ARCHIVUM MATHEMATICUM (BRNO)
Tomus 40 (2004), 47 – 61

THE TANAKA–WEBSTER CONNECTION FOR ALMOST $\mathcal{S}$-MANIFOLDS AND CARTAN GEOMETRY

ANTONIO LOTTA AND ANNA MARIA PASTORE

Abstract. We prove that a CR-integrable almost $\mathcal{S}$-manifold admits a canonical linear connection, which is a natural generalization of the Tanaka–Webster connection of a pseudo-hermitian structure on a strongly pseudoconvex CR manifold of hypersurface type. Hence a CR-integrable almost $\mathcal{S}$-structure on a manifold is canonically interpreted as a reductive Cartan geometry, which is torsion free if and only if the almost $\mathcal{S}$-structure is normal. Contrary to the CR-codimension one case, we exhibit examples of non normal almost $\mathcal{S}$-manifolds with higher CR-codimension, whose Tanaka–Webster curvature vanishes.

1. Introduction

In [3] D. E. Blair initiated the study of the differential geometry of manifolds carrying an $U(k) \times O(s)$-structure. These are exactly the manifolds M which admit an f-structure, i.e. a tensor field φ of type $(1,1)$ with constant rank $2k$, and such that $\varphi^3 + \varphi = 0$. This kind of structure was investigated first by K. Yano in [15]. An f–structure provides a splitting of the tangent bundle

$$TM = \mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\varphi)$$

and the restriction J of φ to $\mathcal{D} = \mathrm{Im}(\varphi)$ is a partial complex structure, that is $J^2 = -\mathrm{Id}$. Hence M is an almost CR manifold having CR-dimension k and CR-codimension $s = n - 2k$, where $n = \dim_{\mathbf{R}} M$. Actually, an f-structure is equivalent to an almost CR structure $(\mathcal{D}, J)$ together with the choice of a complementary subbundle to $\mathcal{D}$ in TM. Here we restrain our attention to the case where the subbundle $\mathrm{Ker}(\varphi)$ is trivial, i.e. the structure group can be further reduced to $U(k) \times I_s$. In this case M is called an f-manifold with parallelizable kernel ($f{\cdot}$pk manifold). From the CR point of view, this is equivalent to the triviality of the annihilator $\mathcal{D}^0 M$ of the analytic tangent bundle $\mathcal{D}$, which is the subbundle of the cotangent bundle T^*M whose fiber is $\mathcal{D}^0_x M = \{\eta \in T^*_x M \mid \eta(X) = 0 \ \forall X \in \mathcal{D}_x\}$.

2000 *Mathematics Subject Classification*: 53C25, 53C15, 53B05, 32V05.
Key words and phrases: almost $\mathcal{S}$-structure, Tanaka–Webster connection, Cartan connection, CR manifold.
Received April 2, 2002.

Notice that $\mathcal{D}^0 M$ is automatically trivial for any orientable almost CR manifold of hypersurface type ($s = 1$); in this case, a trivialization η of $\mathcal{D}^0 M$ is usually called a pseudohermitian structure and $(M, \mathcal{D}, J, \eta)$ is called a pseudohermitian manifold.

A *metric $f{\cdot}pk$ manifold* is an $f{\cdot}$pk manifold endowed with a Riemannian metric g such that

$$g(X,Y) = g(\varphi X, \varphi Y) + \sum_{i=1}^{s} \eta^i(X)\eta^i(Y) \tag{1}$$

where $\{\eta^i\}_{i=1,\dots,s}$ is a fixed trivialization of $\mathcal{D}^0 M$. Notice that φ is then skew--symmetric with respect to g.

In [4] an *almost $\mathcal{S}$-manifold* is defined as a metric $f{\cdot}$pk manifold such that

$$d\eta^i = \Phi \qquad i = 1, \dots, s \tag{2}$$

where Φ is the fundamental 2-form of the $f{\cdot}$pk structure, defined as usual by $\Phi(X,Y) = g(X, \varphi Y)$.

This notion is a natural generalization of the concept of contact metric structure, which corresponds to the case $s = 1$ (cf. [2]).

It is known that an orientable almost CR manifold $(M, \mathcal{D}, J)$ of hypersurface type is an almost $\mathcal{S}$-manifold with underlying almost CR structure $(\mathcal{D}, J)$ if and only if *i*) J is partially integrable, i.e. $[X,Y] - [JX, JY] \in \mathcal{D}$ for all sections X, Y of $\mathcal{D}$, and *ii*) a pseudohermitian structure η can be chosen with positive definite Levi form $\mathcal{L}_\eta$. Recall that $\mathcal{L}_\eta$ is defined by $\mathcal{L}_\eta(X,Y) = d\eta(JX, Y)$ for all $X, Y \in \mathcal{D}$. When these two conditions are satisfied, a pseudohermitian structure η as in *ii*) uniquely determines an f-structure φ extending J and a compatible metric g satisfying the above conditions (1) and (2) with $\eta^1 = \eta$. If moreover *i*) is replaced by CR-integrability, $(M, \mathcal{D}, J)$ is called a strongly pseudoconvex CR manifold (see e.g. [11]).

The strongly pseudoconvex CR manifolds have been investigated by several authors, and one of their fundamental properties is the existence of a unique linear connection $\tilde{\nabla}$ such that the tensors φ, η, g are all $\tilde{\nabla}$-parallel and whose torsion satisfies

$$\tilde{T}(X,Y) = 2\Phi(X,Y)\xi \quad \text{ for all } \quad X, Y \in \mathcal{D}, \tag{3}$$

$$\tilde{T}(\xi, \varphi X) = -\varphi \tilde{T}(\xi, X) \quad \text{ for all } \quad X \in \mathcal{X}(M)\,. \tag{4}$$

Here ξ is the dual vector field of η with respect to the metric g.

This connection was introduced first by N. Tanaka in [10], and independently by Webster in [14]. We remark that $\tilde{\nabla}$ actually depends not only on the CR structure but also on the choice of the pseudohermitian structure η.

In this paper we provide a geometrical characterization of condition (2), showing that a metric $f{\cdot}$pk manifold admits a connection $\tilde{\nabla}$ having the same formal properties as (3)-(4) (cf. (6)–(7) in sec. 2), with the additional requirement that $\tilde{T}$ vanishes on $\mathrm{Ker}(\varphi)$, if and only if (2) holds and the almost CR structure $(\mathcal{D}, J)$ is integrable. This connection is uniquely determined and hence we call it the Tanaka–Webster connection of a CR-integrable almost $\mathcal{S}$-manifold.

This result is also interpreted from the point of view of Cartan's method of equivalence, showing that the datum of a CR-integrable almost $\mathcal{S}$-structure on a manifold admits a canonical interpretation as a reductive Cartan geometry (cf. [9]).

We also obtain that, as a Cartan geometry, a CR-integrable almost $\mathcal{S}$-structure is *torsion free* if and only if the tensor field

$$N = [\varphi, \varphi] + 2d\eta^i \otimes \xi_i$$

vanishes, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ, while $\{\xi_i\}$ is the g-orthonormal frame of $\mathrm{Ker}(\varphi)$ dual of $\{\eta^i\}$. This is the *normality condition* considered by Blair in [3], where an almost $\mathcal{S}$-manifold satisfying $N = 0$ is called an $\mathcal{S}$-manifold.

Finally, we exhibit examples of $\tilde{\nabla}$-flat *non* normal almost $\mathcal{S}$-manifolds with CR codimension $s > 1$. This is interesting since it is easily seen that a strongly pseudoconvex CR manifold of hypersurface type with vanishing Tanaka–Webster curvature is necessarily normal.

Acknowledgement. The authors are grateful to the referee for valuable suggestions and remarks, especially as regards the examples in the last section.

2. The Tanaka–Webster connection of a CR-integrable almost $\mathcal{S}$-manifold

Let M^{2k+s} be a metric f·pk manifold with structure $(\varphi, \xi_i, \eta^i, g)$.

Let ∇ be the Levi–Civita connection of g. Denote by Q the tensor field of type $(1, 2)$ on M defined by

$$\begin{aligned} Q(X,Y) := (\nabla_X \varphi)Y + \Phi(X, \varphi Y)\bar{\xi} - g(h_j X, Y)\xi_j \\ - \bar{\eta}(Y)\varphi^2 X + \eta^j(Y) h_j X \,. \end{aligned} \tag{5}$$

Here and in the following the sum symbol for repeated indices is omitted. In this formula $\bar{\xi} := \sum_{i=1}^{s} \xi_i$, $\bar{\eta} := \sum_{i=1}^{s} \eta_i$, while h_i is the operator $h_i = \frac{1}{2}\mathcal{L}_{\xi_i}\varphi$. Φ denotes the fundamental 2-form defined by $\Phi(X, Y) = g(X, \varphi Y)$.

For basic properties of almost $\mathcal{S}$-manifolds, we refer the reader to [4]. In particular, we have the following:

Proposition 2.1 ([4]). *Assume that M is an almost $\mathcal{S}$-manifold. Then:*

1) *Each h_i is a self-adjoint operator anti-commuting with φ.*
2) *Each h_i vanishes on* $\mathrm{Ker}(\varphi)$ *and takes values in $\mathcal{D}$.*
3) *For each $i, j = 1, \ldots, s$ we have*

$$\nabla_{\xi_i}\varphi = 0,\ \nabla_{\xi_i}\xi_j = 0\,,$$
$$\nabla_X \xi_i = -\varphi(X) - \varphi h_i(X)\,.$$

4) *M is CR-integrable, that is the partial complex structure J induced by φ on $\mathcal{D} = \mathrm{Im}(\varphi)$ is formally integrable, if and only if $Q \equiv 0$.*

In this section we prove the following geometric characterization of the CR--integrable almost $\mathcal{S}$-manifolds:

Theorem 2.2. *Let M be a metric $f\cdot pk$-manifold with structure $(\varphi, \xi_i, \eta^i, g)$. Then M is a CR–integrable almost $\mathcal{S}$-manifold if and only if it admits a linear connection $\tilde{\nabla}$ with the following properties:*

1) $\tilde{\nabla}\varphi = 0$, $\tilde{\nabla} g = 0$ *and* $\tilde{\nabla}\eta^i = 0$ *for each* $i \in \{1, \ldots, s\}$*;*
2) *The torsion $\tilde{T}$ of $\tilde{\nabla}$ satisfies:*

$$\tilde{T}(X, Y) = 2\Phi(X, Y)\bar{\xi} \quad \text{for all} \quad X, Y \in \mathcal{D}\,, \tag{6}$$

$$\tilde{T}(\xi_i, \varphi X) = -\varphi \tilde{T}(\xi_i, X) \quad \text{for all} \quad X \in \mathcal{X}(M)\,, \quad i \in \{1, \ldots, s\}\,, \tag{7}$$

$$\tilde{T}(\xi_i, \xi_j) = 0\,, \quad i, j \in \{1, \ldots, s\}\,. \tag{8}$$

Such a linear connection $\tilde{\nabla}$ is uniquely determined.

Notice that in the case $s = 1$, condition (8) is vacuous, and a CR-integrable almost $\mathcal{S}$-manifold is a strictly pseudoconvex CR manifold of hypersurface type (cf. e.g. [11], [13]); hence $\tilde{\nabla}$ coincides with the Tanaka–Webster connection (cf. [10], [13], [11]). For this reason, we shall adopt the name Tanaka–Webster connection to refer to $\tilde{\nabla}$ also in the higher CR-codimension case.

We remark that the factor 2 in (6) appears since we follow the convention of [5] for the exterior derivative (the same convention is adopted in Blair's book [2]).

To prove Theorem 2.2 we start by defining a tensor field H of type $(1, 2)$, $H : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, such that

$$\begin{aligned} H(X, Y) = {} & \Phi(X, Y)\bar{\xi} + \bar{\eta}(Y)\varphi(X) + \bar{\eta}(X)\varphi(Y) \\ & + \Phi(h_j X, Y)\xi_j + \eta^j(Y)\varphi h_j(X)\,. \end{aligned}$$

Lemma 2.3. *For all $X, Y, Z \in \mathcal{X}(M)$ we have:*

$$g(H(X, Y), Z) + g(H(X, Z), Y) = 0\,; \tag{9}$$

moreover, if M is an almost $\mathcal{S}$-manifold:

$$H(X, Y) - H(Y, X) = 2\Phi(X, Y)\bar{\xi} + \eta^j(Y)\varphi h_j(X) - \eta^j(X)\varphi h_j(Y) \tag{10}$$

Proof. Notice that for all $X, Y, Z \in \mathcal{X}(M)$ we have

$$\begin{aligned} g(H(X, Y), Z) = {} & \Phi(X, Y)\bar{\eta}(Z) + \Phi(Z, X)\bar{\eta}(Y) + \Phi(Z, Y)\bar{\eta}(X) \\ & + \Phi(h_j X, Y)\eta^j(Z) + \Phi(Z, h_j X)\eta^j(Y)\,; \end{aligned}$$

interchanging Y and Z in this formula we get

$$\begin{aligned} g(H(X, Z), Y) = {} & \Phi(X, Z)\bar{\eta}(Y) + \Phi(Y, X)\bar{\eta}(Z) + \Phi(Y, Z)\bar{\eta}(X) \\ & + \Phi(h_j X, Z)\eta^j(Y) + \Phi(Y, h_j X)\eta^j(Z)\,, \end{aligned}$$

and (9) follows. To prove (10), it suffices to observe that, assuming that M is an almost $\mathcal{S}$ manifold, then the operators h_j are self–adjoint and they anti-commute with φ; this yields

$$\Phi(h_j X, Y) = \Phi(h_j Y, X)$$

for all $X, Y \in \mathcal{X}(M)$, and this implies (10). □

Lemma 2.4. *Assume that M admits a linear connection $\tilde\nabla$ satisfying properties* 1), 2) *stated in Theorem 2.2. Then we have:*

i) $\tilde\nabla\xi_i = 0 \quad i \in \{1,\dots,s\}$;

ii) $\tilde\nabla_Z X \in \mathcal{D}$ *for all $X \in \mathcal{D}$ and $Z \in \mathcal{X}(M)$;*

iii) $[\xi_i,\mathcal{D}] \subset \mathcal{D}$;

iv) *For all $X \in \mathcal{X}(M)$ and for each $i \in \{1,\dots,s\}$, we have*

$$\tilde T(\xi_i, X) = -\varphi h_i(X) = -\frac{1}{2}N(X,\xi_i)\,. \tag{11}$$

Proof. i) Since $\tilde\nabla$ is metric, we get, for all $X, Y \in \mathcal{X}(M)$:

$$\begin{aligned} g(\tilde\nabla_X\xi_i, Y) &= X\cdot g(\xi_i, Y) - g(\xi_i, \tilde\nabla_X Y)\\ &= X\cdot\eta^i(Y) - \eta^i(\tilde\nabla_X Y) = (\tilde\nabla_X\eta^i)(Y) = 0\,. \end{aligned}$$

ii) This is clear since

$$\eta^i(\tilde\nabla_Z X) = Z\cdot\eta^i(X) - (\tilde\nabla_Z\eta^i)X = 0\,;$$

iii) Expanding formula (7), and using i), we have

$$\tilde\nabla_{\xi_i}\varphi X - [\xi_i,\varphi X] = -\varphi\tilde\nabla_{\xi_i}X + \varphi[\xi_i, X]\,;$$

using $\tilde\nabla\varphi = 0$, this equation can be rewritten as follows:

$$2\varphi(\tilde\nabla_{\xi_i}X) = [\xi_i,\varphi X] + \varphi[\xi_i, X]\,. \tag{12}$$

Notice that this formula implies that for all $X \in \mathcal{X}(M)$, we have $[\xi_i,\varphi X] \in \mathcal{D}$, thus proving iii). Now, assume that $X \in \mathcal{D}$; applying φ to both sides of (12), we get

$$-2\tilde\nabla_{\xi_i}X = \varphi[\xi_i,\varphi X] - [\xi_i, X]$$

which implies

$$\tilde T(\xi_i, X) = -\frac{1}{2}\{\varphi[\xi_i,\varphi X] + [\xi_i, X]\}\,. \tag{13}$$

On the other hand, by definition

$$h_i(X) = \frac{1}{2}\{[\xi_i,\varphi X] - \varphi[\xi_i, X]\}$$

so that

$$\varphi h_i(X) = \frac{1}{2}\{\varphi[\xi_i,\varphi X] + [\xi_i, X]\}\,.$$

This proves the equality

$$\tilde T(\xi_i, X) = -\varphi h_i(X)$$

for $X \in \mathcal{D}$. Since by hypothesis $\tilde T(\xi_i,\xi_j) = 0$, in force of i) we also have $[\xi_i,\xi_j] = 0$, and this gives $h_i(\xi_j) = 0$. Hence we conclude that the above equality is actually valid for all $X \in \mathcal{X}(M)$. The lemma is proved. □

Proof of Theorem 2.2. Define a linear connection $\tilde\nabla$ on M by

$$\tilde\nabla := \nabla + H \tag{14}$$

where ∇ is the Levi–Civita connection relative to g. We have

$$\begin{aligned}(\tilde\nabla_X\varphi)Y &= (\nabla_X\varphi)Y + H(X,\varphi Y) - \varphi H(X,Y)\\ &= (\nabla_X\varphi)Y + \Phi(X,\varphi Y)\bar\xi + g(h_jX,\varphi^2Y)\xi_j\\ &\quad - \overline{\eta}(Y)\varphi^2X - \eta^j(Y)\varphi^2h_jX\\ &= Q(X,Y) + \eta^k(Y)(\eta^k(h_jX) - \eta^j(h_kX))\xi_j\,.\end{aligned}$$

Notice that when M is an almost $\mathcal{S}$-manifold, according to Prop. 2.1, since the operators h_j take values in $\mathcal{D}$, the above formula simplifies to

$$\tilde\nabla\varphi = Q\,. \tag{15}$$

Now, assume that M is a CR-integrable almost $\mathcal{S}$-manifold. Then $Q = 0$, and (15) yields $\tilde\nabla\varphi = 0$. Moreover, since $\nabla g = 0$, it is an immediate consequence of (9) that $\tilde\nabla g = 0$. Using the formula (Prop. 2.1)

$$\nabla_X\xi_i = -\varphi(X) - \varphi h_i(X)\,,$$

we also get

$$\begin{aligned}(\tilde\nabla_X\,\eta^i)Y &= Xg(Y,\xi_i) - \eta^i(\tilde\nabla_XY)\\ &= g(\nabla_XY,\xi_i) + g(Y,\nabla_X\xi_i) - \eta^i(\nabla_XY) - \eta^i(H(X,Y))\\ &= -g(Y,\varphi X) - g(Y,\varphi h_iX) - g(H(X,Y),\xi_i) = 0\,.\end{aligned}$$

Finally, notice that

$$\tilde T(X,Y) = H(X,Y) - H(Y,X)\,;$$

by virtue of (10), taking into account that each h_i vanishes on $\mathrm{Ker}(\varphi)$, this implies that $\tilde T$ has properties (6)–(8). We have thus proved the existence of a linear connection having the properties stated in the theorem, under the assumption that M is a CR-integrable almost $\mathcal{S}$-manifold. To show the converse, we first prove that the equations

$$d\eta^i(X,Y) = \Phi(X,Y) \quad i\in\{1,\dots,s\} \tag{16}$$

hold as a consequence of the existence of $\tilde\nabla$. Indeed, if $X,Y\in\mathcal{D}$, from (6) we have

$$\tilde\nabla_XY - \tilde\nabla_YX - [X,Y] = 2\Phi(X,Y)\bar\xi\,,$$

which gives

$$g(\tilde\nabla_XY,\xi_i) - g(\tilde\nabla_YX,\xi_i) - \eta^i([X,Y]) = 2\Phi(X,Y)\,.$$

Observe that, since $\tilde\nabla$ is metric and ξ_i is parallel with respect to $\tilde\nabla$, we have $g(\tilde\nabla_XY,\xi_i) = g(\tilde\nabla_YX,\xi_i) = 0$. Hence $\eta^i([X,Y]) = -2\Phi(X,Y)$ and this shows that (16) holds for $X,Y\in\mathcal{D}$. Using iii) in the above lemma, we also get $d\eta^i(\xi_k,X) = 0 = \Phi(\xi_k,X)$ for $X\in\mathcal{D}$ and since $[\xi_k,\xi_j] = 0$ (see the proof of iv) in the same lemma), we also have $d\eta^i(\xi_k,\xi_j) = 0 = \Phi(\xi_k,\xi_j)$. These facts imply (16), that is M is an almost $\mathcal{S}$-manifold. To conclude the proof of the theorem, we make the following

Claim: *Let $\tilde\nabla$ be a linear connection satisfying conditions* 1) *and* 2) *in Theorem 2.2; then $\tilde\nabla$ is given by formula* (14).

Clearly, this implies the uniqueness assertion about $\tilde{\nabla}$. Moreover, since M is an almost $\mathcal{S}$-manifold, using (15) again, we get $Q = 0$, that is M is CR–integrable. To prove the claim, set $\nabla' := \tilde{\nabla} - H$; then ∇' is a linear connection. We just have to verify that ∇' is metric and without torsion. Since $\tilde{\nabla}$ is metric, we obtain

$$Xg(Y,Z) = g(\nabla'_X Y, Z) + g(Y, \nabla'_X Z) + g(H(Z,X),Y) + g(Y, H(X,Z))$$

for all $X, Y, Z \in \mathcal{X}(M)$, and in force of (9) this implies that ∇' is metric. Clearly, the condition that ∇' be torsionless is equivalent to

$$\tilde{T}(X,Y) = H(X,Y) - H(Y,X)\,;$$

taking into account (10), the validity of this equation is an immediate consequence of the formulas

$$\tilde{T}(X,Y) = 2\Phi(X,Y)\bar{\xi}\,, \quad \tilde{T}(\xi_i, Z) = -\varphi h_i(Z)\,, \quad X, Y \in \mathcal{D},\ Z \in \mathcal{X}(M)$$

which hold by assumption on $\tilde{\nabla}$ and by virtue of Lemma 2.4. This completes the proof of Theorem 2.2. □

Corollary 2.5. *Let M be a CR-integrable almost $\mathcal{S}$-manifold with Tanaka–Webster connection $\tilde{\nabla}$. Then M is normal, i.e. the tensor $N = [\varphi,\varphi] + 2d\eta^i \otimes \xi_i$ vanishes, if and only if*

$$\tilde{T}(\xi_i, X) = 0, \quad \textit{for all} \quad X \in \mathcal{D},\ i \in \{1, \dots, s\}\,.$$

We end this section with a remark on the relationship between Theorem 2.2 and a result of R. Mizner [8]. Let M be an almost $\mathcal{S}$-manifold with structure $(\varphi, \xi_i, \eta^i, g)$. Denote by $TM^{\mathbf{C}}$ the complexified tangent bundle of M, and let $\mathcal{H}$ be the complex version of the almost CR structure $(\mathcal{D}, J)$, namely the distribution $\mathcal{H} \subset TM^{\mathbf{C}}$ defined by

$$\mathcal{H}_p = \{Z \in \mathcal{D}_p^{\mathbf{C}} \mid JZ = iZ\} = \{X - iJX \mid X \in \mathcal{D}_p\}\,.$$

It is easily verified that the almost CR structure under consideration is partially integrable, namely $[\mathcal{H},\mathcal{H}] \subset \mathcal{H} \oplus \bar{\mathcal{H}}$. Moreover the 1-forms $\{\eta^1, \dots, \eta^s\}$ make up an *annhilating frame*, i.e. a globally defined frame for the annihilator $\mathcal{D}^0 M$ of $\mathcal{D}$. In the terminology of Mizner ([8], p. 1341), such a frame is *nondegenerate* of type $\{1, \dots, s\}$. This means that at each point $p \in M$, and for each $j \in \{1, \dots, s\}$, $\eta^j \circ \mathcal{L}_p$ is a nondegenerate hermitian form on $\mathcal{H}_p$, where

$$\mathcal{L}_p : \mathcal{H}_p \times \mathcal{H}_p \to T_p M^{\mathbf{C}} / \mathcal{H}_p^{\mathbf{C}}$$

is the Levi form (cf. e.g. [8], p. 1340). We recall that $\mathcal{L}_p$ is defined by

$$\mathcal{L}_p(Z_p, W_p) = i\pi[Z, \bar{W}]_p\,, \quad Z_p, W_p \in \mathcal{H}_p$$

where Z and W are arbitrary extensions of the tangent vectors Z_p, W_p to sections of $\mathcal{H}$. In the present situation, if $Z \in \mathcal{H}_p$, $Z = X - iJX$, with $X \in \mathcal{D}_p$, we have

$$\begin{aligned}(\eta^j \circ \mathcal{L}_p)(Z_p) &= i\eta^j([Z, \bar{Z}]_p) = -2i\, d\eta^j(Z, \bar{Z}) \\ &= -2i\Phi(Z, \bar{Z}) = -2ig(Z, J\bar{Z}) \\ &= -2g(Z, \bar{Z}) = -4g(X, X)\end{aligned}$$

so that $\eta^j \circ \mathcal{L}_p$ is negative definite. The main result in [8] states that a globally defined nondegenerate annhilating frame for a partially integrable almost CR structure canonically determines an affine connection ∇'. This connection is uniquely determined by the following requirements. Consider the decomposition of $TM^{\mathbf{C}}$

$$TM^{\mathbf{C}} = E_1 \oplus E_2 \oplus E_3 \oplus \cdots \oplus E_{s+2}$$

where $E_1 := \mathcal{H}$, $E_2 := \bar{\mathcal{H}}$, and for each $i \in \{1, \dots, s\}$, E_{i+2} is the complex line bundle spanned by ξ_i. Then $\mathcal{E} = \{E_1, \dots, E_{s+2}\}$ is an almost product structure, whose *torsion* is the skew-symmetric bilinear map $\tau : TM^{\mathbf{C}} \times TM^{\mathbf{C}} \to TM^{\mathbf{C}}$ defined as follows:

$$\tau := \frac{1}{2} \sum_{i=1}^{s+2} \pi_i [\pi_i, \pi_i]\,,$$

where $\pi_i : TM^{\mathbf{C}} \to E_i$ denotes the natural projection, and $[\pi_i, \pi_i]$ is the Nijenhuis torsion of π_i. It is known that for all $i, j \in \{1, \dots, s+2\}$ and for all $Z_i \in \Gamma E_i$, $Z_j \in \Gamma E_j$:

$$\tau(Z_i, Z_j) = \sum_{k \neq i,j} [Z_i, Z_j]_k$$

where $[Z_i, Z_j]_k = \pi_k\,[Z_i, Z_j]$. Then Mizner's connection ∇' is the unique affine connection on M whose $\mathbf{C}$-linear extension to $TM^{\mathbf{C}}$ satifies the following conditions:

1. ∇' is a parallelizing connection for $\mathcal{E}$;
2. $T'_{ij} = -\tau_{ij}$ for all distinct $i, j \in \{1, \dots, s+2\}$;
3. $\nabla'_{\xi_i} \xi_i = 0$ for all $i \in \{1, \dots, s\}$
4. $\nabla'_X \tau_{123} = 0$ for any $X \in \Gamma\mathcal{H}$.

Here T' is the torsion of ∇', and we have adopted the following convention: for a map $F : TM^{\mathbf{C}} \times TM^{\mathbf{C}} \to TM^{\mathbf{C}}$, and for all $i, j, k \in \{1, \dots, s+2\}$,

$$F_{ij} : E_i \times E_j \to TM^{\mathbf{C}}\,, \quad F_{ijk} : E_i \times E_j \to E_k$$

denote the maps obtained from F in the obvious way.

Theorem 2.6. *Let M be an almost $\mathcal{S}$-manifold with structure $(\varphi, \xi_i, \eta^i, g)$, and let ∇' be its Mizner's connection according to the above discussion. Then the following conditions are equivalent:*

(a) *M is CR-integrable;*

(b) *$T'(Z, W) = 0$ for all $Z, W \in \Gamma\mathcal{H}$.*

When (a) *or* (b) *holds, ∇' coincides with the Tanaka–Webster connection $\tilde{\nabla}$ of M according to Theorem 2.2.*

Proof. To prove (b)⇒(a), it suffices to use the following relation which holds as a consequence of the conditions defining ∇' (for a proof see [8], p. 1353):

$$T'_{iij} = -\tau_{iij} \quad \text{for all distinct} \quad i, j \in \{1, \dots, s+2\}\,.$$

Assuming (b), applying this relation for $i = 1$ we get $\tau_{11j}(Z, W) = 0$, for all sections Z, W of $\mathcal{H}$, which means $[Z, W]_j = 0$ for all $j \geq 2$. This proves that $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$, i.e. M is CR-integrable.

In order to prove that (a)⇒(b), it suffices to show that if M is CR-integrable, then the Tanaka–Webster connection $\tilde{\nabla}$ coincides with ∇'. After this, (b) follows from

$$\tilde{T}(X, Y) = 2\Phi(X, Y)\bar{\xi}\,, \quad X, Y \in \Gamma\mathcal{D}\,. \tag{17}$$

Indeed, if $X, Y \in \Gamma\mathcal{D}$, then

$$\begin{aligned}&\tilde{T}(X - iJX, Y - iJY)\\ &\quad = 2\{\Phi(X, Y) - i\Phi(X, \varphi Y) - i\Phi(\varphi X, Y) - \Phi(\varphi X, \varphi Y)\}\bar{\xi} = 0\end{aligned}$$

and this yields $\tilde{T}(Z, W) = 0$ for all $Z, W \in \Gamma\mathcal{H}$. Hence we verify that $\tilde{\nabla} = \nabla'$ showing that $\tilde{\nabla}$ satisfies the above conditions 1. – 4. It is clear that, since φ and the ξ_i are all $\tilde{\nabla}$-parallel, then $\tilde{\nabla}$ parallelizes $\mathcal{E}$, and morever $\tilde{\nabla}$ satisfies condition 3. To prove 2, we consider first the case where $i = 1$ and $j = 2$. Let $Z = X - iJX$ and $\bar{W} = Y + iJY$ be arbitrary sections of $\mathcal{H}$ and $\bar{\mathcal{H}}$ respectively, where $X, Y \in \Gamma\mathcal{D}$. Then, using (13):

$$\begin{aligned}\tilde{T}_{12}(Z, \bar{W}) &= 2\{\Phi(X, Y) + i\Phi(X, \varphi Y) - i\Phi(\varphi X, Y) + \Phi(\varphi X, \varphi Y)\}\bar{\xi}\\ &= 4\{\Phi(X, Y) - i\Phi(\varphi X, Y)\}\bar{\xi}\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\tau_{12}(Z, \bar{W}) = \sum_{k \neq 1,2} [Z, \bar{W}]_k &= \sum_{t=1}^{s} \eta^t([X, Y])\xi_t\\ &\quad + i\sum_{t=1}^{s} \eta^t([X, \varphi Y])\xi_t - i\sum_{t=1}^{s} \eta^t([\varphi X, Y])\xi_t + \sum_{t=1}^{s} \eta^t([\varphi X, \varphi Y])\xi_t\\ &= -2\Phi(X, Y)\bar{\xi} - 2i\Phi(X, \varphi Y)\bar{\xi} + 2i\Phi(\varphi X, Y)\bar{\xi} - 2\Phi(\varphi X, \varphi Y)\bar{\xi}\end{aligned}$$

and this implies $\tilde{T}_{12} = -\tau_{12}$. Next we treat the case where $i = 1$ and $j > 2$. Using (16), setting $t = j - 2$, we have

$$\begin{aligned}\tilde{T}_{1j}(Z, \xi_t) &= \tilde{T}_{1j}(X, \xi_t) - i\tilde{T}_{1j}(\varphi X, \xi_t)\\ &= \frac{1}{2}\{\varphi[\xi_t, \varphi X] + [\xi_t, X]\} - \frac{i}{2}\{-\varphi[\xi_t, X] + [\xi_t, \varphi X]\}\\ &= \frac{1}{2}\{[\xi_t, X] + i\varphi[\xi_t, X]\} - \frac{i}{2}\{[\xi_t, \varphi X] + i\varphi[\xi_t, \varphi X]\}\\ &= [\xi_t, X]_2 - i[\xi_t, \varphi X]_2 = [\xi_t, Z]_2\,.\end{aligned}$$

Now, since $[\xi_t, \mathcal{D}] \subset \mathcal{D}$, we have $[Z, \xi_t] \in \Gamma(\mathcal{H} \oplus \bar{\mathcal{H}})$, hence

$$\tau_{1j}(Z, \xi_t) = [Z, \xi_t]_2$$

so that $\tilde{T}_{1j} = -\tau_{1j}$. The verification of 2. when $i = 2$ and $j > 2$ is similar. For the case when $i, j \geq 3$, observe that both sides of 2. vanish. This completes the verification of 2. As to property 4, it is a consequence of $\tilde{\nabla} g = 0$ and $\tilde{\nabla}\xi_1 = 0$, since

$$\begin{aligned}\tau_{123}(Z, \bar{W}) &= [Z, \bar{W}]_3 = \eta^1([Z, \bar{W}])\xi_1 \\ &= -2\Phi(Z, \bar{W})\xi_1 = 2ig(Z, \bar{W})\xi_1 \, .\end{aligned}$$

We conclude that $\tilde{\nabla} = \nabla'$ and this completes the proof. □

We remark that our approach in the determination of the Tanaka–Webster connection of an almost $\mathcal{S}$-manifold provides an explicit formula for $\tilde{\nabla}$ involving the Levi-Civita connection of the metric g (cf. (14)).

3. CR-integrable almost $\mathcal{S}$-structures as Cartan geometries

As an application of Theorem 2.2, in this section we give a canonical interpretation of the notion of CR–integrable almost $\mathcal{S}$-structure on a manifold as a Cartan geometry with an appropriate reductive model Klein geometry. About this notion, we shall follow the terminology and notations in R. Sharpe's book [9], Chap. 5.

Consider the real vector space

$$V := \mathbf{R}^{2k} \oplus \mathbf{R}^s = \mathsf{D} \oplus \mathsf{D}^\perp$$

where $k \geq 1$, $s \geq 1$. We denote by $\{x_1, \dots, x_{2k}, e_1, \dots, e_s\}$ the standard basis and by g_o the standard inner product on V. Moreover, let $J : \mathsf{D} \to \mathsf{D}$ be the complex structure associated to the matrix

$$\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$$

with respect to the basis $\{x_1, \dots, x_{2k}\}$ of D. Let $f : V \to V$ be the endomorphism defined by

$$f(Z) = \begin{cases} JZ & \text{if} \quad Z \in \mathsf{D} \\ 0 & \text{if} \quad Z \in \mathsf{D}^\perp \end{cases} .$$

We also set $e := \sum_{i=1}^s e_i \in \mathsf{D}^\perp$ and we denote by Φ_o the 2-form on V such that

$$\Phi_o(x, y) := g_o(x, fy)$$

for all $x, y \in V$.

Now let M be a smooth manifold of dimension $n = 2k + s$; we denote by $L(M)$ the bundle of frames of M; we think of $L(M)$ as the $GL(V)$-principal fibre bundle over M consisting of all linear isomorphisms $u : V \to T_xM$, $x \in M$. The following proposition is standard:

Proposition 3.1. *There is a natural bijective correspondence between metric $f{\cdot}pk$ structures $\zeta = (\varphi, \xi_i, \eta^i, g)$ of rank $2k$ and $U(k) \times I_s$-reductions Q_ζ of the bundle $L(M)$. A frame $u \in L_x(M)$ belongs to Q_ζ if and only if*

$$\varphi_x \circ u = u \circ f \, , \quad u^*(g_x) = g_o \, , \quad u(e_i) = \xi_i(x) \, .$$

Moreover, a linear connection on M, with covariant differentiation ∇, is reducible to Q_ζ if and only if

$$\nabla\varphi = \nabla g = \nabla\xi_i = 0\,.$$

Next we introduce a Lie algebra structure on the vector space

$$\mathsf{g} = \mathsf{u}(\mathsf{k}) \oplus V \tag{18}$$

as follows. We set

$$[x,y] := -2\Phi_o(x,y)e,\ \ [A,x] := A\cdot x =: -[x,A],\ \ [A,B] := AB - BA$$

for all $x,y \in V$ and $A,B \in \mathsf{u}(\mathsf{k})$; here $A\cdot x$ denotes the natural action of $\mathsf{u}(\mathsf{k})$ on V. We remark that the validity of the Jacobi identity for $[\,,\,]$ is based on the fact that each $A \in \mathsf{u}(\mathsf{k})$ acts as a skew-symmetric endomorphism of V with respect to g_o, commuting with f.

The adjoint representation of $U(k)\times I_s$ on its Lie algebra $\mathsf{u}(\mathsf{k})$ extends to a representation, still denoted by $Ad : U(k)\times I_s \to Aut(\mathsf{g})$ such that

$$Ad(h)(x) = h\cdot x\,, \quad Ad(h)(A) = hAh^{-1} \quad \text{for all} \quad x\in V,\, A\in\mathsf{u}(\mathsf{k})\,.$$

Hence the Klein pair $(\mathsf{g},\mathsf{u}(\mathsf{k}))$ is a *model geometry with group $H = U(k)\times I_s \subset GL(V)$* according to R. Sharpe's definition in [9], page 174. Notice that the representation Ad and the induced representation Ad_V of H on V are faithful, so that the model is *effective* and of *first–order*. Moreover, the decomposition $\mathsf{g} = \mathsf{u}(\mathsf{k})\oplus V$ is a *reductive* one, namely V is an $Ad(H)$-submodule of g. This property implies the following characterization of Cartan connections with model $(\mathsf{g},\mathsf{u}(\mathsf{k}))$ and group H (see e.g. [1] or [9], Appendix A):

Proposition 3.2. *Up to gauge equivalence, every Cartan geometry on M modeled on $(\mathsf{g},\mathsf{u}(\mathsf{k}))$ with group H is given by (Q,ω) where Q is an H-reduction of the bundle $L(M)$, and $\omega = \gamma+\theta$, where $\gamma : TQ \to \mathsf{u}(\mathsf{k})$ is a principal connection form on Q, while $\theta : TQ \to V$ is the canonical form given by*

$$\theta_u(Y) = u^{-1}(\pi_* Y), \quad \pi : Q \to M \quad \textit{natural projection}$$

for each frame $u \in Q$ and $Y \in T_uQ$.

We recall that two Cartan geometries (P,ω) and (Q,ω') on a manifold M, having the same Klein model, are called *gauge equivalent* it there is a bundle isomorphism $\Psi : P \to Q$ covering the identity i_M, such that $\Psi^*\omega' = \omega$.

In order to get a canonical interpretation of CR-integrable almost $\mathcal{S}$-structures as Cartan geometries, we need to restrain our attention to a special class of the latter, which we shall call *normal* Cartan geometries. Their characterization is done by means of the corresponding curvature function. We recall that the *curvature form* Ω of a Cartan geometry (P,ω) modeled on $(\mathsf{g},\mathsf{u}(\mathsf{k}))$ is the g-valued 2-form on P, such that

$$\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega(X),\omega(Y)]\,.$$

Denote by $C^2(V,\mathsf{g})$ the real vector space of alternating bilinear maps $\psi : V\times V \to \mathsf{g}$. This is an H–module under the left action

$$h\cdot\psi(\cdot,\cdot) := Ad(h)\psi\big(Ad_V(h^{-1})\cdot, Ad_V(h^{-1})\cdot\big)\,.$$

The *curvature function* of (P,ω) is the smooth map $K : P \to C^2(V,\mathsf{g})$ defined by

$$K(u)(X,Y) := \Omega_u(\omega^{-1}X, \omega^{-1}Y)\,.$$

A Cartan geometry is called *torsion free* if $K_V = 0$, where $K_V(u) = \mathrm{pr}_V \circ K(u)$. Now consider the subspace $\mathcal{M}$ of $C^2(V,\mathsf{g})$ consisting of the bilinear maps $\psi : V\times V \to \mathsf{g}$ such that

$$\psi_V(x,y) = \psi_V(e_i,e_j) = 0\,,\quad \psi_V(e_i,fx) = -f\psi_V(e_i,x)$$

for all $x,y\in\mathsf{D}$.

Remark 3.3. *$\mathcal{M}$ is an H–submodule of $C^2(V,\mathsf{g})$.*

This is an immediate consequence of the fact that the decomposition $V = \mathbf{C}^k\oplus\mathbf{R}^s$ is H-invariant and that H acts by complex linear maps on $\mathbf{C}^k$.

According to this remark, we define an $\mathcal{M}$-*normal* Cartan geometry on M, modeled on $(\mathsf{g},\mathsf{u}(\mathsf{k}))$ with group H, to be one which is of curvature type $\mathcal{M}$, i.e. $K(P)\subset\mathcal{M}$. This is in accordance with the general prescription in [9], page 201. Notice that normality is preserved under gauge equivalence.

Now we can state the main result of this section.

Theorem 3.4. *Let M be a real manifold of dimension $2k+s$. There is a natural bijection between the set of CR–integrable almost $\mathcal{S}$–structures of rank $2k$ on M and the set of $\mathcal{M}$-normal Cartan geometries on M modeled on $(\mathsf{g},\mathsf{u}(\mathsf{k}))$, with group $H = U(k)\times I_s$, modulo gauge equivalence. Moreover, the $\mathcal{S}$-structures correspond to the torsion free Cartan geometries.*

Before starting the proof, we make the following remark:

Lemma 3.5. *Maintaining the notation in Proposition 3.2, let Q be an H-reduction of $L(M)$, and let $\omega = \gamma+\theta$ be a Cartan geometry on M modeled on $(\mathsf{g},\mathsf{u}(\mathsf{k}))$ with group H. We denote by $\tilde\nabla$ the linear connection induced by the principal connection γ. Let K denote the curvature function of ω, and let $\tilde T$ denote the torsion tensor of $\tilde\nabla$. Then for each frame $u\in Q_x$, we have the following formula:*

$$2uK_V(u)(X,Y) = \tilde T(uX,uY) + u[X,Y]\,,\quad \textit{for all}\quad X,Y\in V\,. \tag{19}$$

Proof. This is a standard computation, cf. [9] or [5]. □

Proof of Theorem 3.4. Fix a CR-integrable almost $\mathcal{S}$-structure $\zeta = (\varphi,\xi_i,\eta^i,g)$; according to Proposition 3.1, ζ gives rises canonically to a reduction Q_ζ of $L(M)$ to the group H. Moreover on M we have the Tanaka–Webster connection $\tilde\nabla$ according to Theorem 2.2. Since the tensor fields φ, g, ξ_i are all parallel with respect to $\tilde\nabla$, this connection reduces to a principal connection γ on Q_ζ. Let θ be the canonical form of Q_ζ and set $\omega_\zeta = \gamma+\theta$. Then (Q_ζ,ω_ζ) is a Cartan geometry modeled on $(\mathsf{g},\mathsf{u}(\mathsf{k}))$ with group H (Proposition 3.2). Using formula (19) we see that (Q_ζ,ω_ζ)

is a normal geometry. Indeed, for all $X, Y \in \mathsf{D}$ we have $[X,Y] = -2\Phi_o(X,Y)e$; if $u \in Q_\zeta(x)$, $x \in M$, it follows

$$u[X,Y] = -2\Phi(x)(uX, uY)\bar{\xi}_x\,,$$

and on the other hand, taking into account property (6) of $\tilde{\nabla}$, since $uX, uY \in \mathcal{D}(x)$, we have

$$\tilde{T}(uX, uY) = 2\Phi(x)(uX, uY)\bar{\xi}_x\,.$$

It follows from (19) that $uK_V(u)(X,Y) = 0$, that is $K_V(u)(X,Y) = 0$. Since $[e_i, e_j] = 0$, in the same way we can verify that $K_V(\mathsf{D}^\perp, \mathsf{D}^\perp) = 0$. Finally, using property (7) of $\tilde{\nabla}$, we get

$$\begin{aligned} 2uK_V(u)(e_i, fX) &= \tilde{T}(\xi_i(x), \varphi(uX)) = -\varphi_x \tilde{T}(\xi_i(x), uX) \\ &= -2\varphi_x u K_V(u)(e_i, X) = -2ufK_V(u)(e_i, X) \end{aligned}$$

whence $K_V(u)(e_i, fX) = -fK_V(u)(e_i, X)$.

Hence to each CR-integrable almost $\mathcal{S}$-structure ζ we have associated a normal Cartan geometry $\mathcal{C}_\zeta = (Q_\zeta, \omega_\zeta)$ modeled on $(\mathsf{g}, \mathsf{u}(\mathsf{k}))$ with group H. Clearly, the map $\zeta \mapsto \mathcal{C}_\zeta$ is injective. Notice that, according to corollary 2.5, ζ is normal, i.e. it is an $\mathcal{S}$-structure, if and only if $\tilde{T}(\xi_i, Z) = 0$ for all $Z \in \mathcal{D}$. Using again (19), we easily see that this is equivalent to $K_V(u)(e_i, X) = 0$ for all $u \in Q_\zeta$ and $X \in \mathsf{D}$. By definition of $\mathcal{M}$, this is equivalent to $K_V = 0$, that is to $\mathcal{C}_\zeta$ being torsion free.

To conclude the proof of the theorem, it suffices to verify that, up to gauge equivalence, every normal Cartan geometry (P, ω) with model $(\mathsf{g}, \mathsf{u}(\mathsf{k}))$ and group H is given by $\mathcal{C}_\zeta$ for some CR-integrable almost $\mathcal{S}$-structure on M. We know from Proposition 3.2 that (P, ω) is gauge equivalent to $\mathcal{C} = (Q, \omega')$ where Q is a reduction of $L(M)$ to H, and $\omega' = \gamma + \theta$, where γ is a principal connection form on Q. There exists a unique metric $f{\cdot}$pk structure $\zeta = (\varphi, \xi_i, \eta^i, g)$ on M such that $Q = Q_\zeta$. To γ there corresponds a linear connection $\tilde{\nabla}$; clearly, $\tilde{\nabla}\varphi = \tilde{\nabla}g = \tilde{\nabla}\xi_i = 0$. Moreover, using the $\mathcal{M}$-normality of (Q, ω'), we see as above that the torsion $\tilde{T}$ of $\tilde{\nabla}$ satisfies the conditions (6)–(8) in Theorem 2.2. Hence ζ is actually a CR-integrable almost $\mathcal{S}$-structure and $\tilde{\nabla}$ is the corresponding Tanaka–Webster connection. In particular, it follows that $\mathcal{C}_\zeta = \mathcal{C}$ and this concludes the proof of the theorem. □

Examples. We end by discussing examples of homogeneous non normal almost $\mathcal{S}$-manifold whose Tanaka–Webster curvature vanishes. Notice that, for the case $s = 1$, a manifold with this properties does not exist. Namely, it can be easily verified by using the Bianchi identity that a contact metric manifold with vanishing Tanaka–Webster curvature is necessarily Sasakian.

Set

$$\mathsf{m} = \mathbf{R}^{2k} \oplus \mathbf{R}^s = V_1 \oplus V_2,\ s \geq 2$$

and denote by $\{X_1, \ldots, X_k, JX_1, \ldots, JX_k\}$ the standard basis of $\mathbf{R}^{2k}$ endowed with the complex structure J associated with the matrix $\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$. Moreover let $\{\xi_1, \ldots, \xi_s\}$ denote the natural basis of V_2 and let g be the inner product on m

obtained by declaring the basis $\{X_i, JX_i, \xi_j\}$ to be orthonormal. Let $\varphi : \mathsf{m} \to \mathsf{m}$ be the natural f-structure on m, i.e. φ is the endomorphism which coincides with J on V_1 and vanishes on V_2.

We also denote by U the endomorphism of m which is associated to the matrix

$$\begin{pmatrix} I_k & 0 & 0 \\ 0 & -I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Notice that $U\varphi = -\varphi U$.

We denote by h the Lie subalgebra of $End(\mathsf{m})$ consisting of all endomorphisms which vanish on V_2 and annihilate the tensors φ, g and U when extended to the tensor algebra of m as derivations. We remark that

$$A \in \mathsf{so(k)} \mapsto \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

provides a Lie-algebra isomorphism $\mathsf{so(k)} \cong \mathsf{h}$. In particular, h is compact semisimple provided $k \geq 3$.

Now we define a Lie algebra structure on $\mathsf{g} := \mathsf{h} \oplus \mathsf{m}$ as follows:

$$[X, Y] := -2g(X, JY)e,\ \ [v, X] := a(v)UX = -[X, v]$$
$$[A, X] = A \cdot X = -[X, A],\ \ [A, v] := 0,\ [v, w] := 0, [A, B] := AB - BA$$

for each $X, Y \in V_1, v, w \in V_2$, $A \in \mathsf{h}$. Here $e := \sum_i \xi_i \in V_2$, and $a : V_2 \to \mathbf{R}$ is a fixed non null linear functional such that $a(e) = 0$.

Let G be the connected and simply connected Lie group with Lie algebra g and let H denote the analytic subgroup corresponding to the subalgebra h. Assuming $k \geq 3$, we have that H is compact, so that $M = G/H$ is a reductive homogeneous space. The tensors φ and g on the reductive summand m are $Ad(H)$-invariant, and $Ad(h)\xi_i = \xi_i$, for each $h \in H$. Then $(\varphi, \xi_i, \eta^i, g)$, where the η^i are the dual forms of the ξ_i, canonically determine a G-invariant metric $f{\cdot}$pk structure on M. The canonical G-invariant linear connection $\tilde{\nabla}$ satisfies the conditions 1), 2) in Theorem 2.2. Indeed, since the structure is G-invariant, the tensor fields φ, η^i and g are all parallel with respect to $\tilde{\nabla}$. Moreover at the point $o = H$, under the natural identification $T_oM \cong \mathsf{m}$, we have the formula $\tilde{T}_o(Z, W) = -[Z, W]$ for the torsion of $\tilde{\nabla}$, which implies that $\tilde{\nabla}$ satisfies properties (6)–(8), according to the definition of the Lie bracket $\mathsf{m} \times \mathsf{m} \to \mathsf{m}$; in particular notice that (7) holds since $[v, JX] = a(v)UJX = -J[v, X]$, for each $v \in V_2$ and $X \in V_1$. Hence M is a homogeneous almost $\mathcal{S}$-manifold, which is not normal according to Corollary 2.5. Finally, $\tilde{\nabla}$ has vanishing Tanaka–Webster curvature because $[\mathsf{m}, \mathsf{m}] \subset \mathsf{m}$.

References

[1] Alekseevsky, D. V., Michor, P. W., *Differential Geometry of Cartan connections*, Publ. Math. Debrecen **47** (1995), 349–375.

[2] Blair, D. E., *Contact manifolds in Riemannian Geometry*, Lecture Notes in Math. **509**, 1976, Springer–Verlag.

[3] Blair D. E., Geometry of manifolds with structural group $\mathcal{U}(n)\times\mathcal{O}(s)$ *J. Differential Geom.* **4** (1970), 155–167.

[4] Duggal, K. L., Ianus, S., Pastore, A. M., *Maps interchanging f-structures and their harmonicity*, Acta Appl. Math. **67** (2001), 91–115.

[5] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry*, Vol. I, Interscience, New-York, 1963.

[6] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry*, Vol. II, Interscience, New-York, 1969.

[7] Lotta, A., *Cartan connections on CR manifolds*, PhD Thesis, University of Pisa, 2000.

[8] Mizner, R. I., *Almost CR structures, f-structures, almost product structures and associated connections*, Rocky Mountain J. Math. **23**, no. 4 (1993), 1337–1359.

[9] Sharpe, R. W., *Differential geometry. Cartan's generalization of Klein's Erlangen program*, Graduate Texts in Mathematics **166**, Springer-Verlag, New York, 1997.

[10] Tanaka, N., *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan. J. Math. **20** (1976), 131–190.

[11] Tanno, S., *Variational problems on contact Riemannian manifolds*, Trans. Amer. Math. Soc., Vol. **314** (1989), 349–379.

[12] Tanno, S., *The automorphism groups of almost contact Riemannian manifolds*, Tohoku Math. J. **21** (1969), 21–38.

[13] Urakawa, H., *Yang-Mills connections over compact strongly pseudoconvex CR manifolds*, Math. Z. **216** (1994), 541–573.

[14] Webster, S. M., *Pseudo-hermitian structures on a real hypersurface*, J. Differential Geom. **13** (1978), 25–41.

[15] Yano, K., *On a structure defined by a tensor field f of type $(1,1)$ satisfying $f^3+f=0$*, Tensor (N.S.) **14** (1963), 99–109.

Dipartimento di Matematica
Università di Bari, Via E. Orabona 4
70125 Bari, Italy
E-mail: lotta@dm.uniba.it, pastore@dm.uniba.it