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A SINGULAR VERSION OF LEIGHTON'S COMPARISON THEOREM FOR FORCED QUASILINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. We extend the classical Leighton comparison theorem to a class of quasilinear forced second order differential equations

$$(r(t)|x'|^{\alpha-2}x')' + c(t)|x|^{\beta-2}x = f(t)\,, \quad 1 < \alpha \le \beta,\ t \in I = (a,b)\,, \tag{*}$$

where the endpoints a, b of the interval I are allowed to be singular. Some applications of this statement in the oscillation theory of (*) are suggested.

1. Introduction

In this paper we deal with oscillatory properties of the quasilinear forced second order differential equation

$$L_{\alpha\beta}[x] := (r(t)|x'|^{\alpha-2}x')' + c(t)|x|^{\beta-2}x = f(t)\,, \tag{1.1}$$

where $t \in I = (a,b)$, $-\infty \le a < b \le \infty$, r, c, f are real-valued continuous functions with $r(t) > 0$, $c(t) \ge 0$ for $t \in I$ and $1 < \alpha \le \beta$ are real constants.

The classical Leighton comparison theorem [12] (see also [17]) concerns the pair of second order Sturm-Liouville equations

$$(r(t)x')' + c(t)x = 0\,, \tag{1.2}$$

$$(R(t)y')' + C(t)y = 0\,, \tag{1.3}$$

and reads as follows.

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Proposition 1.1. *Suppose that the interval $I=[a,b]$ is compact, the functions r, R, c, C are continuous on I, $r(t)>0$, $R(t)>0$ in this interval and there exists a nontrivial solution $\tilde y$ of* (1.3) *such that $\tilde y(a)=0=\tilde y(b)$. If*

$$\int_a^b \left[(r(t)-R(t))\tilde y'^2(t)-(c(t)-C(t))\tilde y^2(t)\right]\,dt\le 0\,,$$

Then every solution of (1.2) *has a zero point in (a,b) or it is a multiple of $\tilde y$.*

Since 1962, when the original Leighton theorem was published, this statement has been extended in various directions, see [9-11,15,16] and the references given therein. Our investigation is mainly motivated by the papers [2,10]. In the first paper conjugacy criteria for the half-linear second order equation

$$(r(t)|x'|^{\alpha-2}x')'+c(t)|x|^{\alpha-2}x=0 \tag{1.4}$$

are established and the second one studies oscillation properties of equation (1.1) on a compact interval. In our treatment we combine ideas of these two papers and we prove a Leightone-type comparison theorem for (1.1) where the end points a,b of the interval I are allowed to be singular. The solution $\tilde y$ from the classical Leighton theorem (i. e., the solution for which $\tilde y(a)=0=\tilde y(b)$) is replaced by the principal solution at a and b of a certain half-linear equation. An important role in the proofs of our results is played by the recently established Picone's identity for half-linear equations, see [8].

2. Picone's identity and principal solution of half-linear equations

Denote, for convenience, $\varphi(u):=|u|^{\alpha-2}u$ and rewrite half-linear equation (1.4) into the form

$$(r(t)\varphi(x'))'+c(t)\varphi(x)=0\,. \tag{2.1}$$

It is well known that the classical Sturmian oscillation theory extends almost verbatim to half-linear equation (2.1), see e. g. [1,3,6,13]. In particular, the Riccati-type equation (related to (2.1) by the substitution $w=r\varphi(x')/\varphi(x)$)

$$w'+c(t)+(\alpha-1)r^{\frac{1}{1-\alpha}}(t)|w|^{\frac{\alpha}{\alpha-1}}=0\,, \tag{2.2}$$

and the α degree functional

$$\mathcal{I}(y):=\int_a^b \left[r(t)|y'|^\alpha-c(t)|y|^\alpha\right]\,dt \tag{2.3}$$

play the same role as the classical Riccati equation and quadratic functionals in the linear oscillation theory. The proof of the relationship between (2.1), (2.2) and (2.3) is based on the recently established Picone's identity, see [8]. Here we present this identity in a modified form as used in [10].

Lemma 2.1. *Let $x, y, \varphi(y') \in C^1(I)$ and $y(t) \neq 0$ for $t \in I$. Then*

$$\left[\frac{r(t)|x|^\alpha}{\varphi(y)}\varphi(y')\right]' = r(t)|x'|^\alpha - \left[c(t)|y|^{\beta-\alpha} - \frac{f(t)}{\varphi(y)}\right]|x|^\alpha - r(t)P_\alpha\left(x', xy'/y\right) + \frac{|x|^\alpha}{\varphi(y)}\left\{L_{\alpha\beta}[y] - f(t)\right\},$$

where

$$P_\alpha(u,v) := |u|^\alpha - \alpha u\varphi(v) + (\alpha-1)|v|^\alpha \geq 0$$

for every $u, v \in \mathbb{R}$ with equality holding if and only if $u = v$.

We will need also the following statement proved in [10].

Lemma 2.2. *Suppose that the interval $I = [a,b]$ is compact, there exists a nontrivial piecewise differentiable function $y \in C[a,b]$ such that $y(a) = 0 = y(b)$ and*

$$\mathcal{F}(y; a, b) := \int_a^b \left[r(t)|y'|^\alpha - M_{\alpha\beta}[c,f](t)|y|^\alpha\right] dt \leq 0,$$

where

$$M_{\alpha\beta}[c,f](t) := (\alpha-1)^{-\frac{\alpha-1}{\beta-1}}(\beta-1)(\beta-\alpha)^{\frac{\alpha-\beta}{\beta-1}}[c(t)]^{\frac{\alpha-1}{\beta-1}}|f(t)|^{\frac{\beta-\alpha}{\beta-1}} \tag{2.4}$$

(with tne convention that $0^0 = 1$ in case $\alpha = \beta$). Then every solution of (1.1) defined on $[a,b]$ and satisfying $x(t)f(t) \leq 0$ in this interval has a zero in $[a,b]$.

Note that Lemma 2.2. is proved in [10] under the (stronger) assumption $y \in C^1(I)$, but the proof directly extends to the case when $y \in C^1(I)$ only piecewise.

Now recall the concept of the principal solution of half-linear equation (2.1) as introduced by Mirzov [14]. Suppose that (2.1) is nonoscillatory, then among all proper solutions of the associated Riccati type equation (2.2) (i. e. solutions which are extensible up to ∞) one can distinguish the so-called *eventually minimal* solution $\tilde{w}$, a solution with the property that any other proper solution w satisfies the inequality $w(t) > \tilde{w}(t)$ eventually. The solution $\tilde{x}$ of (2.1) given by the formula

$$x(t) = \exp\left\{\int^t r^{\frac{1}{1-\alpha}}(s)\varphi^{-1}(\tilde{w}(s))\,ds\right\}, \quad \text{i.e.} \quad \tilde{w}(t) = r(t)\frac{\varphi(\tilde{x}')}{\varphi(\tilde{x})},$$

where φ^{-1} is the inverse function of φ, is said to be the *principal* solution (at ∞) of (2.1). Clearly, the principal solution is determined uniquely up to a multiple by a nonzero real constant. Observe that if b is a regular point of (2.1) (i. e. the initial value problem $x(b) = x_0$, $r(b)\varphi(x'(b)) = x_1$ has the unique solution for any $x_0, x_1 \in \mathbb{R}$) and x_b is a solution given by the initial condition $x_b(b) = 0$, $r(b)\varphi(x_b'(b)) \neq 0$, then the associated solution $w_b = r\varphi(x_b')/\varphi(x_b)$ of (2.2) satisfies $w_b(b-) = -\infty$ and w_b is minimal solution of (2.2) in a left neighbourhooh of b. Hence, x_b can be regarded as the principal solution at (the regular point) b and the condition $\tilde{y}(a) = 0 = \tilde{y}(b)$ from Proposition 1.1 can be reformulated "$\tilde{y}$ is the principal solution at a and b". From this point of view the main result of our paper can be regarded as a singular version of Leighton's comparison theorem. For more details concerning principal solution of half-linear differential equations see [2, 4, 5, 7].

3. Singular Leighton's theorem

The main result of our paper reads as follows.

Theorem 3.1. *Suppose that the half-linear equation*

$$(R(t)\varphi(y'))' + C(t)\varphi(y) = 0 \tag{3.1}$$

is nonoscillatory both at a and b and the principal solutions at a an b of this equation coincide, denote this solution by $\tilde{y}$. Further suppose that $0 < r(t) \le R(t)$ near a and b and

$$\begin{aligned}\limsup_{t_1\downarrow a, t_2\uparrow b} \Big\{ \int_{t_1}^{t_2} (r(t) - R(t))|\tilde{y}'|^\alpha \, dt \\ - \inf_{s_1\le t_1, s_2\ge t_2} \int_{s_1}^{s_2} (M_{\alpha\beta}[c,f](t) - C(t))|\tilde{y}|^\alpha)\, dt\Big\} < 0\,.\end{aligned} \tag{3.2}$$

Then every solution of (1.1) *satisfying $x(t)f(t) \le 0$ has a zero in I.*

Proof. According to Lemma 2.2 it suffices to find $a_1, b_1 \in I$, $a_1 < b_1$, and a nontrivial piecewise C^1 function y such that $y(a_1) = 0 = y(b_1)$ and

$$\mathcal{F}(y; a_1, b_1) = \int_{a_1}^{b_1} [r(t)|y'|^\alpha - M_{\alpha\beta}[c,f](t)|y|^\alpha]\, dt \le 0.$$

In our proof we borrow some ideas used in the proof of the conjugacy criterion of [2].

Let $a < a_1 < t_1 < t_2 < b_1 < b$ (these values will be specified later) and let f, g be the solutions of (3.1) satisfying the boundary conditions

$$f(a_1) = 0\,,\ f(t_1) = \tilde{y}(t_1)\,, \quad g(t_2) = \tilde{y}(t_2)\,,\ g(b_1) = 0\,.$$

Note that such solutions exist if t_1 and t_2 are sufficiently close to a and b respectively (due to nonoscillation of (3.1) near a and b and the fact that the solution space of this equation is homogeneous). Define the function y as follows

$$y(t) = \begin{cases} 0 & t \in (a, a_1]\,, \\ f(t) & t \in [a_1, t_1]\,, \\ \tilde{y}(t) & t \in [t_1, t_2]\,, \\ g(t), & t \in [t_2, b_1]\,, \\ 0 & t \in [b_1, b)\,. \end{cases}$$

Then we have

$$\mathcal{F}(y;a_1,b_1)=\int_{a_1}^{b_1}[r(t)|y'|^\alpha-M_{\alpha\beta}[c,f](t)|y|^\alpha]\ dt$$

$$=\int_{a_1}^{b_1}[R(t)|y'|^\alpha-C(t)|y|^\alpha]\ dt+\int_{a_1}^{b_1}[(r(t)-R(t))|y'|^\alpha-(M_{\alpha\beta}[c,f](t)-C(t))|y|^\alpha]\ dt$$

$$=\int_{a_1}^{t_1}[R(t)|f'|^\alpha-C(t)|f|^\alpha]\ dt+\int_{a_1}^{t_1}[(r(t)-R(t))|f'|^\alpha-(M_{\alpha\beta}[c,f](t)-C(t))|f|^\alpha]\ dt$$

$$+\int_{t_1}^{t_2}[R(t)|\tilde{y}'|^\alpha-C(t)|\tilde{y}|^\alpha]\ dt+\int_{t_1}^{t_2}[(r(t)-R(t))|\tilde{y}'|^\alpha-(M_{\alpha\beta}[c,f](t)-C(t))|\tilde{y}|^\alpha]\ dt$$

$$+\int_{t_2}^{b_1}[R(t)|g'|^\alpha-C(t)|g|^\alpha]\ dt+\int_{t_2}^{b_1}[(r(t)-R(t))|g'|^\alpha-(M_{\alpha\beta}[c,f](t)-C(t))|g|^\alpha]\ dt.$$

Denote by v_f, v_g and $\tilde{v}$ the solutions of Riccati equation associated with (3.1)

$$v'+C(t)+(\alpha-1)R^{\frac{1}{1-\alpha}}(t)|v|^{\frac{\alpha}{\alpha-1}}=0 \tag{3.3}$$

generated by f, g and $\tilde{y}$, respectively, i.e.,

$$v_f=\frac{R\varphi(f')}{\varphi(f)}\,,\quad v_g=\frac{R\varphi(g')}{\varphi(g)}\,,\quad \tilde{v}=\frac{R\varphi(\tilde{y}')}{\varphi(\tilde{y})}\,.$$

Then using integration by parts

$$\int_{a_1}^{t_1}[R(t)|f'|^\alpha-C(t)|f|^\alpha]\ dt=Rf\varphi(f')\Big|_{a_1}^{t_1}-\int_{a_1}^{t_1}f\,[(R(t)\varphi(f'))'+C(t)\varphi(f)]\ dt$$
$$=v_f|f|^\alpha\Big|_{a_1}^{t_1}\ ,$$

similarly,

$$\int_{t_1}^{t_2}[R(t)|\tilde{y}'|^\alpha-C(t)|\tilde{y}|^\alpha]\ dt=\tilde{v}|\tilde{y}|^\alpha\Big|_{t_1}^{t_2}\ ,$$
$$\int_{t_2}^{b_1}[R(t)|g'|^\alpha-C(t)|g|^\alpha]\ dt=v_g|g|^\alpha\Big|_{t_2}^{b_1}\ .$$

Consequently,

$$\int_{a_1}^{b_1}[R(t)|y'|^\alpha\ -C(t)|y|^\alpha]\ dt=v_f|f|^\alpha\Big|_{a_1}^{t_1}+\tilde{v}|\tilde{y}|^\alpha\Big|_{t_1}^{t_2}+v_g|g|^\alpha\Big|_{t_2}^{b_1}$$
$$=|\tilde{y}(t_1)|^\alpha(v_f(t_1)-\tilde{v}(t_1))+|\tilde{y}(t_2)|^\alpha(\tilde{v}(t_2)-v_g(t_2))\,.$$

Next we deal with the integral $\int_{a_1}^{t_1}(M_{\alpha\beta}[c,f](t)-C(t))\,|f|^\alpha\,dt$. The function $\frac{f}{\tilde{y}}$ is monotonically increasing in (a_1,t_1) since $\frac{f}{\tilde{y}}(a_1)=0$, $\frac{f}{\tilde{y}}(t_1)=1$ and $\left(\frac{f}{\tilde{y}}\right)'=$

$\frac{f'\tilde{y}-f\tilde{y}'}{\tilde{y}^2} \neq 0$ in (a_1, t_1). Indeed, if $f'\tilde{y} - f\tilde{y}' = 0$ at some point $\tilde{t} \in (a_1, t_1)$, i.e. $\frac{f'}{f}(\tilde{t}) = \frac{\tilde{y}'}{\tilde{y}}(\tilde{t})$ then $v_f(\tilde{t}) = \tilde{v}(\tilde{t})$ which contradicts the unique solvability of (3.3). By the second mean value theorem of integral calculus now there exists $\xi_1 \in (a_1, t_1)$ such that

$$\int_{a_1}^{t_1} (M_{\alpha\beta}[c,f](t) - C(t))\,|f|^\alpha\,dt = \int_{a_1}^{t_1} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde{y}|^\alpha \frac{|f|^\alpha}{|\tilde{y}|^\alpha}\,dt$$
$$= \int_{\xi_1}^{t_1} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde{y}|^\alpha\,dt\,.$$

By the same argument $\frac{g}{\tilde{y}}$ is monotonically decreasing in (t_2, b_1) and

$$\int_{t_2}^{b_1} (M_{\alpha\beta}[c,f](t) - C(t))\,|g|^\alpha\,dt = \int_{t_2}^{\xi_2} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde{y}|^\alpha\,dt$$

for some $\xi_2 \in (t_2, b_1)$.

Therefore

$$\int_{a_1}^{b_1} (M_{\alpha\beta}[c,f](t) - C(t))\,|y|^\alpha\,dt = \int_{\xi_1}^{\xi_2} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde{y}|^\alpha\,dt\,.$$

Summarizing our computations and using the fact that $r(t) - R(t) \le 0$ on $[a_1, t_1]$ and $[t_2, b_1]$ if t_1, t_2 are sufficiently close to a and b, respectively, i.e.,

$$\int_{a_1}^{t_1} (r(t) - R(t))|f'|^\alpha\,dt \le 0\,, \qquad \int_{t_2}^{b_1} (r(t) - R(t))|g'|^\alpha\,dt \le 0\,,$$

we have

$$\mathcal{F}(y; a_1, b_1) \le |\tilde{y}(t_1)|^\alpha\,(v_f(t_1) - \tilde{v}(t_1)) + |\tilde{y}(t_2)|^\alpha\,(\tilde{v}(t_2) - v_g(t_2))$$
$$+ \int_{t_1}^{t_2} (r(t) - R(t))|\tilde{y}'|^\alpha\,dt - \int_{\xi_1}^{\xi_2} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde{y}|^\alpha\,dt\,.$$

According to (3.2) there exists $\varepsilon > 0$ such that

$$\int_{t_1}^{t_2} (r(t) - R(t))|\tilde{y}'|^\alpha\,dt - \int_{s_1}^{s_2} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde{y}|^\alpha\,dt < -\varepsilon$$

whenever $s_1 \in (a, t_1)$, $s_2 \in (t_2, b)$, if t_1, t_2 are sufficiently close to a and b, respectively. Further, since $\tilde{v}$ is generated by the solution $\tilde{y}$ of (3.1) which is principal both at $t = a$ and $t = b$, according to the “Riccati equation” construction of the principal solution mentioned in the previous section, we have (for t_1, t_2 fixed for a moment)

$$\lim_{a_1 \to a+} [v_f(t_1) - \tilde{v}(t_1)] = 0\,, \qquad \lim_{b_1 \to b-} [v_g(t_2) - \tilde{v}(t_2)] = 0\,,$$

(see [2] for details) observe that solutions v_f, v_g actually depend also on a_1 and b_1, respectively. Hence

$$|\tilde{y}(t_1)|^\alpha\,[v_f(t_1)-\tilde{v}(t_1)]<\frac{\varepsilon}{2}\,,\quad |\tilde{y}(t_2)|^\alpha\,[\tilde{v}(t_2)-v_g(t_2)]<\frac{\varepsilon}{2}$$

if $a_1 < t_1$, $b_1 > t_2$ are sufficiently close to a and b, respectively.

Consequently, for the above specified choice of $a_1 < t_1 < t_2 < b_1$ we have

$$\mathcal{F}(y;a_1,b_1)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}-\varepsilon=0$$

what we needed to prove. The statement now follows from Lemma 2.2. □

If $r(r) \le R(t)$ in the *whole* interval I (not only near a and b as supposed in Theorem 3.1), assumptions of Theorem 3.1 can be modified as follows.

Theorem 3.2. *Suppose that $r(t) \le R(t)$ on I, $\tilde{y}$ is the same as in Theorem 3.1,*

$$\liminf_{s_1\downarrow a,\, s_2\uparrow b}\int_{s_1}^{s_2}\{M_{\alpha\beta}[c,f](t)-C(t)\}\,|\tilde{y}|^\alpha\,dt\ge 0 \tag{3.4}$$

and

$$M_{\alpha\beta}[c,f](t)\not\equiv C(t)\quad in\quad I. \tag{3.5}$$

Then every solution of (1.1) *satisfying $x(t)f(t) \le 0$ in I has a zero in this interval.*

Proof. We will proceed similarly as in the proof of the previous theorem. Continuity of the functions $C, M_{\alpha\beta}[c,f]$ and (3.5) imply the existence of $\bar{t} \in I$ and $d, \varrho > 0$ such that $(M_{\alpha\beta}[c,f](t) - C(t))|\tilde{y}(t)|^\alpha > d$ for $(\bar{t}-\varrho, \bar{t}+\varrho)$, and let Δ be any positive differentiable function with the compact support in $(\bar{t}-\varrho, \bar{t}+\varrho)$. Further let $a < t_1 < \bar{t}-\varrho < \bar{t}+\varrho < t_2 < b_1 < b$ and define the function y as follows

$$y(t)=\begin{cases}0 & t\in(a,a_1]\,,\\ f(t) & t\in[a_1,t_1]\,,\\ \tilde{y}(t) & t\in[t_1,t_2]\backslash[\bar{t}-\varrho,\bar{t}+\varrho]\,,\\ \tilde{y}(t)(1+\delta\Delta(t)), & t\in[\bar{t}-\varrho,\bar{t}+\varrho]\,,\\ g(t), & t\in[t_2,b_1]\,,\\ 0 & t\in[b_1,b)\,,\end{cases}$$

where δ is a real parameter. We have

$$\begin{aligned}\mathcal{F}(y;a_1,b_1)&=\int_{a_1}^{b_1}[r(t)|y'|^\alpha-M_{\alpha\beta}[c,f](t)|y|^\alpha]\,dt\\ &=\int_{a_1}^{b_1}[R(t)|y'|^\alpha-C(t)|y|^\alpha]\,dt\\ &\quad+\int_{a_1}^{b_1}[(r(t)-R(t))|y'|^\alpha-(M_{\alpha\beta}[c,f](t)-C(t))|y|^\alpha]\,dt\\ &\le\int_{a_1}^{b_1}[R(t)|y'|^\alpha-C(t)|y|^\alpha]\,dt-\int_{a_1}^{b_1}(M_{\alpha\beta}[c,f](t)-C(t))|y|^\alpha\,dt\,.\end{aligned}$$

Computation of the integrals over $[a_1, t_1]$ and $[t_2, b_1]$ is the same as in the proof of Theorem 3.1. Concerning the interval $[t_1, t_2]$, we have by Lemma 2.2 with $\beta = \alpha$ and $f \equiv 0$

$$
\begin{aligned}
&\int_{t_1}^{t_2} [R(t)|y'|^\alpha - C(t)|y|^\alpha]\, dt \\
&\quad = \tilde v|\tilde y|^\alpha \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \Big[R(t)|y'|^\alpha - \alpha\tilde v(t) y'\varphi(y) + (\alpha-1)R^{\frac{1}{1-\alpha}}(t)|\tilde v(t)|^{\frac{\alpha}{\alpha-1}}|y|^\alpha\Big]\, dt \\
&\quad = \tilde v|\tilde y|^\alpha \Big|_{t_1}^{t_2} + \int_{\bar t-\varrho}^{\bar t+\varrho} \Big\{R(t)|\tilde y' + \delta(\Delta\tilde y)'|^\alpha - \alpha R(t)\frac{\varphi(\tilde y')}{\varphi(\tilde y)}(\tilde y' + \delta(\tilde y\Delta)')\tilde y^{\alpha-1} \\
&\qquad \times (1+\delta\Delta)^{\alpha-1} + (\alpha-1)R^{\frac{1}{1-\alpha}}(t)\left|\frac{R(t)\varphi(\tilde y')}{\varphi(\tilde y)}\right|^{\frac{\alpha}{\alpha-1}} \tilde y^\alpha(1+\delta\Delta)^\alpha\Big\}\, dt \\
&\quad = \tilde v|\tilde y|^\alpha \Big|_{t_1}^{t_2} + \int_{\bar t-\varrho}^{\bar t+\varrho} R(t)\,\{|\tilde y'|^\alpha + \alpha\delta(\Delta\tilde y)'\varphi(\tilde y') + o(\delta) - \alpha(\tilde y' + \delta(\Delta\tilde y)')\varphi(\tilde y') \\
&\qquad \times (1 + (\alpha-1)\delta\Delta + o(\delta)) + (\alpha-1)|\tilde y'|^\alpha(1 + \alpha\delta\Delta + o(\delta))\}\, dt \\
&\quad = \tilde v|\tilde y|^\alpha \Big|_{t_1}^{t_2} + \int_{\bar t-\varrho}^{\bar t+\varrho} R(t)\,\{|\tilde y'|^\alpha + \alpha\delta(\Delta\tilde y)'\varphi(\tilde y') - \alpha|\tilde y'|^\alpha - \alpha\delta\varphi(\tilde y')(\Delta\tilde y)' \\
&\qquad - \alpha(\alpha-1)\delta\Delta|\tilde y'|^\alpha + (\alpha-1)|\tilde y'|^\alpha + (\alpha-1)\alpha\delta\Delta|\tilde y'|^\alpha + o(\delta)\}\, dt \\
&\quad = \tilde v|\tilde y|^\alpha \Big|_{t_1}^{t_2} + o(\delta)\,.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\int_{a_1}^{b_1} [R(t)|y'|^\alpha - C(t)|y|^\alpha]\, dt &= w_f|f|^\alpha \Big|_{a_1}^{t_1} + \tilde v|\tilde y|^\alpha \Big|_{t_1}^{t_2} + w_g|g|^\alpha \Big|_{t_2}^{b_1} + o(\delta) \\
&= |\tilde y(t_1)|^\alpha(w_f(t_1) - \tilde v(t_1)) + |\tilde y(t_2)|^\alpha(\tilde v(t_2) - w_g(t_2)) + o(\delta)
\end{aligned}
$$

as $\delta \to 0+$. Further,

$$
\begin{aligned}
\int_{t_1}^{t_2} (M_{\alpha\beta}[c,f](t) - C(t))\,|y|^\alpha\, dt &= \int_{t_1}^{\bar t-\varrho} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde y|^\alpha \\
&\quad + \int_{\bar t-\varrho}^{\bar t+\varrho} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde y|^\alpha(1+\delta\Delta)^\alpha\, dt \\
&\quad + \int_{\bar t+\varrho}^{t_2} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde y|^\alpha\, dt \\
&= \int_{t_1}^{t_2} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde y|^\alpha\, dt \\
&\quad + \delta\alpha\int_{\bar t-\varrho}^{\bar t+\varrho} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde y|^\alpha\Delta(t)\, dt + o(\delta) \\
&\ge \int_{t_1}^{t_2} (M_{\alpha\beta}[c,f](t) - C(t))\,|\tilde y|^\alpha\, dt + \delta K + o(\delta)\,,
\end{aligned}
$$

where $K = \alpha \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} (M_{\alpha\beta} - C)|\tilde{y}|^{\alpha}\, dt > 0$.

Summarizing the previous computations, we have

$$\begin{aligned}\mathcal{F}(y; a_1, b_1) &\le |\tilde{y}(t_1)|^{\alpha}(v_f(t_1) - \tilde{v}(t_1)) + |\tilde{y}(t_2)|^{\alpha}(\tilde{v}(t_2) - v_g(t_2)) \\ &\quad - \int_{\xi_1}^{\xi_2} \left(M_{\alpha\beta}[c,f](t) - C(t)\right) |\tilde{y}|^{\alpha}\, dt - (K\delta + o(\delta)) .\end{aligned}$$

Now, let $\delta > 0$ (sufficiently small) be such that $K\delta + o(\delta) =: \varepsilon > 0$. According to (3.4) the points t_1, t_2 can be chosen in such a way that

$$\int_{s_1}^{s_2} \left(M_{\alpha\beta}[c,f](t) - C(t)\right) |\tilde{y}|^{\alpha}\, dt > -\frac{\varepsilon}{4}$$

whenever $s_1 \in (a, t_1)$, $s_2 \in (t_2, b)$. Consequently, for suitably chosen $a_1 < t_1 < t_2 < b_1$, using the estimates from the proof of Theorem 3.1 we have then

$$\mathcal{F}(y; a_1, b_1) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} - \varepsilon < 0$$

and the statement follows from Lemma 2.2. □

Corollary 3.1. *Suppose that* $I = \mathbb{R} = (-\infty, \infty)$*,*

$$\int_{-\infty} r^{\frac{1}{1-\alpha}}(t)\, dt = \infty = \int^{\infty} r^{\frac{1}{1-\alpha}}(t)\, dt\,, \tag{3.6}$$

and

$$\liminf_{s_1 \downarrow -\infty, s_2 \uparrow \infty} \int_{s_1}^{s_2} M_{\alpha\beta}[c,f](t)\, dt \ge 0\,, \quad M_{\alpha\beta}[c,f](t) \not\equiv 0 \quad \textit{for} \quad t \in \mathbb{R}\,.$$

Then every solution of (1.1) *satisfying* $x(t)f(t) \le 0$ *for* $t \in \mathbb{R}$ *has at least one zero in* $\mathbb{R}$*.*

Proof. Let $R(t) \equiv r(t)$ and $C(t) \equiv 0$ in (3.1). Divergence of the integrals in (3.6) implies that $\tilde{y} \equiv 1$ is the principal solution of $(R(t)\varphi(y'))' = 0$ at $\pm\infty$, see e. g. [2]. The statement now follows from Theorem 3.2. □

Corollary 3.2. *Suppose that* $I = (0, \infty)$*,*

$$\liminf_{s_1 \downarrow 0, s_2 \uparrow \infty} \int_{s_1}^{s_2} \left[M_{\alpha\beta}[c,f](t) - \left(\frac{\alpha - 1}{\alpha}\right)^{\alpha} t^{-\alpha} \right] t^{\alpha - 1}\, dt \ge 0$$

and

$$M_{\alpha\beta}[c,f](t) - \left(\frac{\alpha - 1}{\alpha}\right)^{\alpha} t^{-\alpha} \not\equiv 0 \quad \textit{in} \quad (0, \infty)\,.$$

Then every solution of (1.1) *with* $r(t) \equiv 1$ *satisfying* $x(t)f(t) \le 0$ *for* $t \in (0, \infty)$ *has a zero in this interval.*

Proof. Let $R(t) \equiv 1$ and $C(t) = \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} t^{-\alpha}$. Then (3.1) is the generalized Euler equation

$$(\varphi(y'))' + \frac{\gamma_\alpha}{t^\alpha}\varphi(y) = 0, \quad \gamma_\alpha = \left(\frac{\alpha - 1}{\alpha}\right)^{\alpha} .$$

This equation is disconjugate on $I = (0, \infty)$ and $\tilde{y} = t^{\frac{\alpha-1}{\alpha}}$ is its principal solution at $t = 0$ and $t = \infty$, see [3]. The statement now again follows from Theorem 3.2. □

Corollary 3.3. *Suppose that* $r(t) \geq (1+|t|^\alpha)^{\alpha-1}$ *on* $\mathbb{R}$,

$$\liminf_{t_1\downarrow -\infty, t_2\uparrow\infty} \int_{t_1}^{t_2} \left\{ M_{\alpha\beta}[c,f](t) - \frac{\alpha-1}{1+|t|^\alpha} \right\} \frac{1}{(1+|t|^\alpha)^{\alpha-1}}\, dt \geq 0$$

and

$$M_{\alpha\beta}[c,f](t) - \frac{\alpha-1}{1+|t|^\alpha} \not\equiv 0 \quad in \quad \mathbb{R}\,.$$

Then every solution of (1.1) *satisfying* $x(t)f(t) \leq 0$ *in* $\mathbb{R}$ *has a zero point in* $\mathbb{R}$.

Proof. Consider the equation

$$\left((1+|t|^\alpha)^{\alpha-1}\varphi(y')\right)' + \frac{\alpha-1}{1+|t|^\alpha}\varphi(y) = 0\,. \tag{3.7}$$

The transformation of independent variable $x(t) = y(\arctan_\alpha t)$, where

$$\arctan_\alpha t = \int_0^t \frac{ds}{1+|s|^\alpha}\,,$$

transforms (3.7) into the equation

$$(\varphi(x'))' + (\alpha-1)\varphi(x) = 0\,.$$

The last equation has been extensively studied by Elbert [6]. Denote by $\sin_\alpha t$ its unique solution given by the initial condition $y(0)=0$, $y'(0)=1$. Further denote $\cos_\alpha t := (\sin_\alpha t)'$, $\tan_\alpha t := \frac{\sin_\alpha t}{\cos_\alpha t}$. The first positive zero of the function $\sin_\alpha$ is $\pi_\alpha := 2\frac{\pi/\alpha}{\sin(\pi/\alpha)}$, $\cos_\alpha t = 0$ for $t = \pm\frac{\pi_\alpha}{2}$ and the function $\tan_\alpha$ (which is the inverse function of $\arctan_\alpha$) maps the interval $\left(-\frac{\pi_\alpha}{2}, \frac{\pi_\alpha}{2}\right)$ onto $\mathbb{R}$. This implies that $\tilde{y}(t) = \cos_\alpha(\arctan_\alpha t) = (1+|t|^\alpha)^{-1}$ is the principal solution of (3.7) at $\pm\infty$. The statement now follows from Theorem 3.2. □

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