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FIXED POINTS OF FUZZY MONOTONE MULTIFUNCTIONS

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Abstract. Under suitable conditions we prove the existence of fixed points of fuzzy monotone multifunctions.

1. Introduction and preliminaries

Since Zadeh's invention of the concept of fuzzy sets it has been extensively investigate in mathematics science and engineering. Many authors studied the existence of fixed point in fuzzy setting: Heilpern [7], Hadzic [6], Fang [5], Beg [2, 3]. In the present note we first establish the existence of a maximal fixed point of fuzzy monotone multifunctions (see Theorem 2.1). In the second time we prove the existence of fixed points of fuzzy monotone multifunctions by using iteration method (see Theorem 2.7).

Let X be a space of points, with generic element of X denoted by x. A fuzzy subset A of X is characterized by its membership function

$$\mu_A : X \to [0,1]$$

and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy subset A for each $x \in X$.

Let A and B be two fuzzy subsets of X. We say that A is included in B and we write $A \subseteq B$ if $\mu_A(x) \le \mu_B(x)$, for all $x \in X$. In particular, if $x \in X$ and A is a fuzzy set in X, then $\{x\} \subseteq A$ if $\mu_A(x) = 1$.

In [8], Zadeh gave the definition of fuzzy order relation. In this note we shall use the Claude Ponsard's definition of order relation (see [4]).

Definition 1.1. Let X be a crisp set. A fuzzy order relation on X is a fuzzy subset R of $X \times X$ satisfying the following three properties

(i) for all $x \in X$, $r(x,x) \in [0,1]$ (reflexivity);
(ii) for all $x, y \in X$, $r(x,y) + r(y,x) > 1$ implies $x = y$ (antisymmetry)
(iii) for all $(x,y,z) \in X^3$, $[r(x,y) \ge r(y,x)$ and $r(y,z) \ge r(z,y)]$ implying $r(x,z) \ge r(z,x)$ (f-transitivity).

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A set with fuzzy order defined on its is called a fuzzy ordered set. A fuzzy order is said to be total if for all $x \neq y$ we have either $r(x,y) > r(y,x)$ or $r(y,x) > r(x,y)$. A fuzzy ordered set on which fuzzy order is total is called fuzzy chain.

Let A be a subset of X. We say that $x \in X$ is an upper bound of A if $r(y,x) \geq r(x,y)$ for all $y \in A$. If x is an upper bound of A and $x \in A$ then x is called a greatest element of A. An $x \in A$ is called a maximal element of A if there is no $y \neq x$ in A for which $r(x,y) \geq r(y,x)$. Similarly, we can define lower bound, minimal and least element of A. As usual,

$\sup(A) =$ least element of upper bound of A (if it exists),

$\inf(A) =$ greatest element of lower bound of A (if it exists).

Let X be a fuzzy ordered set. A fuzzy multifunction is any map $T : X \to [0,1]^X \setminus \{\emptyset\}$ such that for every $x \in X$, $T(x)$ is a nonempty fuzzy subset of X. The fuzzy multifunction $T : X \to [0,1]^X \setminus \{\emptyset\}$ is said to be fuzzy monotone if and only if for every $x, y \in X$ $r(x,y) \geq r(y,x)$ implies that for all $\{a\} \subseteq T(x)$ there exists $\{b\} \subseteq T(y)$ such that $r(a,b) \geq r(b,a)$.

A point $x \in X$ is called a fixed point of a fuzzy multifunction $T : X \to [0,1]^X \setminus \{\emptyset\}$ if $\{x\} \subseteq T(x)$.

In [1], I. Beg established the following fuzzy Zorn's lemma by using Claude Ponsard's definition of order relation.

Theorem 1.2. *Let X be a fuzzy ordered set. If every fuzzy chain in X has an upper bound, then X has a maximal element.*

2. The results

First, we establish the existence of a maximal fixed point of fuzzy monotone multifunctions.

Theorem 2.1. *Let X be a fuzzy ordered set with the property that every fuzzy chain in X has a supremum. Let $T : X \to [0,1]^X \setminus \{\emptyset\}$ be a fuzzy monotone multifunction such that $\sup(T(x))$ exists and $\{\sup(T(x))\} \subseteq T(x)$, for all $x \in X$. If there exist $a, b \in X$ such that $\{b\} \subseteq T(a)$ and $r(a,b) \geq r(b,a)$, then the set of fixed points of T is nonempty and has a maximal element.*

Proof. Let P be the fuzzy ordered subset defined by $P = \{x \in X :$ there exists $y \in X$ such that $\{y\} \subseteq T(x)$ and $r(x,y) \geq r(y,x)\}$. The subset P is nonempty because $a \in P$.

Claim 1. The subset P has a maximal element. Indeed, if C is a chain in P and $b = \sup(C)$, then for every $c \in C$, $r(c,b) \geq r(b,c)$. On the other hand there exists $\{d\} \subseteq T(c)$ such that $r(c,d) \geq r(d,c)$. Then there exists $\{l\} \subseteq T(b)$ such that $r(d,l) \geq r(l,d)$. So $r(c,l) \geq r(l,c)$. Let $m = \sup\big(T(b)\big)$ Then $r(l,m) \geq r(m,l)$. Thus $r(c,m) \geq r(m,c)$ for all $c \in C$. So m is an upper bound of C. Since b is a supremum of C, then we have $r(b,m) \geq r(m,b)$. So $b \in P$. Therefore any chain in P has an upper bound in P. From Theorem 1.2 P has a maximal element x (say).

Claim 2. The element x is a fixed point of T. Indeed, since $x \in P$, then there exists $\{y\} \subseteq T(x)$ such that $r(x,y) \geq r(y,x)$. Let $z = \sup\big(T(x)\big)$. So, $r(y,z) \geq r(z,y)$. Thus, $r(x,z) \geq r(z,x)$. Since T is a monotone fuzzy multifunction and $\{z\} \subseteq T(x)$, then there exists $\{t\} \subseteq T(z)$ satisfying $r(z,t) \geq r(t,z)$. Thus, $z \in P$. Since x is a maximal element of P, then $x = z$ and $\{x\} \subseteq T(x)$. So x is a fixed point of T.

Claim 3. The element x is a maximal element of the set of fixed points of T. Indeed, if y a fixed point of T, then $y \in P$. Thus x is a maximal element of the set of all fixed points of T. □

As a consequence of Theorem 2.1, we reobtain the Beg's result [2, Theorem 1].

Corollary 2.2. *Let X be a fuzzy ordered set with the property that every fuzzy chain in X has a supremum. Let $f : X \to X$ be a fuzzy monotone map and assume that there exists some $a \in X$ with $r\big(a, f(a)\big) \geq r\big(f(a), a\big)$. Then the set of fixed points of f is nonempty and has a maximal element.*

By using iteration method, we study the existence of fixed points of fuzzy monotone multifunctions. First, we give some definitions.

Definition 2.3. Let X be a fuzzy ordered set and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $(x_n)_{n\in\mathbb{N}}$ is a fuzzy increasing sequence if $r(x_n, x_{n+1}) \geq r(x_{n+1}, x_n)$ and $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Similarly we can define a fuzzy decreasing sequence.

Definition 2.4. Let X be a fuzzy ordered set. We say that X is a fuzzy order complete if for every fuzzy increasing sequence $(x_n)_{n\in\mathbb{N}}$ in X, $\sup((x_n)_{n\in\mathbb{N}})$ exists in X.

Definition 2.5. Let X be a fuzzy ordered set. Let $(x_n)_{n\in\mathbb{N}}$ be a fuzzy increasing sequence and let $T : X \to [0,1]^X \setminus \{\emptyset\}$ be a fuzzy multifunction. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ is an increasing T-orbit if $\{x_{n+1}\} \subseteq T(x_n)$ for every $n \in \mathbb{N}$.

Definition 2.6. Let X be a fuzzy order complete set and let $T : X \to [0,1]^X \setminus \{\emptyset\}$ be a fuzzy monotone multifunction. We say that T is fuzzy order continuous if for all increasing T-orbit $(x_n)_{n\in\mathbb{N}}$, we have

$$\{\sup((x_n)_{n\in\mathbb{N}})\} \subseteq T\left(\sup((x_n)_{n\in\mathbb{N}})\right) .$$

Theorem 2.7. *Let X be a fuzzy ordered set and let $T : X \to [0,1]^X \setminus \{\emptyset\}$ be a fuzzy order continuous multifunction. Assume that there is $a \in X$ such that:*

(i) *there exists $b \in X$ such that $\{b\} \subseteq T(a)$ and $r(a,b) \geq r(b,a)$, and*

(ii) *the fuzzy ordered set $\{x \in X : r(a,x) \geq r(x,a)\}$ is fuzzy order complete.*

Then, T has a fixed point.

Proof. First, we construct an increasing T-orbit $(x_n)_{n\in\mathbb{N}}$ by setting: $x_0 = a$ and $x_1 = b$. Since T is fuzzy monotone, $\{b\} \subseteq T(a)$ and $r(a,b) \geq r(b,a)$, then there exists $c \in X$ such that $\{c\} \subseteq T(b)$ and $r(b,c) \geq r(c,b)$. Then we set $x_2 = c$. By induction we construct the sequence $(x_n)_{n\in\mathbb{N}}$ as follows:
suppose that $x_0, x_1, \ldots, x_n$ exist such that $r(x_p, x_{p+1}) \geq r(x_{p+1}, x_p)$ and $\{x_{p+1}\} \subseteq T(x_p)$ for all $p \in \{0, 1, \ldots, n-1\}$. Since T is fuzzy monotone, $\{x_n\} \subseteq T(x_{n-1})$ and $r(x_{n-1}, x_n) \geq r(x_n, x_{n-1})$, then there exists $x_{n+1} \in X$ such that $\{x_{n+1}\} \subseteq T(x_n)$

and $r(x_n, x_{n+1}) \geq r(x_{n+1}, x_n)$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. Otherwise, the sequence $(x_n)_{n\in\mathbb{N}}$ is an increasing T-orbit. Since the supremum c of the sequence $(x_n)_{n\in\mathbb{N}}$ exists in X and T is fuzzy order continuous then $\{c\} \subseteq T(c)$. Therefore T has a fixed point. $\square$

References

[1] Beg, I., *On fuzzy Zorn's lemma*, Fuzzy Sets and Systems **101** (1999), 181–183.

[2] Beg, I., *Fixed points of fuzzy monotone maps*, Arch. Math. (Brno) **35** (1999), 141–144.

[3] Beg, I., *A general theorem on selector of fuzzy multifunctions*, Jour. Fuzzy Math. **9**(1) (2001).

[4] Billot, A., *Economic theory of fuzzy equilibria*, Lecture Notes in Economics and Mathematical Systems-373, Springer-Verlag, Berlin 1992.

[5] Fang, J. X., *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems **46** (1992), 107–113.

[6] Hadzic, O., *Fixed point theorems for multivalued mapping in some classes of fuzzy metric spaces*, Fuzzy Sets and Systems **29** (1989), 115–125.

[7] Heilpern, S., *Fuzzy mapping and fixed point theorem*, Jour. Math. Anal. Appl. **83** (1981), 566–569.

[8] Zadeh, L. A., *Fuzzy sets*, Information and Control **8** (1965), 338–353.

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