsen (see [1], [4], [9]) use conditions on the integral of the Ricci curvature along geodesics. Notice that Theorem 3.3 is an integral version of Equation (8), although it is not a full generalization. We will make some comments about the lack of sharpness of Theorem 3.3 in Remark 5.4.

Controlling several geometric properties of Riemannian manifolds through some integrals of the curvature has been an important issue nowadays. For instance, many works have used some local-global integral of the lowest eigenvalue of the Ricci tensor to study geometrical and topological properties of Riemannian manifolds (See [6], [7], [11], [12], [13] among others). In particular, we can get bounds for the diameter and a generalization of Bishop's volume comparison theorem using these integral invariants (See [11], [12], [13]).

4 Symmetrization of starlike domains and a generalization of Bishop's volume comparison theorem

In this section, we describe the symmetrization process on starlike domains. In Subsection 4.1, we define the 1-2 symmetrization and we get a qualitative generalization of Bishop's volume comparison theorem. In Subsection 4.2, we define the 2-3 symmetrization.

4.1 Symmetrization along radial geodesics and a qualitative generalization of Bishop's volume comparison theorem. We begin justifying the name *starlike domain with scalar radial curvature operator*.

Proposition 4.1. *Let* $(D_f(p,c), dt^2 + f^2(t,\theta).d\theta^2)$ *be a starlike domain with radial scalar curvature operator. Then, for every* $(t,\theta) \in D_f(p,c) - \{p\}$*, its radial curvature operator is given by*

$$\mathbf{R}(t,\theta) = -\frac{f''(t,\theta)}{f(t,\theta)} I, \tag{12}$$

where I denotes the identity operator.

Proof. Fix $(\bar{t},\bar{\theta}) \in (D_f(p,c) - \{p\})$. In spherical coordinates $(r, \xi_1, \dots, \xi_{n-1})$, we have that

$$ds_f^2 = dt^2 + f^2(t,\theta).d\xi_1^2 + f^2(t,\theta).\cos^2\xi_1.d\xi_2^2$$
$$+ f^2(t,\theta).\cos^2\xi_1.\cos^2\xi_2.d\xi_3^2 + \cdots + f^2(t,\theta).\cos^2\xi_1\ldots\cos^2\xi_{n-2}.d\xi_{n-1}^2.$$

Observe that we can always choose spherical coordinates without singularities at $(\bar{t},\bar{\theta})$. Now we can calculate the Christoffel symbols and the components of the curvature operator at $(\bar{t},\bar{\theta})$ explicitly, and the result follows. □

Now we describe the 1-2 symmetrization. Denote the space of symmetric $(n-1)\times(n-1)$ matrices over $\mathbb{R}$ by M_{n-1} and the starshaped Euclidean domain with respect to the origin $\{(t,\theta)\in\mathbb{E}^n; \theta\in\mathcal{S}^{n-1}; 0\leqslant t<c(\theta)\}$ by $\mathcal{D}(0,c)$.

Theorem 4.2. *Let $D_g(p,c)$ be a general starlike domain. For each θ fixed, let $f(t,\theta)$ be the solution of the equation $f''(t,\theta)+\frac{\mathrm{Ricr}(t,\theta)}{(n-1)}f(t,\theta)=0$ satisfying the initial conditions $f(0,\theta)=0$ and $f'(0,\theta)=1$. Consider the punctured starshaped Euclidean domain $\mathcal{D}(0,c)-\{0\}$ endowed with the symmetric 2-form*

$$ds_f^2 = dt^2 + f^2(t,\theta)\,d\theta^2, \tag{13}$$

and denote it by $D_f(p,c)-\{t=0\}$. Then ds_f^2 is a smooth metric in $[D_f(p,c)-\{t=0\}]$ extendable to a continuous metric at $(t=0)\in D_f(p,c)$. Moreover, $D_f(p,c)$ has the same radial Ricci curvature as $D_g(p,c)$.

Proof. We will divide the proof in three parts:

First part: The 2-form ds_f^2 is a positive definite symmetric 2-form on $\mathcal{D}(0,c)-\{0\}$. It is a consequence of Lemma 4.3 below. Notice that its proof is similar to the proof of Bishop's volume comparison theorem (compare [5]).

Lemma 4.3. *Let $\mathcal{R}:[0,r)\to M_{n-1}$ be a continuous map and $\mathcal{A}:[0,r)\to M_{n-1}$ the solution of the matricial ordinary differential equation $\mathcal{A}''(t)+\mathcal{R}(t).\mathcal{A}(t)=0$ with initial conditions $\mathcal{A}(0)=0$ and $\mathcal{A}'(0)=I$. Consider the symmetrization of this problem, that is, $\mathcal{A}_f''(t)+\mathcal{R}_f(t).\mathcal{A}_f(t)=0$ with initial conditions $\mathcal{A}_f(0)=0$ and $\mathcal{A}_f'(0)=I$, where $\mathcal{R}_f(t):=\mathrm{trace}[\mathcal{R}(t)/(n-1)].I$. If $\det\mathcal{A}(t)$ is non-singular for every $t\in(0,r)$, then*

$$\left(\frac{\det\mathcal{A}(t)}{\det\mathcal{A}_f(t)}\right)'\leqslant 0 \tag{14}$$

on $(0,r)$ and

$$\det\mathcal{A}(t)\leqslant\det\mathcal{A}_f(t). \tag{15}$$

on $[0,r)$. In particular, $\det\mathcal{A}_f(t)$ is also non-singular on $(0,r)$.

Equality is achieved in (14) (as well as in (15)) at $t_0\in(0,r)$ if and only if $\mathcal{R}=\mathcal{R}_f$ on $[0,t_0]$.

Proof of Lemma 4.3. Consider $T \in M_{n-1}$. The Cauchy–Schwarz inequality implies that

$$\operatorname{trace}(T^2) \geqslant \frac{(\operatorname{trace}(T))^2}{n-1} \tag{16}$$

with equality if and only if T is a scalar multiple of the identity.

Define $U := \mathscr{A}'\mathscr{A}^{-1}$. Then U is self-adjoint and satisfies the matricial Riccati equation

$$U' + U^2 + \mathscr{R} = 0. \tag{17}$$

By (16) we have that

$$0 = (\operatorname{trace}(U))' + \operatorname{trace}(U^2) + \operatorname{trace}\mathscr{R} \geqslant (\operatorname{trace}(U))' + \frac{(\operatorname{trace}(U))^2}{n-1} + \operatorname{trace}(\mathscr{R}_f)$$

and $\phi := \operatorname{trace}(U) = (\det\mathscr{A})'/\det\mathscr{A}$ satisfies the differential inequality

$$\phi' + \frac{\phi^2}{n-1} + \operatorname{trace}(\mathscr{R}_f) \leqslant 0. \tag{18}$$

Let us study $\mathscr{A}_f$. If φ is the solution of

$$\varphi''(t) + \frac{\operatorname{trace}(\mathscr{R}_f(t))}{n-1}\varphi(t) = 0 \tag{19}$$

with initial conditions $\varphi(0) = 0$ and $\varphi'(0) = 1$, then $\varphi^{n-1}(t) = \det\mathscr{A}_f(t)$. Set

$$\psi := \frac{(\det\mathscr{A}_f)'}{(\det\mathscr{A}_f)} = (n-1)\frac{\varphi'}{\varphi}. \tag{20}$$

Using (19) and (20) we have that

$$\psi' + \frac{\psi^2}{n-1} + \operatorname{trace}(\mathscr{R}_f) = 0. \tag{21}$$

Let $(0,\beta) \subset (0,r)$ be the maximal interval such that $\det \mathrm{A}_f(t) > 0$ for every $t \in (0,\beta)$. We will compare ϕ and ψ in $(0,\beta)$. They satisfy (18) and (21) respectively and $\lim_{t\to 0^+}\phi(t) = \lim_{t\to 0^+}\psi(t) = +\infty$. In order to compare them near $t = 0$ (let us say, in the interval $(0,\varepsilon]$), consider its inverses $\tilde{\phi} = 1/\phi$ and $\tilde{\psi} = 1/\psi$. They satisfy respectively

$$\tilde{\phi}'(t) \geqslant \frac{1}{n-1} + \operatorname{trace}(\mathscr{R}_f(t)).\tilde{\phi}^2(t) \tag{22}$$

and

$$\tilde{\psi}'(t) = \frac{1}{n-1} + \operatorname{trace}(\mathscr{R}_f(t)).\tilde{\psi}^2(t) \tag{23}$$

with $\phi(0) = \psi(0) = 0$. Subtracting (22) from (23) we have that:

$$(\tilde{\psi}(t) - \tilde{\phi}(t))' \leqslant \operatorname{trace}(\mathscr{R}_f(t))(\tilde{\psi}(t) + \tilde{\phi}(t))(\tilde{\psi}(t) - \tilde{\phi}(t)).$$

Set $a_1(t) := (\tilde{\psi}(t) - \tilde{\phi}(t))$ and $b_1(t) := \operatorname{trace}(\mathscr{R}_f(t))(\tilde{\psi}(t) + \tilde{\phi}(t))$. Then

$$\frac{d}{dt}(a_1(t)e^{-\int_0^t b_1(\zeta)\,d\zeta}) = e^{-\int_0^t b_1(\zeta)\,d\zeta}(a_1'(t) - b_1(t)a_1(t)) \leqslant 0$$

and it follows that

$$a_1(t)e^{-\int_0^t b_1(\zeta)\,d\zeta} \leqslant 0 \Rightarrow a_1(t) \leqslant 0$$

with equality if and only if $\mathscr{R}(\zeta) = \mathscr{R}_f(\zeta)$ for every $\zeta \in [0, t]$. Therefore $\phi(t) \leqslant \psi(t)$ in $(0, \varepsilon]$, with $\phi(t_0) = \psi(t_0)$ if and only if $\mathscr{R}(t) = \mathscr{R}_f(t)$ for every $t \in [0, t_0]$.

The comparison between ϕ and ψ in $[\varepsilon, \beta)$ follows in a similar fashion. Subtracting (21) from (18), we have that

$$(\phi - \psi)' \leqslant -\frac{(\psi + \phi)}{n-1}(\phi - \psi).$$

Set $a_2(t) := (\phi(t) - \psi(t))$ and $b_2(t) := -(\psi(t) + \phi(t))/(n-1)$. Then

$$\frac{d}{dt}(a_2(t).e^{-\int_\varepsilon^t b_2(\zeta)\,d\zeta}) = e^{-\int_\varepsilon^t b_2(\zeta)\,d\zeta}(a_2'(t) - a_2(t)b_2(t)) \leqslant 0$$

hence

$$a_2(t) \leqslant a_2(\varepsilon)e^{\int_\varepsilon^t b_2(\zeta)\,d\zeta}$$

with equality if and only if $\mathscr{R}(\zeta) = \mathscr{R}_f(\zeta)$ for every $\zeta \in [\varepsilon, t]$. Therefore $\phi(t) \leqslant \psi(t) + a_2(\varepsilon)e^{\int_\varepsilon^t b_2(\zeta)\,d\zeta}$ in $[\varepsilon, r)$, with $\phi(t_0) = \psi(t_0)$ if and only if $\phi(\varepsilon) = \psi(\varepsilon)$, and $\mathscr{R}(t) = \mathscr{R}_f(t)$ for every $t \in [0, t_0]$.

Joining the estimates in $(0, \varepsilon]$ and $[\varepsilon, \beta)$, we have that $\phi(t) \leqslant \psi(t)$ in $(0, \beta)$, with $\phi(t_0) = \psi(t_0)$ if and only if $\mathscr{R}(t) = \mathscr{R}_f(t)$ for every $t \in [0, t_0]$.

Thus

$$\frac{\det(\mathscr{A}_f(t))}{\det(\mathscr{A}(t))}\left(\frac{\det \mathscr{A}(t)}{\det \mathscr{A}_f(t)}\right)' = \frac{(\det \mathscr{A}(t))'}{\det \mathscr{A}(t)} - \frac{(\det \mathscr{A}_f(t))'}{\det \mathscr{A}_f(t)} \leqslant 0$$

what implies that

$$\left(\frac{\det \mathscr{A}(t)}{\det \mathscr{A}_f(t)}\right)' \leqslant 0.$$

Therefore $\det \mathscr{A}(t) \leqslant \det \mathscr{A}_f(t)$ in $(0, \beta)$ due to

$$\lim_{t\to 0+} \frac{\det \mathscr{A}(t)}{\det \mathscr{A}_f(t)} = 1,$$

and consequently we have that $\beta = r$, which settles Lemma 4.3. □

Equation (15) implies that (13) is positive definite in $\mathscr{D}(0,c) - \{0\}$, which settles the first part of the proof of Theorem 4.2.

Second part: The 2-form ds_f^2 is a smooth metric defined in $D_f(p,c) - \{t = 0\}$, continuously extendable at $(t = 0) \in D_f(p,c)$. The smoothness of $ds_f^2 = dt^2 + f^2(t,\theta).d\theta^2$ in $D_f(p,c) - \{t = 0\}$ follows from a classical result on ordinary differential equations (smooth dependence of f in terms of the parameter θ).

We prove now that ds_f^2 can be continuously extended to $(t = 0)$. Let X and Y be two continuous vector fields in the ball $\mathscr{B}(0,r) \subset \mathscr{D}(0,c)$, where $r > 0$ is a sufficiently small positive number. Consider the Euclidean metric $ds_E^2 = dt^2 + t^2.d\theta^2$ in $\mathscr{B}(0,r)$. Decomposing X and Y in its radial and angular components in $\mathscr{B}(0,r) - \{t = 0\}$, we have that

$$X = X_t + X_\theta \quad \text{and} \quad Y = Y_t + Y_\theta. \tag{24}$$

The Euclidean scalar product $\langle X, Y\rangle_E$ can be written as

$$\langle X, Y\rangle_E = \langle X_t, Y_t\rangle_E + \langle X_\theta, Y_\theta\rangle_E. \tag{25}$$

Now consider the 2-form $ds_f^2 = dt^2 + f^2(t,\theta)\,d\theta^2$ in $\mathscr{B}(0,r) - \{t = 0\}$. Denote ds_f^2 by $\langle \cdot, \cdot\rangle_f$. We will prove that $\langle X, Y\rangle_f$ is continuously extendable at $(t = 0)$, with $\lim_{t\to 0}\langle X(t,\theta), Y(t,\theta)\rangle_f = \langle X(0), Y(0)\rangle_E$.

The decomposition of X and Y in its radial and angular part with respect to ds_f^2 is also given by $X = X_t + X_\theta$ and $Y = Y_t + Y_\theta$. Indeed the radial-angular decomposition of the tangent spaces coincide for ds_f^2 and ds_E^2. Thus the scalar product of X and Y in ds_f^2 is given by

$$\langle X, Y\rangle_f = \langle X_t, Y_t\rangle_f + \langle X_\theta, Y_\theta\rangle_f. \tag{26}$$

Now observe that

$$\langle X_t, X_t\rangle_E = \langle X_t, X_t\rangle_f$$

and

$$\langle X_\theta, X_\theta \rangle_f = \frac{f^2(t,\theta)}{t^2} \langle X_\theta, X_\theta \rangle_E.$$

Hence

$$\langle X, Y \rangle_f = \langle X_t, Y_t \rangle_E + \frac{f^2(t,\theta)}{t^2} \langle X_\theta, Y_\theta \rangle_E. \tag{27}$$

For fixed θ, the Taylor series of $f(t,\theta)$ is given by $f(t,\theta) = t + f''(\bar{t}_\theta, \theta)t^2/2$ for some $\bar{t}_\theta \in (0,t)$. We know that $f''(t,\theta) = -\mathrm{Ricr}(t,\theta)f(t,\theta)/(n-1)$ is bounded in $\mathscr{B}(0,r) - \{t = 0\}$. Hence $\lim_{t\to 0} f^2(t,\theta)/t^2 = 1$ uniformly with respect to θ, what implies that $\lim_{t\to 0}\langle X(t,\theta), Y(t,\theta)\rangle_f = \langle X(0), Y(0)\rangle_E$. Therefore ds_f^2 can be extended continuously to $(t = 0)$.

Third part: The starlike domain $D_f(p,c)$ has the same radial Ricci curvature as $D_g(p,c)$. It follows from the definition of f and Proposition 4.1. This settles Theorem 4.2. □

Therefore the symmetrization along radial geodesics is a well defined operation. The only problem, as observed in the introduction, is that the metric is not necessarily smooth at the origin, but this is a minor problem because it does not harm the volume calculations.

Definition 4.4. The starlike domain $(D_f(p,c), dt^2 + f^2(t,\theta)\, d\theta^2)$ constructed in Theorem 4.2 is called the 1-2 symmetrization of $D_g(p,c)$.

We have the following qualitative generalization of Bishop's volume comparison theorem as a direct consequence of Proposition 4.1, Theorem 4.2 and Lemma 4.3.

Theorem 4.5. *Let (M^n, g) be a Riemannian manifold, $D_g(p,c) \subset M$ a starlike domain, and let $D_f(p,c)$ be the* 1-2 *symmetrization of $D_g(p,c)$. Fix θ. If $\sqrt{\det \mathbf{g}(t,\theta)}.dt.dA$ and $\sqrt{\det \mathbf{f}(t,\theta)}.dt.dA$ are respectively the volume element of $D_g(p,c)$ and $D_f(p,c)$ in a polar coordinate system, then*

$$\left(\frac{\sqrt{\det \mathbf{g}(t,\theta)}}{\sqrt{\det \mathbf{f}(t,\theta)}}\right)' \leqslant 0 \tag{28}$$

on $(0, c(\theta))$ and

$$\sqrt{\det \mathbf{g}(t,\theta)}.dt.dA \leqslant \sqrt{\det \mathbf{f}(t,\theta)}.dt.dA. \tag{29}$$

on $[0, c(\theta))$.

In particular, $\mathrm{Vol}(D_g(p,c)) \leqslant \mathrm{Vol}(D_f(p,c))$.
Equality is achieved in (28) (*as well as in* (29)) *at* $t_0 \in (0, r)$, *if and only if*

$$\mathcal{R}_g(t,\theta) = \mathcal{R}_f(t,\theta)$$

for every $t \in [0, t_0]$, *where* $\mathcal{R}_g$ *and* $\mathcal{R}_f$ *are identified in the natural way.*

Remark 4.6. A geodesic ball $B(p,r) \subset M$ is a starlike domain $D_g(p,c)$ with some of its closure points included. Therefore we can generalize Theorem 2.2 replacing (2) by

$$\mathrm{Vol}(B(p,r)) \leqslant \mathrm{Vol}(D_f(p,c)),$$

where $D_f(p,c)$ is the 1-2 symmetrization of $D_g(p,c)$.

4.2 Symmetrization along spheres that are equidistant to the origin. Let $(B_f(r), dt^2 + f^2(r,\theta)\,d\theta^2)$ be a geodesic ball within the injectivity radius. Define $\overline{\mathrm{Ricr}}_f(t)$ as the average of the radial Ricci curvature in $\partial B_f(t)$. Observe that $\overline{\mathrm{Ricr}}_f(t)$ can be extended continuously to $(t = 0)$ as the scalar curvature of $B_f(r)$ at the origin, even if $B_f(r)$ is a 1-2 symmetrization of another geodesic ball. We will create a radially symmetric geodesic ball $(B_h(r_h), dt^2 + h^2(t)\,d\theta^2)$, $r_h \leqslant r$, such that the average of the radial Ricci curvature on $\partial B_h(t)$ is equal to $\overline{\mathrm{Ricr}}_f(t)$ for every $t \in [0, r_h)$.

Using Proposition 4.1, the radial Ricci curvature of a radially symmetric geodesic ball with metric $ds_h^2 = dt^2 + h^2(t)\,d\theta^2$ is given by

$$\mathrm{Ricr}_h(t) = -(n-1)\frac{h''(t)}{h(t)},$$

which does not depend on θ. Thus the expression above is also the average of the radial Ricci curvature on $\partial B_h(t)$. Therefore h must satisfy

$$h''(t) + \frac{\overline{\mathrm{Ricr}}_f(t)}{n-1}h(t) = 0; \quad h(0) = 0; \quad h'(0) = 1. \tag{30}$$

The existence and uniqueness of h is assured by the theory of ordinary differential equations. The 2-form $ds_h^2 = dt^2 + h^2(t)\,d\theta^2$ is obviously smooth in $B_h(r) - \{t = 0\}$ and it can be extended continuously to $(t = 0)$, because we can prove that $\lim_{t\to 0} ds_h^2 = ds_E^2$ in the same fashion as in the second part of the proof of Theorem 4.2. Finally, restrict the domain of h to the maximal interval $[0, r_h) \subseteq [0, r)$ such that $h(t) > 0$ for every $t \in (0, r_h)$, and we have the following definition:

Definition 4.7. The geodesic ball $(B_h(r_h), dt^2 + h^2(t)\,d\theta^2)$ is called the 2-3 *symmetrization* of $B_f(r)$.

Remark 4.8. The 2-3 symmetrization can be extended to starlike domains such that

$\mathrm{Vol}(\partial D_f(p,t))$ is a smooth function of t. But we will restrict this symmetrization to geodesic balls within the injectivity radius due to the artificiality of the general situation.

5 Volume estimates

In Subsection 5.1, we get some upper bounds for the volume using the 1-2 symmetrization. Afterwards, in Subsection 5.2, we prove a volume comparison theorem related to the 2-3 symmetrization.

5.1 Upper bounds for the volume related to the 1-2 symmetrization.

Theorem 5.1. *Let (M^n, g) be a Riemannian manifold, $D_g(p,c) \subset M$ a starlike domain, and $\sqrt{\det \mathbf{g}(t,\theta)}.dt.dA$ its volume element in a polar coordinate system. Fix $\theta \in \mathscr{S}^{n-1}$ and define $\|\mathrm{Ricr}^-(s,\theta)\|_{L^1} := \int_0^s \mathrm{Ricr}^-(t,\theta)\,dt$. Then there exist constants A_1, A_2, B_1 and B_2 such that the following (equivalent) estimates hold:*

$$\sqrt{\det \mathbf{g}(s,\theta)} \leqslant \left[A_1.s.\frac{\sinh\left(\frac{B_1.\|\mathrm{Ricr}^-(s,\theta)\|_{L^1}.s}{(n-1)}\right)}{\left(\frac{B_1.\|\mathrm{Ricr}^-(s,\theta)\|_{L^1}.s}{(n-1)}\right)} \right]^{n-1} \tag{31}$$

and

$$\sqrt{\det \mathbf{g}(s,\theta)} \leqslant A_2^{n-1} s^{n-1} e^{\|\mathrm{Ricr}^-(s,\theta)\|_{L^1}.B_2.s}. \tag{32}$$

If $\|\mathrm{Ricr}^-(s,\theta)\|_{L^1} = 0$, then (31) has the obvious meaning.

Proof. We begin with some estimates on the solution of the equation

$$\begin{cases} f''(t) + K(t).f(t) = 0 \\ f(0) = 0 \\ f'(0) = 1 \end{cases} \tag{33}$$

defined on the interval $[0,s]$, where K is a continuous function on $[0,s]$ and $f(t) > 0$ in $(0,s]$. Set $K^-(t) = \max(-K(t), 0)$. We are looking for an upper bound of $f(s)$ in terms of $\|K^-\|_{L^1}$. Thus we can consider

$$\begin{cases} \tilde{f}''(t) - K^-(t).\tilde{f}(t) = 0 \\ \tilde{f}(0) = 0 \\ \tilde{f}'(0) = 1 \end{cases}$$

instead of (33) because $f \leqslant \tilde{f}$. Observe that $\tilde{f}$ is increasing and convex on $[0,s]$.

Assume that $\|K^-\|_{L^1} \neq 0$ (otherwise there is nothing to prove). Take $s_0 = 0 < s_1 < \cdots < s_{N-1} < s_N \in \mathbb{R}$ such that $\Delta s := s_{i+1} - s_i = 1/(4\|K^-\|_{L^1})$ and $s_{N-1} < s \leqslant s_N$. Suppose that $\Delta s < s$, which is equivalent to supposing that $N \geqslant 2$ (The other case will be considered afterwards). Fix $[s_i, s_{i+1}]$ for some $i = 0, \ldots, N-2$. We intend to estimate $\tilde{f}(s_{i+1})$ and $\tilde{f}'(s_{i+1})$ in terms of $\tilde{f}(s_i)$ and $\tilde{f}'(s_i)$.

We have that

$$\frac{1}{2}(\tilde{f}'(s_{i+1}))^2 - \frac{1}{2}(\tilde{f}'(s_i))^2 = \frac{1}{2}(\tilde{f}'(t))^2|_{s_i}^{s_{i+1}} = \frac{1}{2}\int_{s_i}^{s_{i+1}} ((\tilde{f}'(t))^2)'.dt$$

$$= \int_{s_i}^{s_{i+1}} \tilde{f}''(t).\tilde{f}'(t).dt = \int_{s_i}^{s_{i+1}} K^-(t).\tilde{f}(t).\tilde{f}'(t).dt \leqslant \tilde{f}(s_{i+1})\tilde{f}'(s_{i+1})\|K^-\|_{L^1},$$

which gives the following quadratic inequality in terms of $\tilde{f}'(s_{i+1})$:

$$(\tilde{f}'(s_{i+1}))^2 - 2\tilde{f}(s_{i+1})\|K^-\|_{L_1}\tilde{f}'(s_{i+1}) - (\tilde{f}'(s_i))^2 \leqslant 0. \tag{34}$$

Solving (34), we have the estimate

$$\tilde{f}'(s_{i+1}) \leqslant 2\tilde{f}(s_{i+1})\|K^-\|_{L^1} + \tilde{f}'(s_i). \tag{35}$$

We know that $\tilde{f}$ is convex, what gives

$$\frac{\tilde{f}(s_{i+1}) - \tilde{f}(s_i)}{\Delta s} \leqslant \tilde{f}'(s_{i+1}). \tag{36}$$

Joining (35), (36) and $\Delta s = 1/(4\|K^-\|_{L^1})$, we have after some calculations that

$$\tilde{f}(s_{i+1}) \leqslant \frac{1}{2\|K^-\|_{L^1}}\tilde{f}'(s_i) + 2\tilde{f}(s_i). \tag{37}$$

Combining (35) and (37), we can finally estimate $\tilde{f}(s_{i+1})$ and $\tilde{f}'(s_{i+1})$ in terms of $\tilde{f}(s_i)$ and $\tilde{f}'(s_i)$:

$$\tilde{f}(s_{i+1}) \leqslant 2\tilde{f}(s_i) + \frac{1}{\|K^-\|_{L^1}}\tilde{f}'(s_i) \tag{38}$$

$$\tilde{f}'(s_{i+1}) \leqslant 4\|K^-\|_{L^1}\tilde{f}(s_i) + 2\tilde{f}'(s_i). \tag{39}$$

A priori, the estimates (38) and (39) are valid only for $i = 0, \ldots, N-2$. We claim that we can estimate $\tilde{f}(s)$ and $\tilde{f}'(s)$ from $\tilde{f}(s_{N-1})$ and $\tilde{f}'(s_{N-1})$ using (38) and (39). Indeed, we only have to repeat all the calculations replacing $\Delta s = 1/(4\|K^-\|_{L^1})$ by $\Delta s \leqslant 1/(4\|K^-\|_{L^1})$.

Now we use (38) and (39) N times in order to estimate $\tilde{f}(s)$ from $\tilde{f}(0) = 0$ and $\tilde{f}'(0) = 1$. This is a Linear Algebra problem: The matrix

$$M = \begin{bmatrix} 2 & \frac{1}{\|K^-\|_{L^1}} \\ 4\|K^-\|_{L^1} & 2 \end{bmatrix}$$

has eigenvalues 0 and 4, and their respective eigenvectors are

$$v_0 = \begin{bmatrix} -\frac{1}{2\|K^-\|_{L^1}} \\ 1 \end{bmatrix} \quad \text{and} \quad v_4 = \begin{bmatrix} \frac{1}{2\|K^-\|_{L^1}} \\ 1 \end{bmatrix}.$$

The vector

$$\begin{bmatrix} \tilde{f}(0) \\ \tilde{f}'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is written as $\frac{1}{2}v_0 + \frac{1}{2}v_4$. After N iterations, we get

$$f(s) \leqslant \tilde{f}(s) \leqslant \left[M^N . \left(\frac{1}{2}v_0 + \frac{1}{2}v_4 \right) \right]_1 = \frac{4^{N-1}}{\|K^-\|_{L^1}} \leqslant 4^{(4\|K^-\|_{L^1}s)} 4s, \tag{40}$$

where $[\cdot]_1$ represents the first line of the vector. Thus (40) is the desired estimate of (33) if $\Delta s < s$.

If $\Delta s \geqslant s$, then we make the same estimates directly on the interval $[0, s]$ instead of making on each interval $[s_i, s_{i+1}]$. As a result, Inequality (37) gives $\tilde{f}(s) \leqslant 2s$, which is included in (40). Hence (40) is the desired estimate of (33).

Let us return to the theorem. In order to get an upper bound for the volume element of $D_g(p, c)$, we can consider its 1-2 symmetrization $D_f(p, c)$ instead of the original domain (due to Theorem 4.5). If we write the metric of $D_f(p, c)$ as $ds_f^2 = dt^2 + f^2(t, \theta)\, d\theta$, then f is exactly the solution of (33) when $K(t)$ is replaced by $\mathrm{Ricr}(t)/(n-1)$. But the volume element can be written as $f^{n-1}(t, \theta).dt.dA$, and (32) follows.

In order to get (31), apply (32) in the formula $x.e^x \leqslant \sinh 2x$. Finally, estimates (31) and (32) are equivalent because $\sinh x \leqslant x.e^x$. □

Let us make some remarks about Theorem 5.1.

Remark 5.2. We can get the explicit estimates $A_1 = 8$, $A_2 = 4$, $B_1 = 8 \ln 4$ and $B_2 = 4 \ln 4$ using the proof above, but they certainly are not close to the sharpest ones.

Remark 5.3. Let us point out the importance of the term $(s.\|\mathrm{Ricr}^-(s, \theta)\|_{L^1})$ in this kind of estimates.

Consider a starlike domain $(D_f(p, c), dt^2 + f^2(t, \theta)\, d\theta^2)$. Its volume element in a polar coordinate system is given by $f^{n-1}(t, \theta).dt.dA$, where $f(\cdot\,, \theta) : [0, t] \to \mathbb{R}$ is the solution of the equation

$$f''(t, \theta) + \frac{\mathrm{Ricr}(t, \theta)}{n-1}.f(t, \theta) = 0$$

satisfying the initial conditions $f(0, \theta) = 0$ and $f'(0, \theta) = 1$.

We want to compare $D_f(p,c)$ and $(D_{f_\lambda}(p,\lambda.c), dt^2 + f_\lambda^2(t,\theta)\,d\theta^2)$, where $\lambda > 0$ and $f_\lambda(t,\theta) := \lambda.f(t/\lambda,\theta)$. We have that

$$f_\lambda''(t,\theta) + \frac{1}{\lambda^2}\frac{\operatorname{Ricr}(t/\lambda,\theta)}{(n-1)} f_\lambda(t,\theta) = 0.$$

If we identify $D_f(p,c)$ and $D_{f_\lambda}(p,\lambda.c)$ via dilation by λ, then we can see that every pair of identified points has the same value of $(s.\|\operatorname{Ricr}^-(s,\theta)\|_{L^1})$. Moreover the volume element of $D_{f_\lambda}(p,\lambda.c)$ in polar coordinate system is λ^{n-1} times the correspondent volume element of $D_f(p,c)$. Therefore the estimate (31) is good because it considers this kind of symmetries.

Remark 5.4. Unfortunately, estimates (31) and (32) are not sharp because their growth rate when s goes to infinity are larger than the growth rate of the volume element of geodesic balls in space forms with constant negative curvature. We could try to improve (31) considering estimates like

$$\sqrt{\det \mathbf{g}(s,\theta)} \leqslant \left[A_1.\frac{\sinh\left(\frac{B_1.\|\operatorname{Ricr}^-(s,\theta)\|_{L^1}.s}{(n-1)}\right)}{\left(\frac{B_1.\|\operatorname{Ricr}^-(s,\theta)\|_{L^1}.s}{(n-1)}\right)}\right]^{n-1}, \tag{$*$}$$

$$\sqrt{\det \mathbf{g}(s,\theta)} \leqslant \left[A_1.\frac{\sinh\left(\frac{B_1.\|\operatorname{Ricr}^-(s,\theta)\|_{L^1}}{(n-1)}\right)}{\left(\frac{B_1.\|\operatorname{Ricr}^-(s,\theta)\|_{L^1}}{(n-1)}\right)}\right]^{n-1} \tag{$**$}$$

or

$$\sqrt{\det \mathbf{g}(s,\theta)} \leqslant \left[A_1.s.\frac{\sinh\left(\frac{B_1.\|\operatorname{Ricr}^-(s,\theta)\|_{L^1}}{(n-1)}\right)}{\left(\frac{B_1.\|\operatorname{Ricr}^-(s,\theta)\|_{L^1}}{(n-1)}\right)}\right]^{n-1}, \tag{$***$}$$

but they do not work: Estimates $(*)$ and $(**)$ do not work because we can take a suitable sequence of starlike domains with negative curvature $D_{f_\lambda}(p,\lambda.c)$, $\lambda \to \infty$ (see Remark 5.3), and see that $(*)$ and $(**)$ fail. Estimate $(***)$ fails because for every A_1 and B_1, we can choose a space form with small negative curvature (in absolute value) such that $(***)$ does not hold for a sufficiently large geodesic ball in it. These examples show that it is not easy to improve (31) using similar estimates that depend on $\|\operatorname{Ricr}^-(s,\theta)\|_{L^1}$.

A natural candidate for a sharp estimate is an inequality of the form

$$\sqrt{\det g(t,\theta)} \leqslant A_1.s.\frac{\sinh\left(B_1.\left\|\frac{\sqrt{\operatorname{Ricr}^-(s,\theta)}}{n-1}\right\|_{L^1}\right)}{\left(B_1.\left\|\frac{\sqrt{\operatorname{Ricr}^-(s,\theta)}}{n-1}\right\|_{L^1}\right)}.$$

As a consequence of Theorem 5.1, now we prove the following upper bound for the volume of a geodesic ball, which is not necessarily within the injectivity radius.

Corollary 5.5. *Let M^n be a complete Riemannian manifold and $B(p,r) \subset M$ a geodesic ball. Then there exist constants $A_3, B_3 > 0$ such that*

$$\mathrm{Vol}[B(p,r)] \leqslant A_3^{n-1} r^{n-1} \int_{\mathscr{S}^{n-1}} \int_0^r e^{B_3 . r^2 . \mathrm{Ricr}^-(t,\theta)}\, dt . dA. \tag{41}$$

Proof. Using Formula (32), we have the following estimate:

$$\mathrm{Vol}[B(p,r)] \leqslant \int_{\mathscr{S}^{n-1}} \int_0^{\tilde{c}(\theta)} A_2^{n-1} s^{n-1} e^{B_2 . s . \int_0^s \mathrm{Ricr}^-(t,\theta)\, dt} . ds . dA.$$

Substituting s in $\int_0^s \mathrm{Ricr}^-(t,\theta)\, dt$ and $\tilde{c}(\theta)$ by r, and using the Hölder inequality, we have that

$$\mathrm{Vol}[B(p,r)] \leqslant A_2^{n-1} \int_{\mathscr{S}^{n-1}} \left(\int_0^r s^{2n-2}\, ds \right)^{1/2} \left(\int_0^r e^{2 . B_2 . s \int_0^r \mathrm{Ricr}^-(t,\theta)\, dt}\, ds \right)^{1/2} dA.$$

Using the inequality $\int_0^r e^{as}\, ds \leqslant r e^{ar}$, we have that

$$\mathrm{Vol}[B(p,r)] \leqslant A_3^{n-1} r^n \int_{\mathscr{S}^{n-1}} (e^{\int_0^r \mathrm{Ricr}^-(t,\theta) . B_3 . r . dt})\, dA,$$

which can be written as

$$\mathrm{Vol}[B(p,r)] \leqslant A_3^{n-1} r^n \int_{\mathscr{S}^{n-1}} e^{(\int_0^r B_3 . r^2 . \mathrm{Ricr}^-(t,\theta)\, dt / r)}\, dA.$$

Finally use the Jensen inequality to get

$$\mathrm{Vol}[B(p,r)] \leqslant A_3^{n-1} r^{n-1} \int_{\mathscr{S}^{n-1}} \int_0^r e^{B_3 . r^2 . \mathrm{Ricr}^-(t,\theta)}\, dt . dA$$

and the result follows. □

5.2 Volume comparison theorem related to the 2-3 symmetrization.

Theorem 5.6. *Let $(B_f(r), dt^2 + f^2(t,\theta)\, d\theta^2) \subset M$ be a geodesic ball within the injectivity radius. If $B_h(r_h)$ is the 2-3 symmetrization of $B_f(r)$, then we have that $\mathrm{Vol}[\partial B_h(t)] \leqslant \mathrm{Vol}[\partial B_f(t)]$ for every $t \in [0, r_h)$. In particular, $\mathrm{Vol}[B_h(r_h)] \leqslant \mathrm{Vol}[B_f(r_h)]$. If $n = 2$, then we have that $r_h = r$, $\mathrm{Vol}[\partial B_h(t)] = \mathrm{Vol}[\partial B_f(t)]$ for every $t \in [0, r)$, and $\mathrm{Vol}[B_h(r)] = \mathrm{Vol}[B_f(r)]$. This theorem is still valid if $B_f(r)$ is the 1-2 symmetrization of another geodesic ball.*

Proof. Assume $n \geqslant 3$ (The case $n = 2$ is simpler and it can be proved in the same fashion as in the case $n \geqslant 3$).

The proof will follow an indirect approach. We construct a radially symmetric geodesic ball $(B_u(r), dt^2 + u^2(t)\, d\theta^2)$ such that $\mathrm{Vol}[\partial B_u(t)] = \mathrm{Vol}[\partial B_f(t)]$ for every $t \in [0, r)$. Then we compare B_u with B_h.

Let us see that $(B_u(r), dt^2 + u^2(t)\, d\theta^2)$ is a well defined object. The explicit expression of u is

$$u^{n-1}(t) = \frac{\int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\, dA}{\int_{\mathscr{S}^{n-1}} dA}. \tag{42}$$

We will show that $dt^2 + u^2(t)\, d\theta^2$ is a smooth metric in $(B_u(r) - \{t = 0\})$ that admits a continuous extension at $(t = 0)$ in the same fashion as in the second part of the proof of Theorem 4.2.

The smoothness in $(B_u(r) - \{t = 0\})$ is straightforward. Let us see the continuity at $(t = 0)$. By (42) we have that

$$u(t) = \left(\frac{\int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\, dA}{\int_{\mathscr{S}^{n-1}} dA}\right)^{1/(n-1)}.$$

Take the Taylor series of f. We get

$$\begin{aligned} u(t) &= \left(\frac{\int_{\mathscr{S}^{n-1}} (t + f''(\bar{t}_\theta,\theta).t^2/2)^{n-1}(t,\theta)\, dA}{\int_{\mathscr{S}^{n-1}} dA}\right)^{1/(n-1)} \\ &= t\left(\frac{\int_{\mathscr{S}^{n-1}} (1 + f''(\bar{t}_\theta,\theta).t/2)^{n-1}(t,\theta)\, dA}{\int_{\mathscr{S}^{n-1}} dA}\right)^{1/(n-1)} \end{aligned}$$

where $\bar{t}_\theta \in (0, t)$.

Now take two continuous vector fields X and Y and make exactly the same calculations as in the second part of Theorem 4. Instead of (27), we have that

$$\langle X, Y\rangle_u = \langle X_t, Y_t\rangle_E + \frac{u^2(t)}{t^2}\langle X_\theta, Y_\theta\rangle_E. \tag{43}$$

and now is clear that $dt^2 + u^2(t)\, d\theta^2$ converges to ds_E^2 when t goes to 0. Thus $dt^2 + u^2(t)\, d\theta^2$ is continuous at $(t = 0)$.

The next step is to compare the average of the radial Ricci curvature on $\partial B_f(t)$ with its correspondent on $\partial B_u(t)$. Taking derivatives with respect to t in (42), we have that

$$u^{n-2}(t).u'(t) = \frac{\int_{\mathscr{S}^{n-1}} f^{n-2}(t,\theta).f'(t,\theta)\, dA}{\int_{\mathscr{S}^{n-1}} dA}. \tag{44}$$

Taking another derivative in t, we get

$$(n-2).u^{n-3}(t)(u'(t))^2 + u^{n-2}(t)u''(t) = \frac{\int_{\mathscr{S}^{n-1}}[(n-2)f^{n-3}(t,\theta)(f'(t,\theta))^2 + f^{n-2}(t,\theta)f''(t,\theta)]\,dA}{\int_{\mathscr{S}^{n-1}} dA}. \tag{45}$$

Using (42), (44) and (45) we can isolate u'' from the rest:

$$\begin{aligned} u''(t) = {} & \frac{(2-n)(\int_{\mathscr{S}^{n-1}} f^{n-2}(t,\theta)f'(t,\theta)\,dA)^2}{(\int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\,dA)^{(2n-3)/(n-1)}(\int_{\mathscr{S}^{n-1}} dA)^{1/(n-1)}} \\ & + \frac{(\int_{\mathscr{S}^{n-1}} f^{n-3}(t,\theta)[f'(t,\theta)]^2.dA)(\int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\,dA)}{(\int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\,dA)^{(2n-3)/(n-1)}(\int_{\mathscr{S}^{n-1}} dA)^{1/(n-1)}} \\ & + \frac{(\int_{\mathscr{S}^{n-1}} f^{n-2}(t,\theta)f''(t,\theta)\,dA)(\int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\,dA)}{(\int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\,dA)^{(2n-3)/(n-1)}(\int_{\mathscr{S}^{n-1}} dA)^{1/(n-1)}}. \end{aligned} \tag{46}$$

Now we can calculate the radial Ricci curvature $\mathrm{Ricr}_u(t)$ of $B_u(r)$:

$$\mathrm{Ricr}_u(t) = -(n-1)\frac{u''(t)}{u(t)}$$

$$= \frac{(n-1)(n-2)(\int_{\mathscr{S}^{n-1}} f^{n-2}(t,\theta)f'(t,\theta)\,dA)^2}{(\int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\,dA)^2} \tag{47}$$

$$- \frac{(n-1)(n-2)(\int_{\mathscr{S}^{n-1}} f^{n-3}(t,\theta)(f'(t,\theta))^2\,dA)}{\int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\,dA} \tag{48}$$

$$- \frac{(n-1)(\int_{\mathscr{S}^{n-1}} f^{n-2}(t,\theta)f''(t,\theta)\,dA)}{\int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\,dA}. \tag{49}$$

Observe that (49) is the average of the radial Ricci curvature of $B_f(r)$ on the geodesic sphere of radius t. We claim that the sum of the terms (47) and (48) is nonpositive. In fact, the Hölder inequality gives

$$\left(\int_{\mathscr{S}^{n-1}} f^{n-2}(t,\theta)f'(t,\theta)\,dA\right)^2 \leqslant \int_{\mathscr{S}^{n-1}} f^{n-3}(t,\theta)(f'(t,\theta))^2\,dA \int_{\mathscr{S}^{n-1}} f^{n-1}(t,\theta)\,dA.$$

Therefore the average of the radial Ricci curvature on $\partial B_u(t)$ is less or equal than the correspondent average on $\partial B_f(t)$ for every $t \in [0,r)$.

Finally we compare $\mathrm{Vol}[\partial B_h(t)]$ and $\mathrm{Vol}[\partial B_f(t)]$.

Let $(B_h(r_h), dt^2 + h^2(t)\,d\theta^2)$ be the 2-3 symmetrization of $B_f(r,\theta)$. The calculations made just before and the definition of the 2-3 symmetrization implies that the geodesic ball B_u is a radially symmetric geodesic ball, with radial Ricci curvature less or equal than the radial Ricci curvature of B_h. Thus

$$u''(t) + \frac{\mathrm{Ricr}_u(t)}{n-1} u(t) = 0; \quad u(0) = 0; \quad u_t(0) = 1$$
$$h''(t) + \frac{\overline{\mathrm{Ricr}}_f(t)}{n-1} h(t) = 0; \quad h(0) = 0; \quad h_t(0) = 1$$

with $\mathrm{Ricr}_u(t) \leqslant \overline{\mathrm{Ricr}}_f(t)$. Therefore $h(t) \leqslant u(t)$ for every $t \in [0, r_h]$, which implies

$$\begin{aligned}\mathrm{Vol}[\partial B_h(t)] &= \int_{\mathscr{S}^{n-1}} h^{n-1}(t)\, dA \leqslant \int_{\mathscr{S}^{n-1}} u^{n-1}(t)\, dA \\ &= \mathrm{Vol}[\partial B_u(t)] = \mathrm{Vol}[\partial B_f(t)],\end{aligned}$$

and this is the desired result. Observe that the proof works even if B_f is the 1-2 symmetrization of another geodesic ball, which settles the theorem. □

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