ARCHIVUM MATHEMATICUM (BRNO)
Tomus 39 (2003), 179 – 185

THE ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF THE N–TH ORDER NEUTRAL DIFFERENTIAL EQUATIONS

DÁŠA LACKOVÁ

Abstract. The aim of this paper is to deduce oscillatory and asymptotic behavior of the solutions of the $n-$th order neutral differential equation

$$(x(t) - px(t-\tau))^{(n)} - q(t)x(\sigma(t)) = 0\,,$$

where $\sigma(t)$ is a delayed or advanced argument.

We consider the $n-$th order differential equation with a deviating argument of the form

$$(x(t) - px(t-\tau))^{(n)} - q_1(t)x(\sigma_1(t)) = 0\,, \tag{1}$$

where

(i) n is even,
(ii) p and τ are positive numbers,
(iii) $q_1(t), \sigma_1(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $q_1(t)$ is positive, $\lim\limits_{t\to\infty} \sigma_1(t) = \infty$.

By a solution of Eq. (1) we mean a function $x : [T_x, \infty) \to \mathbb{R}$ which satisfies (1) for all sufficiently large t. Such a solution is called oscillatory if it has a sequence of zeros tending to infinity; otherwise it is called nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

We introduce the notation

$$Q_j(t) = q_j(t)\sum_{i=0}^{m} p^i\,, \quad \text{where } m \text{ is a positive integer}, \quad j = 1, 2\,. \tag{2}$$

2000 *Mathematics Subject Classification*: 34C10, 34K11.
Key words and phrases: neutral differential equation, delayed argument, advanced argument.
Received June 12, 2001.

Lemma 1. *Let $z(t)$ be an n times differentiable function on $\mathbb{R}_+$ of constant sign, $z^{(n)}(t) \not\equiv 0$ on $[T_0, \infty)$ which satisfies $z^{(n)}(t)z(t) \geq 0$. Then there is an integer l, $0 \leq l \leq n$ and $t_1 \geq T_0$ such that $n + l$ is even and for all $t \geq t_1$*

$$\begin{aligned} z(t)z^{(i)}(t) &> 0, \qquad 0 \leq i \leq \ell\,, \\ (-1)^{i-\ell} z(t)z^{(i)}(t) &> 0, \qquad \ell \leq i \leq n\,. \end{aligned} \tag{3}$$

Lemma 1 is a well-known lemma of Kiguradze [5].

A function $z(t)$ satisfying (3) is said to be a function of degree l. The set of all functions of degree l is denoted by $\mathcal{N}_l$. If we denote by $\mathcal{N}$ the set of all functions satisfying $z^{(n)}(t)z(t) \geq 0$ then the set $\mathcal{N}$ has the following decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_n\,.$$

Lemma 2. *Let $y(t)$ be a positive function of degree ℓ, $\ell \geq 2$. Then*

$$y(t) \geq \int_{t_1}^{t} y^{(\ell-1)}(s) \frac{(t-s)^{\ell-2}}{(\ell-2)!}\, ds\,. \tag{4}$$

The proof of this lemma is immediate from integration the identity $y^{(l-1)}(t) = y^{(l-1)}(t)$.

Theorem 1. *Assume that m is a positive integer. Let*

$$\sigma_1(t) < t - \tau\,, \quad \sigma_1(t) \in C^1\,, \quad \sigma_1'(t) \geq 0\,. \tag{5}$$

Further assume that the differential equation

$$y^{(n)}(t) + \frac{1}{p} q_1(t) y(\sigma_1(t) + \tau) = 0 \tag{6}$$

is oscillatory and the differential inequality

$$z^{(n)}(t) - Q_1(t) z(\sigma_1(t)) \geq 0 \tag{7}$$

has no solution of degree 0. Then every nonoscillatory solution of Eq. (1) tends to ∞ as $t \to \infty$.

Proof. Without loss of generality let $x(t)$ be an eventually positive solution of Eq. (1) and define

$$z(t) = x(t) - px(t - \tau)\,. \tag{8}$$

It is easy to see that

$$z(t) < x(t)\,. \tag{9}$$

From Eq. (1) we have $z^{(n)}(t) > 0$ for all large t, say $t \geq t_0$. Thus $z^{(i)}(t)$ are monotonous, $i = 0, 1, \ldots, n-1$. If $z(t) < 0$ eventually, then we set $u(t) = -z(t)$. In the view of (8)

$$x(t-\tau) > \frac{1}{p} u(t),$$

that is

$$x(t) > \frac{1}{p} u(t+\tau).$$

One gets that $u(t)$ is a positive solution of the inequality

$$u^{(n)}(t) + \frac{1}{p} q_1(t) u(\sigma_1(t) + \tau) \leq 0$$

and by Kusano and Naito [1] the corresponding equation

$$u^{(n)}(t) + \frac{1}{p} q_1(t) u(\sigma_1(t) + \tau) = 0$$

has a positive solution $u(t)$. This contradicts that (6) is oscillatory.

Therefore $z(t) > 0$. According to Lemma 1 we have two possibilities for $z'(t)$:

(a) $z'(t) > 0$, for $t \geq t_1 \geq t_0$,
(b) $z'(t) < 0$, for $t \geq t_1$.

For case (a) by Lemma 1 we obtain $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$. It implies that $\lim_{t\to\infty} z(t) = \infty$ and from (9) also $\lim_{t\to\infty} x(t) = \infty$.

For case (b) Eq. (1) can be written in the form

$$z^{(n)}(t) - q_1(t) x(\sigma_1(t)) = 0.$$

Using (8) we have

$$z^{(n)}(t) - q_1(t) z(\sigma_1(t)) - p q_1(t) x(\sigma_1(t) - \tau) = 0.$$

Repeating this procedure $m-$times we arrive at

$$z^{(n)}(t) - q_1(t) \sum_{i=0}^{m} p^i z(\sigma_1(t) - i\tau) - p^{m+1} q_1(t) x(\sigma_1(t) - (m+1)\tau) = 0.$$

Since $z(t)$ is decreasing, we get

$$z^{(n)}(t) - q_1(t) z(\sigma_1(t)) \sum_{i=0}^{m} p^i \geq 0.$$

In the view of (2) we have

$$z^{(n)}(t) - Q_1(t) z(\sigma_1(t)) \geq 0. \tag{10}$$

Hence $z(t)$ is a solution of degree 0 of the inequality (10). This is a contradiction. $\square$

Corollary 1. *Let m be a positive integer. Further assume that* (5) *holds, differential equation* (6) *is oscillatory and there exists $k \in \{0, 1, \dots, n-1\}$ such that*

$$\limsup_{t\to\infty} \frac{1}{k!(n-k-1)!} \int\limits_{\sigma_1(t)}^{t} [s-\sigma_1(t)]^k \, [\sigma_1(t)-\sigma_1(s)]^{n-k-1} \, Q_1(s)\, ds > 1 \,. \tag{11}$$

Then every nonoscillatory solution of Eq. (1) *tends to ∞ as $t \to \infty$.*

Proof. By [2, Theorem 1] it follows from (11) that the differential inequality (7) has no solution of degree 0. Our assertion follows from Theorem 1. □

Let us consider the n–th order differential equation with an advanced argument of the form

$$(x(t)-px(t-\tau))^{(n)} - q_2(t)x(\sigma_2(t)) = 0 \,, \tag{12}$$

where (i), (ii) holds and moreover

(iv) $q_2(t), \sigma_2(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $q_2(t)$ is positive, $\lim\limits_{t\to\infty} \sigma_2(t) = \infty$.

We introduce the notation

$$A_\ell(t) = \int\limits_t^\infty q_2(s) \frac{(s-t)^{n-\ell-1}}{(n-\ell-1)!} \times \left[\int\limits_t^{\sigma_2(s)} \frac{(t-u)^{\ell-2}}{(\ell-2)!} du \right] ds \,, \qquad \text{for} \quad \ell = 2, 4, \dots, n-2 \,. \tag{13}$$

Theorem 2. *Assume that m is a positive integer and*

$$\sigma_2(t) - m\tau > t \,, \quad \sigma_2(t) \in C^1 \,, \quad \sigma_2'(t) \ge 0 \,, \qquad 0 < p < 1 \,. \tag{14}$$

Further assume that

$$A_\ell(t)(t-t_1) > 1 \quad \textit{for} \quad \ell = 2, 4, \dots, n-2 \tag{15}$$

and the differential inequality

$$z^{(n)}(t) - Q_2(t) z(\sigma_2(t) - m\tau) \ge 0 \tag{16}$$

has no solution of degree n. Then every nonoscillatory solution of Eg. (12) *is bounded.*

Proof. Without loss of generality let $x(t)$ be an eventually positive solution of Eq. (12) and define

$$z(t) = x(t) - px(t-\tau) \,. \tag{17}$$

From Eq. (12) we have $z^{(n)}(t) > 0$ for all large t, say $t \geq t_0$. Thus $z^i(t)$ are monotonous, $i = 0, 1, \dots, n-1$. If $z(t) < 0$ eventually, then

$$x(t) < px(t-\tau) < p^2 x(t-2\tau) < \cdots < p^k x(t-k\tau)$$

for all large t, which implies $\lim\limits_{t\to\infty} x(t) = 0$.

If $z(t) > 0$, then according to a Lemma 1 we have two possibilities for $z'(t)$:

(a) $z'(t) > 0$, for $t \geq t_1 \geq t_0$,
(b) $z'(t) < 0$, for $t \geq t_1$.

For case (a) we have two possibilities:

(i) $\exists\ \ell \in 2, 4, \dots, n-2$, such that $z(t) \in \mathcal{N}_\ell$,
(ii) $\ell = n$, i.e. $z(t) \in \mathcal{N}_n$.

For case (i) Eq. (12) can be written in the form

$$z^{(n)}(t) = q_2(t) x(\sigma_2(t))\,.$$

Integrating this equation from t to ∞ $\;n-\ell$ times and taking Lemma 2 into account, one gets

$$z^{(\ell)}(t) \geqslant \int_t^\infty q_2(s) x(\sigma_2(s)) \frac{(s-t)^{n-\ell-1}}{(n-\ell-1)!}\, ds \geqslant \int_t^\infty q_2(s) z(\sigma_2(s)) \frac{(s-t)^{n-\ell-1}}{(n-\ell-1)!}\, ds$$

$$\geqslant \int_t^\infty q_2(s) \frac{(s-t)^{n-\ell-1}}{(n-\ell-1)!} \times \left[\int_{t_1}^{\sigma_2(s)} z^{(\ell-1)}(u) \frac{(t-u)^{\ell-2}}{(\ell-2)!}\, du \right] ds$$

Taking into account that $\sigma_2(t)$ is nondecreasing, $t \geq t_1$ and $z^{(\ell-1)}(t)$ is increasing, the above inequalities led to

$$z^{(\ell)}(t) \geq z^{(\ell-1)}(t) A_\ell(t)\,. \tag{18}$$

Integration of the identity $z^{(\ell)}(t) = z^{(\ell)}(t)$ from t_1 to t provides

$$z^{(\ell-1)}(t) \geqslant \int_{t_1}^t z^{(\ell)}(s)\, ds \geqslant z^{(\ell)}(t)(t-t_1), \qquad t \geq t_1\,,$$

which in the view of (18) implies

$$1 \geq (t-t_1) A_\ell(t)\,.$$

This contradicts (15).

For case (ii) Eq. (12) can be written in the form

$$z^{(n)}(t) - q_2(t) x(\sigma_2(t)) = 0\,.$$

Using (17) we have

$$z^{(n)}(t) - q_2(t)z(\sigma_2(t)) - pq_2(t)x(\sigma_2(t) - \tau) = 0\,.$$

Repeating this procedure $m-$times we arrive at

$$z^{(n)}(t) - q_2(t)\sum_{i=0}^{m} p^i z(\sigma_2(t) - i\tau) - p^{m+1}q_2(t)x(\sigma_2(t) - (m+1)\tau) = 0\,.$$

Since $z(t)$ is increasing, we get

$$z^{(n)}(t) - q_2(t)z(\sigma_2(t) - m\tau)\sum_{i=0}^{m} p^i \geq 0\,.$$

In the view of (2) we have

$$z^{(n)}(t) - Q_2(t)z(\sigma_2(t) - m\tau) \geq 0\,. \tag{19}$$

Hence $z(t)$ is a solution of degree n of the inequality (19). This is a contradiction.

For case (b) we have $z(t) > 0$, $z'(t) < 0$. Hence there exists

$$\lim_{t\to\infty} z(t) = c \geq 0\,. \tag{20}$$

If $x(t)$ is unbounded eventually, then we can define the sequence $\{t_n\}$ where $t_n \to \infty$ as $n \to \infty$ as follows. Let us choose t_m for every $m \in \mathbb{N}$ such that

$$x(t_m) = \max\{x(s), t_0 \leq s \leq t_m\}\,.$$

Since

$$x(t_m - \tau) = \max\{x(s), t_0 \leq s \leq t_m - \tau\} \leq \max\{x(s), t_0 \leq s \leq t_m\} = x(t_m)\,,$$

we have

$$z(t_m) = x(t_m) - px(t_m - \tau) \geq x(t_m) - px(t_m) = (1-p)x(t_m)\,.$$

This implies $\lim\limits_{t\to\infty} z(t) = \infty$. This contradicts (20). □

Corollary 2. *Let m be a positive integer. Further assume that* (14) *and* (15) *hold and there exists $k \in \{0, 1, \ldots, n-1\}$ such that*

$$\limsup_{t\to\infty} \frac{1}{k!(n-k-1)!} \int_t^{\sigma_2(t)} [\sigma_2(s) - \sigma_2(t)]^k \, [\sigma_2(t) - s]^{n-k-1} \, Q_2(s)\, ds > 1\,. \tag{21}$$

Then every nonoscillatory solution of Eq. (12) *is bounded.*

Proof. By [2, Theorem 4] it follows from (21) that the differential inequality (16) has no solution of degree n. Our assertion follows from Theorem 2. □

Now we want to extend our previous results to more general differential equation. So let us consider the $n-$th order differential equation with both arguments (advanced and delayed) of the form

$$(x(t) - px(t-\tau))^{(n)} - q_1(t)x(\sigma_1(t)) - q_2(t)x(\sigma_2(t)) = 0\,, \tag{22}$$

where (i), (ii), (iii), (iv) hold.

Theorem 3. *Let m be a positive integer. Further assume that* (5), (14) *and* (15) *hold, differential equality* (6) *is oscillatory, differential inequality* (7) *has no solution of degree 0 and differential inequality* (16) *has no solution of degree n.*

Then every solution of Eg. (22) *is oscillatory.*

Proof. Without loss of generality let $x(t)$ be an eventually positive solution of Eq. (22). Then $x(t)$ is solution of the inequality

$$(x(t) - px(t-\tau))^{(n)} - q_1(t)x(\sigma_1(t) \geq 0\,.$$

Using the same arguments as in Theorem 1 we can prove that $x(t)$ tends to ∞ as $t \to \infty$.

On the other hand, $x(t)$ is also solution of the inequality

$$(x(t) - px(t-\tau))^{(n)} - q_2(t)x(\sigma_2(t) \geq 0\,.$$

Now arguing exactly as in the proof of Theorem 2 we get that $x(t)$ is bounded. This is a contradiction. □

In a paper [2, Theorem 7] Kusano has presented conditions when the functional differential equation

$$y^{(n)}(t) - q_1(t)y(\sigma_1(t)) - q_2(t)y(\sigma_2(t)) = 0$$

is oscillatory. We have extended these conditions also for the neutral differential equation of the form (22). In a paper [6] Džurina and Mihalíková have presented sufficient conditions for all bounded solutions of the second order neutral differential equation with a delayed argument to be oscillatory. We have extended these conditions also for the n-th order neutral differential equation involving both delayed and advanced arguments.

References

[1] Kusano, T., Naito, M., *Comparison theorems for functional differential equations with deviating argument*, J. Math. Soc. Japan **33** (1981).

[2] Kusano, T., *On even functional differential equations with advanced and retarded arguments*, Differential Equations **45** (1985), 75–84.

[3] Džurina, J., Mihalíková, B., *Oscillation criteria for second order neutral differential equations*, Math. Bohem. **2** (2000), 145–153.

[4] Džurina, J., Mihalíková, B., *Oscillations of advanced differential equations*, Fasc. Math. **25** (1995), 95–103.

[5] Kiguradze, I. T., *On the oscillation of solutions of the equation* $d^m u/dt^m + a(t)|u|^n \mathit{sign}\, u = 0$, Mat. Sb **65** (1964). (Russian)

[6] Džurina, J., Mihalíková, B., *A note of unstable neutral differential equations of the second order*, Fasc. Math. **29** (1999), 17–22.

Department of Applied Mathematics and Business Informatics
Faculty of Economics of Technical University Košice
ul. B. Němcovej 32
040 01 Košice, Slovak Republic
E-mail: Dasa.Lackova@tuke.sk