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ON THE BOUNDARY CONDITIONS ASSOCIATED WITH SECOND-ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

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Abstract. The ideas of the present paper have originated from the observation that all solutions of the linear homogeneous differential equation (DE) $y''(t) + y(t) = 0$ satisfy the non-trivial linear homogeneous boundary conditions (BCs) $y(0) + y(\pi) = 0$, $y'(0) + y'(\pi) = 0$. Such a BC is referred to as a *natural BC* (NBC) with respect to the given DE, considered on the interval $[0, \pi]$. This observation suggests the following queries : (i) Will each second-order linear homogeneous DE possess a natural BC ? (ii) How many linearly independent natural BCs can a DE possess ? The present paper answers these queries. It also establishes that any non-trivial homogeneous mixed BC, which is not a NBC with respect to the given linear homogeneous DE, determines uniquely (up to a constant multiplier), the solution of the DE. Two BCs are said to be *compatible with respect to a given DE* if both of them determine the same solution of the DE. Conditions for the compatibility of sets of two and three BCs with respect to a given DE have also been determined.

1. Introduction

We consider the following second-order linear homogeneous ordinary differential equation

$$L[y] \equiv p_0(t)y''(t) + p_1(t)y'(t) + p_2(t)y(t) = 0 \tag{1.1}$$

where $p_0, p_1, p_2 : [a, b] \to \mathbb{C}$ (the set of complex numbers) are continuous and $p_0(t) \neq 0$ for all $t \in [a, b]$.

To set up regular Sturm-Liouville boundary value problem (SLP), the DE 1.1 is generally associated with two linear homogeneous BCs of the form

$$U_\alpha[y] = \alpha_1 y(a) + \alpha_2 y'(a) + \alpha_3 y(b) + \alpha_4 y'(b) = 0\,, \tag{1.2}$$

$$U_\beta[y] = \beta_1 y(a) + \beta_2 y'(a) + \beta_3 y(b) + \beta_4 y'(b) = 0\,, \tag{1.3}$$

where α_i, β_i $(i = 1, 2, 3, 4)$ are real numbers.

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However, we note that all solutions of the DE $y''(t) + y(t) = 0$ satisfy the BC $y(0) + y(\pi) = 0$. Thus, for a given DE of the form (1.1), there may exist a non-trivial BC of the form (1.2), which is satisfied by all solutions of the given DE. Let us name such a BC a *natural BC* (NBC) with respect to the given DE. In this case the DE (1.1) is said to possess a NBC. Having given a DE, one may ask the following questions:

(i) Is there any NBC with respect to the given DE?

(ii) How many linearly independent NBCs are there with respect to the given DE?

The answers to these queries are obtained in Theorem I. In Theorem II, given a BC of the form (1.2) which is not a NBC with respect to the given DE (1.1), we determine the non-trivial solution of (1.1), uniquely up to a constant multiplier, that satisfies the given BC. As a result, the SLP (1.1)–(1.2)–(1.3) will have a non-trivial solution provided the BCs (1.2) and (1.3) determine the same non-trivial solution of (1.1). In that case the BCs (1.2) and (1.3) are said to be *compatible* with respect to the DE (1.1). Theorem III determines the conditions for the BCs (1.2) and (1.3) to be compatible with respect to the DE (1.1). In [1] it has been shown that a non-trivial solution of the DE (1.1) may satisfy, at the most, three linearly independent BCs of the form (1.2). Hence, in Theorem IV, conditions have been obtained for three linearly independent BCs of the form (1.2) to be compatible with respect to the given DE (1.1).

2. Some necessary preliminaries

Let $\xi, \eta : [a, b] \to \mathbb{C}$ denote the solutions of (1.1) that satisfy

$$\xi(a) = 1\,, \qquad \xi'(a) = 0\,, \tag{2.1}$$

$$\eta(a) = 0\,, \qquad \eta'(a) = 1\,. \tag{2.2}$$

As the coefficients p_0, p_1, p_2 in (1.1) are complex-valued, the solutions ξ, η are complex-valued. Let

$$\xi(b) = \xi_1 + i\xi_2\,, \qquad \xi'(b) = \xi_1' + i\xi_2'\,, \tag{2.3}$$

$$\eta(b) = \eta_1 + i\eta_2\,, \qquad \eta'(b) = \eta_1' + i\eta_2'\,, \tag{2.4}$$

where ξ_i, η_i, ξ_i', η_i' $(i = 1, 2)$ are real numbers.
We further note that

$$U_\alpha[\xi] = (\alpha_1 + \alpha_3\xi_1 + \alpha_4\xi_1') + i(\alpha_3\xi_2 + \alpha_4\xi_2')\,, \tag{2.5}$$

and

$$U_\alpha[\eta] = (\alpha_1 + \alpha_3\eta_1 + \alpha_4\eta_1') + i(\alpha_3\eta_2 + \alpha_4\eta_2')\,, \tag{2.6}$$

as α_1, α_2, α_3, α_4 are real numbers.

3. Conditions for the given DE (1.1) to possess NBC

Let the given DE (1.1) possess a NBC, $U_\alpha[y] = 0$ for some $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \neq (0, 0, 0, 0)$. Then we must have

$$U_\alpha[\xi] = 0 = U_\alpha[\eta]\,. \tag{3.1}$$

These imply that the following algebraic equations in α have a nontrivial solution:

$$\alpha_1 + \alpha_3\xi_1 + \alpha_4\xi_1' = 0 = \alpha_2 + \alpha_3\eta_1 + \alpha_4\eta_1' \,, \tag{3.2}$$

$$\alpha_3\xi_2 + \alpha_4\xi_2' = 0 = \alpha_3\eta_2 + \alpha_4\eta_2' \,, \tag{3.3}$$

We note that $(\alpha_3, \alpha_4) = (0, 0)$ imply $(\alpha_1, \alpha_2) = (0, 0)$. Hence we should have $(\alpha_3, \alpha_4) \neq (0, 0)$ and this demands $\xi_2\eta_2' - \xi_2'\eta_2 = 0$, as can be seen from (3.3).

Conversely, if $\xi_2\eta_2' - \xi_2'\eta_2 = 0$, there exists at least one solution $(\alpha_3, \alpha_4) \neq (0, 0)$ of (3.3), and this (α_3, α_4) will determine (α_1, α_2) from (3.2). In other words, there is at least one NBC with respect to the DE (1.1).

Our next job is to find the NBCs with respect to the DE (1.1). As the DE (1.1) possesses a NBC, we have $\xi_2\eta_2' - \xi_2'\eta_2 = 0$. Two cases are to be considered:

$$\begin{aligned} &(i) \quad (\xi_2, \xi_2', \eta_2, \eta_2') \neq (0, 0, 0, 0) \,, \\ &(ii) \quad (\xi_2, \xi_2', \eta_2, \eta_2') = (0, 0, 0, 0) \,. \end{aligned}$$

In case (i), the equations (3.3) possess a unique non-trivial solution (save a constant multiplier). Using (3.2), the corresponding NBC with respect to the DE (1.1) can be exhibited as

$$(\xi_1\xi_2' - \xi_1'\xi_2)y(a) + (\xi_2'\eta_1 - \xi_2\eta_1')y'(a) - \xi_2'y(b) + \xi_2y'(b) = 0 \,. \tag{3.4}$$

In case (ii), (3.3) is satisfied by any $(\alpha_3,\, \alpha_4)$. Choosing $(1, 0)$ and $(0, 1)$ for $(\alpha_3,$ $\alpha_4)$ we find two linearly independent NBCs with respect to the DE (1.1), viz,

$$\xi_1y(a) + \eta_1y'(a) - y(b) = 0 \tag{3.5}$$

$$\xi_1'y(a) + \eta_1'y'(a) - y'(b) = 0 \,. \tag{3.6}$$

Hence we have the following theorem:

Theorem I.

(a) *The DE* (1.1) *possesses NBC if and only if* $\xi_2\eta_2' - \xi_2'\eta_2 = 0$.
(b) *If* $\xi_2\eta_2' - \xi_2'\eta_2 = 0$ *but* $(\xi_2, \xi_2', \eta_2, \eta_2') \neq (0, 0, 0, 0)$, *the DE* (1.1) *possesses only one NBC, which is given by* (3.4).
(c) *If* $(\xi_2, \xi_2', \eta_2, \eta_2') = (0, 0, 0, 0)$, *there are two linearly independent NBCs with respect to the DE* (1.1), *and they can be exhibited as in* (3.5)–(3.6).

Example 1. For the DE $y''(t) - iy'(t) = 0$, $t \in [0, \pi]$, it can be verified that $\xi(t) = 1$, $\eta(t) = i(1 - e^{\mathrm{it}})$. Hence $\xi_2\eta_2' - \xi_2'\eta_2 = 0$ but $\eta_2 \neq 0$. Here the NBC with respect to the given DE is $y'(0) + y'(\pi) = 0$.

Example 2. For the DE $y''(t) + y(t) = 0$, $t \in [0, \pi]$, $\xi(t) = cost$, $\eta(t) = sint$. So $\xi_2 = \xi_2' = \eta_2 = \eta_2' = 0$. The two linearly independent NBCs with respect to the given DE can be exihibited as

$$y(0) + y(\pi) = 0 \,, \qquad y'(0) + y'(\pi) = 0 \,.$$

4. The unique solution of the DE (1.1) determined by a BC $U_\alpha[y] = 0$ that is not natural with respect to (1.1)

We first note that, as the DE (1.1) is homogeneous, the uniqueness of its solution is to be understood up to a constant multiplier. That such a unique solution exists has been proved in [1].

As the BC $U_\alpha[y] = 0$ is not natural w.r. to the DE (1.1), we can not have $U_\alpha[\xi] = 0 = U_\alpha[\eta]$.

If $U_\alpha[\eta] = 0$, then $U_\alpha[\xi] \neq 0$ and the required unique solution of (1.1) satysfiing $U_\alpha[y] = 0$ is η.

If $U_\alpha[\eta] \neq 0$, the required unique solution ψ of (1.1) should be of the form $\psi = \xi + (u + iv))\eta$, where u, v are real numbers. Then $U_\alpha[\psi] = 0$ if and only if

$$\alpha_1 + \alpha_2 u + \alpha_3 (\xi_1 + u\eta_1 - v\eta_2) + \alpha_4 (\xi_1' + u\eta_1' - v\eta_2') = 0 \tag{4.1}$$

and

$$\alpha_2 v + \alpha_3 (\xi_2 + u\eta_2 + v\eta_1) + \alpha_4(\xi_2' + u\eta_2' + v\eta_1') = 0\,. \tag{4.2}$$

The equations (4.1)–(4.2) determine u, v uniquely (and so, ψ uniquely), since $U_\alpha[\eta] \neq 0$ implies $(\alpha_2 + \alpha_3\eta_1 + \alpha_4\eta_1')^2 + (\alpha_3\eta_2 + \alpha_4\eta_2')^2 \neq 0$. These observations lead to the following theorem:

Theorem II. *Let the BC $U_\alpha[y] = 0$ be not natural with respect to the DE* (1.1).

(i) *If $U_\alpha[\eta] = 0$, then the required unique solution of* (1.1)*, that satisfies $U_\alpha[y] = 0$ is η.*

(ii) *If $U_\alpha[\eta] \neq 0$, then the required unique solution of* (1.1)*, that satisfies $U_\alpha[y] = 0$ is $\psi = \xi + (u + iv)\eta$ (u, v : real numbers), where u and v are uniquely determined by the equations* (4.1)–(4.2).

5. Compatibility of boundary conditions

Let $U_\alpha[y] = 0$, $U_\beta[y] = 0$, be two linearly independent BCs, none of which is a NBC with respect to the DE (1.1). Then, each of these BCs will determine a non-trivial solution of the DE (1.1), uniquely up to a constant multiplier; in general, the solutions determined by them will be different. In case the two BCs determine the same non-trivial solution of the DE (1.1), then they are said to be *compatible* with respect to the DE (1.1). In this section we shall determine the condition that will guarantee that the two BCs $U_\alpha[y] = 0 = U_\beta[y]$ are compatible with respect to the DE (1.1). Equivalently we determine the condition under which the Sturm-Liouville problem (SLP) $\pi : L[y] = 0 = U_\alpha[y] = U_\beta[y]$ has a non-trivial solution, unique up to a constant multiplier.

In this connection, we need to use the following result which has been proved in [1].

5.1. Proposition. *For any solution ψ of the SLP* (1.1), (1.2), (1.3) *, let $B(\psi)$ denote the set of vectors $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of $\mathbb{R}^4$ such that*

$$U_\lambda[\psi] = \lambda_1\psi(a) + \lambda_2\psi^{(1)}(a) + \lambda_3\psi(b) + \lambda_4\psi^{(1)}(b) = 0\,. \tag{5.1}$$

Then
(A) *for* $\psi = \xi + (u+iv)\eta$,

$$
\begin{array}{llll}
\dim B(\psi) = 2 & \textit{if either} & (\mathbf{A_1}) & \xi_2\eta_2' - \xi_2'\eta_2 \neq 0\,, \\
& \textit{or} & (\mathbf{A_2}) & \xi_2\eta_2' - \xi_2'\eta_2 = 0\,, \\
& & & (\xi_2, \xi_2', \eta_2, \eta_2') \neq (0,0,0,0)\,, \\
& & & v \neq 0\,, \\
& \textit{or} & (\mathbf{A_3}) & \xi_2\eta_2' - \xi_2'\eta_2 = 0\,, \\
& & & (\xi_2, \xi_2', \eta_2, \eta_2') \neq (0,0,0,0)\,, \\
& & & v = 0, \xi_2 + u\eta_2 \neq 0\,, \quad \textit{or}, \ \xi_2' + u\eta_2 \neq 0\,, \\
& \textit{or} & (\mathbf{A_4}) & \xi_2 = \xi_2' = \eta_2 = \eta_2' = 0\,, \quad v \neq 0\,;
\end{array}
$$

and

$$
\begin{array}{llll}
\dim B(\psi) = 3 & \textit{if either} & (\mathbf{A_5}) & \xi_2\eta_2' - \xi_2'\eta_2 = 0\,, \\
& & & (\xi_2, \xi_2', \eta_2, \eta_2') \neq (0,0,0,0)\,, \\
& & & v = 0\,, \xi_2 + u\eta_2 = 0 = \xi_2' + u\eta_2' \\
& \textit{or} & (\mathbf{A_6}) & \xi_2 = \xi_2' = \eta_2 = \eta_2' = 0\,, \quad v = 0\,;
\end{array}
$$

(B) *for* $\psi = \eta$,

$$
\begin{array}{lll}
\dim B(\eta) = 2 & \text{if } (\mathbf{B_1}) & \eta_1\eta_2' - \eta_1'\eta_2 \neq 0\,, \\
\dim B(\eta) = 3 & \text{if } (\mathbf{B_2}) & \eta_1\eta_2' - \eta_1'\eta_2 = 0\,,
\end{array}
$$

5.2. Compatibility of two boundary conditions

Let $U_\alpha[y] = 0 = U_\beta[y]$ be two linearly independent BCs. We are to find conditions under which they are compatible with respect to the DE (1.1), assuming that none of them is a NBC for the DE (1.1). We shall actually prove the following:

Theorem III. *The two linearly independent BCs* $U_\alpha[y] = 0 = U_\beta[y]$*, none of which is a NBC for the DE* (1.1)*, are compatible with respect to the DE* (1.1) *if and only if*

(5.2) $$A_{34}(\xi_1'\eta_2 - \xi_1\eta_2' + \xi_2'\eta_1 - \xi_2\eta_1') - A_{13}\eta_2 - A_{14}\eta_2' + A_{23}\xi_2 + A_{24}\xi_2' = 0\,,$$

where $A_{ij} = \alpha_i\beta_j - \alpha_j\beta_i$ $(i, j = 1, 2, 3, 4)$.

Proof. First suppose that the BCs

(5.3) $$U_\alpha[y] = 0 = U_\beta[y]$$

are linearly independent and are compatible with respect to the DE (1.1). Then, both the BCs determine the same solution ψ of the DE (1.1). We are to consider the following two cases :

$$\text{(a)} \quad \psi = \xi + (u+iv)\eta\,, \qquad\qquad \text{(b)} \quad \psi = \eta\,.$$

CASE (a):

From the proposition given in §5.1, it is clear that the following six cases are to be considered separately:

(a_1) $\xi_2\eta_2' - \xi_2'\eta_2 \neq 0$,
(a_2) $\xi_2\eta_2' - \xi_2'\eta_2 = 0$, $(\xi_2, \xi_2', \eta_2, \eta_2') \neq (0,0,0,0), v \neq 0$,
(a_3) $\xi_2\eta_2' - \xi_2'\eta_2 = 0$, $(\xi_2, \xi_2', \eta_2, \eta_2') \neq (0,0,0,0), v = 0$, $\xi_2 + u\eta_2 \neq 0$,
(a_4) $\xi_2\eta_2' - \xi_2'\eta_2 = 0$, $(\xi_2, \xi_2', \eta_2, \eta_2') \neq (0,0,0,0), v = 0$, $\xi_2 + u\eta_2 = 0$,
(a_5) $\xi_2 = \xi_2' = \eta_2 = \eta_2' = 0, v \neq 0$,
(a_6) $\xi_2 = \xi_2' = \eta_2 = \eta_2' = 0, v = 0$,

SUBCASE (a_1) : In this case we know that dim $B(\psi) = 2$.
Therefore, the equation $U_\lambda[\psi] = 0$ being equivalent to

$$\lambda_1 + \lambda_2 u + \lambda_3(\xi_1 + u\eta_1 - v\eta_2) + \lambda_4(\xi_1' + u\eta_1' - v\eta_2') = 0\,, \tag{5.4}$$

$$\lambda_2 v + \lambda_3(\xi_2 + u\eta_2 + v\eta_1) + \lambda_4(\xi_2' + u\eta_2' - v\eta_1') = 0\,, \tag{5.5}$$

the system of equations (5.4)–(5.5) in $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ have exactly two linearly independent solutions, and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ form one such pair. This requires that the rank of the coefficient matrix of (5.4)–(5.5) is two. Therefore, we are to further subdivide the subcase (a_1) into the following:

$(a_1 - i)$ $\quad v \neq 0\,,$
$(a_1 - ii)$ $\quad v = 0\,, \quad \xi_2 + u\eta_2 \neq 0\,,$
$(a_1 - iii)$ $\quad v = 0\,, \quad \xi_2 + u\eta_2 = 0\,.$

SUBCASE $(a_1 - i)$: $\boldsymbol{\xi_2\eta_2' - \xi_2'\eta_2 \neq 0\,,\ v \neq 0}$.

Here, two linearly independent solution vectors of (5.4)–(5.5) can be obtained by taking $(\lambda_3, \lambda_4) = (1,0)$ and (0,1)as

$$\mu = \big(u\xi_2 - v\xi_1 + (u^2+v^2)\eta_2,\ \ -(\xi_2 + u\eta_2 + v\eta_1), v, 0\big)\,, \tag{5.6}$$

$$\nu = \big(u\xi_2' - v\xi_1' + (u^2+v^2)\eta_2',\ \ -(\xi_2' + u\eta_2' + v\eta_1'), 0, v\big)\,. \tag{5.7}$$

If $\psi = \xi + (u+iv)\eta$ is the non-trivial solution of the SLP (1.1), (1.2), (1.3), the vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ must belong to $B(\psi)$. As α, β are linearly independent, the vectors μ, ν (given in (5.6), (5.7)) of $B(\psi)$ should be expressible in terms of α, β. Hence there exist real numbers A, B, C, D $\big((A,B) \neq (0,0), (C,D) \neq (0,0)\big)$ such that

$$\mu = A\alpha + B\beta \qquad \text{and} \qquad \nu = C\alpha + D\beta\,. \tag{5.8}$$

Eliminating u, v, u^2+v^2, A, B, C, D from the eight equations of (5.8) we have, if the BCs $U_\alpha[y]=0=U_\beta[y]$ are compatible with respect to the DE (1.1), then (5.2) holds.

SUBCASE $(a_1 - ii)$: $\boldsymbol{\xi_2\eta_2' - \xi_2'\eta_2 \neq 0\,,\ v=0,\ \xi_2+u\eta_2 \neq 0}$.

In this case equations (5.4)–(5.5) become

$$(5.9)\qquad \begin{cases} \lambda_1+\lambda_2 u+\lambda_3(\xi_1+u\eta_1)+\lambda_4(\xi_1'+u\eta_1') &= 0\,,\\ \lambda_3(\xi_2+u\eta_2)+\lambda_4(\xi_2'+u\eta_2') &= 0\,. \end{cases}$$

Since $B(\psi)=B(\xi+u\eta)=2$, equations (5.9) yield two linearly independent solutions, which can be taken to be

$$\mu_1=\big(K,\,0,\,-(\xi_2'+u\eta_2'),\,\xi_2+u\eta_2\big)\,,\qquad \nu_1=(-u,1,0,0)\,,$$

where

$$K=(\xi_1+u\eta_1)(\xi_2'+u\eta_2')-(\xi_1'+u\eta_1')(\xi_2+u\eta_2)\,.$$

If the BCs $U_\alpha[y]=0=U_\beta[y]$ are compatible with respect to the DE (1.1), there must exist real numbers A, B, C, D (in general, different from those used in (5.8)), $\big((A,B)\neq(0,0),(C,D)\neq(0,0)\big)$, such that

$$(5.10)\qquad \mu_1=A\alpha+B\beta\,,\qquad \nu_1=C\alpha+D\beta\,.$$

Then $(C,D)\neq(0,0)$ implies $A_{34}=0$ from the second set of equations in (5.10). Hence

$$uA_{23}+A_{13}=0\,,\qquad (\xi_2+u\eta_2)A_{23}+(\xi_2'+u\eta_2')A_{24}=0\,,$$

and

$$uA_{24}+A_{14}=0\,.$$

These imply

$$A_{23}A_{14}=A_{24}A_{13}$$

and

$$A_{23}(A_{13}\eta_2-A_{23}\xi_2-A_{24}\xi_2')+A_{13}A_{24}\eta_2'=0\,.$$

Hence

$$(5.11)\qquad A_{23}(A_{13}\eta_2-A_{23}\xi_2-A_{24}\xi_2'+A_{14}\eta_2')=0\,.$$

Now, if $A_{23}=0$, it follows that $A_{13}=0$.
Then $A_{13}=0=A_{23}=A_{34}$ will lead to the fact that α and β are linearly dependent, which is contrary to our hypothesis. Hence $A_{23}\neq 0$. So, $A_{34}=0$, and (5.11) then implies that (5.2) holds.

SUBCASE $(a_1 - iii)$: $\boldsymbol{\xi_2\eta_2' - \xi_2'\eta_2 \neq 0,\ v=0,\ \xi_2+u\eta_2 = 0}$.

In this case we have $\xi_2'+u\eta_2'\neq 0$. Then (5.9) implies that $\lambda_4=0$. So

$$(5.12)\qquad \alpha_4=\beta_4=0\,.$$

As $\alpha=(\alpha_1\alpha_2,\alpha_3,\alpha_4)$, $\beta=(\beta_1,\beta_2,\beta_3,\beta_4)$ satisfy the first equation of (5.9), and $\xi_2+u\eta_2=0$, we get

$$(5.13)\qquad \xi_2A_{23}-\eta_2A_{13}=0\,.$$

(5.12)–(5.13) will then imply that (5.2) is satisfied.

SUBCASE (a_2): $\boldsymbol{\xi_2\eta_2' - \xi_2'\eta_2 = 0}$, $\boldsymbol{(\xi_2, \xi_2', \eta_2, \eta_2') \neq (0,0,0,0)}$, $\boldsymbol{v \neq 0}$.

In this case we know from Theorem I that there is a unique NBC for DE (1.1). Proceeding as in subcase $(a_1 - i)$, we obtain real numbers A, B, C, D such that (5.8) hold.

Using $\xi_2\eta_2' - \xi_2'\eta_2 = 0$, the eight equations in (5.8) can be reduced to eight equations in $A\xi_2' - C\xi_2$, $B\xi_2' - D\xi_2$, $A\eta_2' - C\eta_2$, $B\eta_2' - D\eta_2$ and v. As $v \neq 0$, it will then follow that

$$A_{34}(\xi_1'\eta_2 - \xi_1\eta_2') - A_{14}\eta_2' - A_{13}\eta_2 = 0 \tag{5.14}$$

and

$$A_{34}(\xi_2'\eta_1 - \xi_2\eta_1') + A_{24}\eta_2' + A_{23}\xi_2 = 0 \tag{5.15}$$

(5.14) and (5.15) then imply that (5.2) is satisfied.

SUBCASE (a_3): $\boldsymbol{\xi_2\eta_2' - \xi_2'\eta_2 = 0}$, $\boldsymbol{(\xi_2, \xi_2', \eta_2, \eta_2') \neq (0,0,0,0)}$, $\boldsymbol{v = 0}$, $\boldsymbol{\xi_2 + u\eta_2 \neq 0}$.

As $v = 0$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ are two linearly independent solutions of (5.9). From the four equations so determined, as $\xi_2 + u\eta_2 \neq 0$ we can prove that

$$A_{34} = 0\,, \tag{5.16}$$

$$A_{13} + A_{23}u = 0\,, \tag{5.17}$$

$$A_{14} + A_{24}u = 0\,. \tag{5.18}$$

Using (5.17)–(5.18), it is then easy to show that

$$A_{23}\xi_2 + A_{24}\xi_2' - A_{13}\eta_2 - A_{14}\eta_2' = 0\,. \tag{5.19}$$

Then (5.16) and (5.19) imply that (5.2) is satisied.

SUBCASE (a_4): $\boldsymbol{\xi_2\eta_2' - \xi_2'\eta_2 = 0}$, $\boldsymbol{(\xi_2, \xi_2', \eta_2, \eta_2') \neq (0,0,0,0)}$, $\boldsymbol{v = 0}$, $\boldsymbol{\xi_2 + u\eta_2 = 0}$.

Two cases arise: (i) $\xi_2' + u\eta_2' \neq 0$ (ii) $\xi_2' + u\eta_2' = 0$.

In both cases, proceeding as before, it may be shown that (5.2) holds.

SUBCASE (a_5): $\boldsymbol{\xi_2 = \xi_2' = \eta_2 = \eta_2' = 0}$, $\boldsymbol{v \neq 0}$.

In this case we shall show that there is no non-trivial BC, other than the NBCs, which is satisfied by $\psi = \xi + (u + iv)\eta$. If possible, suppose that the solution $\psi = \xi + (u + iv)\eta$, $(v \neq 0)$ of the DE (1.1) satisfies the BC $U_\lambda[y] = 0$ which is not a NBC of the DE (1.1). Then we get

$$\lambda_2 + \lambda_3\eta_1 + \lambda_4\eta_1' = 0\,, \tag{5.20}$$

$$\lambda_1 + \lambda_3\xi_1 + \lambda_4\xi_1' = 0\,. \tag{5.21}$$

As $U_\lambda[y]=0$ is not a NBC of the DE (1.1), $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ must be orthogonal to the vectors $(\xi_1,\eta_1,-1,0)$ and $(\xi_1',\eta_1',0,-1)$, see Theorem I. So

$$\lambda_1\xi_1+\lambda_2\eta_1-\lambda_3=0\,, \tag{5.22}$$

$$\lambda_1\xi_1'+\lambda_2\eta_1'-\lambda_4=0\,. \tag{5.23}$$

We find that the determinant of the coefficient matrix of the linear equations (5.20)–(5.21)–(5.22)–(5.23) is non-zero.

Hence $\lambda_1=\lambda_2=\lambda_3=\lambda_4=0$.

SUBCASE (a_6): $\boldsymbol{\xi_2=\xi_2'=\eta_2=\eta_2'=0}$, $\boldsymbol{v=0}$.

In this case we know that dim $B(\psi)=3$ and there are two linearly independent NBCs for DE (1.1), which may be given by (3.5), (3.6).

Hence, if $U_\alpha[y]=0$ is a BC satisfied by ψ, and if it is not a NBC for the DE (1.1), we have

$$\begin{aligned}\lambda_1+\lambda_2u+\lambda_3(\xi_1+u\eta_1)+\lambda_4(\xi_1'+u\eta_1')&=0\\ \lambda_1\xi_1+\lambda_2\eta_1-\lambda_3&=0\\ \lambda_1\xi_1'+\lambda_2\eta_1'-\lambda_4&=0\,.\end{aligned}$$

It can be easily proved that the above system of linear equations in λ_1, λ_2, λ_3, λ_4 has a unique solution (up to a constant multiplier).

Hence the BCs $U_\alpha[y]=0=U_\beta[y]$ can not be compatible unless either they are linearly dependent or at least one of them is a NBC for the DE (1.1), both of which are contrary to our hypothesis. So this case does not arise.

CASE (b) : $\boldsymbol{\psi=\eta}$

We note that $U_\lambda[\eta]=0$ implies

$$\lambda_2+\lambda_3\eta_1+\lambda_4\eta_1'=0\,,\qquad \lambda_3\eta_2+\lambda_4\eta_2'=0\,.$$

Clearly two cases are to be considered :

$$(b_1)\quad \eta_1\eta_2'-\eta_1'\eta_2\neq 0\,,\qquad (b_2)\quad \eta_1\eta_2'-\eta_1'\eta_2=0\,.$$

SUBCASE (b_1) : $\boldsymbol{\eta_1\eta_2'-\eta_1'\eta_2\neq 0}$.

From the above system of equations we can determine $\lambda_2:\lambda_3:\lambda_4$. So, if the BCs $U_\alpha[y]=0=U_\beta[y]$ are compatible, we must have

$$\frac{\alpha_2}{\beta_2}=\frac{\alpha_3}{\beta_3}=\frac{\alpha_4}{\beta_4}\,.$$

Hence,

$$A_{23}=A_{24}=A_{34}=0\,. \tag{5.24}$$

Also,

$$\begin{aligned} A_{13}\eta_2 + A_{14}\eta_2' &= (\alpha_1\beta_3 - \alpha_3\beta_1)\eta_2 + (\alpha_1\beta_4 - \alpha_4\beta_1)\eta_2' \\ &= \alpha_1(\beta_3\eta_2 + \beta_4\eta_2') - \beta_1\,(\alpha_3\eta_2 + \alpha_4\eta_2') \\ &= 0\,, \end{aligned}$$

since

$$U_\alpha[y] = 0 = U_\beta[y]\,. \tag{5.25}$$

These imply that (5.2) is satisfied.

SUBCASE (b_2) : $\boldsymbol{\eta_1\eta_2' - \eta_1'\eta_2 = 0}$.

We first note that $(\eta_1, \eta_2, \eta_1', \eta_2') \neq (0,0,0,0)$, for, otherwise, we have $\eta(b) = \eta'(b) = 0$, which is contrary to our hypothesis, as η is a non-trivial solution.

Two further subcases are to be considered :

$$(i) \quad (\eta_2, \eta_2') \neq (0,0)\,, \qquad\qquad (ii) \quad (\eta_2, \eta_2') = (0,0)\,.$$

SUBCASE $(b_2 - i)$: Here $U_\lambda[\eta] = 0$ will imply $\lambda_2 = 0$. Hence $U_\alpha[y] = 0 = U_\beta[y]$ will imply $\alpha_2 = \beta_2 = 0$, $A_{34} = 0$,

$$A_{13}\eta_2 + A_{14}\eta_2' = 0\,. \tag{5.26}$$

Then it follows that (5.2) holds.

SUBCASE $(b_2 - ii)$: $\boldsymbol{\eta_2 = \eta_2' = 0}$.

Here $U_\alpha[y] = 0 = U_\beta[y]$ imply

$$A_{23} - \eta_1' A_{34} = 0 = A_{24} + \eta_1 A_{34}\,.$$

It then follows that (5.2) is satisfied.

Now we prove the necessity part of Theorem III. We suppose that (5.2) holds, where $U_\alpha[y] = 0 = U_\beta[y]$ are two linearly independent BCs, none of which is a NBC for the DE (1.1). We are to show that the two BCs are compatible, i.e.,

$$L[\psi] = 0\,, \quad U_\alpha[\psi] = 0 \quad \text{and (5.2) must imply} \quad U_\beta[\psi] = 0\,.$$

Once again we are to consider separately the following cases :

$$\text{(I)} \quad \psi \;=\; \xi + (u+iv)\eta, \qquad \text{(II)} \quad \psi = \eta\,.$$

CASE (I) : $\boldsymbol{\psi \;=\; \xi + (u+iv)\eta}$.

Here $U_\alpha[\psi] = 0$ implies

$$\alpha_1 + \alpha_2 u + \alpha_3(\xi_1 + u\eta_1 - v\eta_2) + \alpha_4(\xi_1' + u\eta_1' - v\eta_2') = 0$$

and

$$\alpha_2 v + \alpha_3(\xi_2 + u\eta_2 + v\eta_1) + \alpha_4(\xi_2' + u\eta_2' + v\eta_1') = 0$$

or

$$(\alpha_1 + \alpha_3\xi_1 + \alpha_4\xi_1') + u(\alpha_2 + \alpha_3\eta_1 + \alpha_4\eta_1') - v(\alpha_3\eta_2 + \alpha_4\eta_2') = 0$$

and

$$(\alpha_3\xi_2 + \alpha_4\xi_2') + u(\alpha_3\eta_2 + \alpha_4\eta_2') + v(\alpha_2 + \alpha_3\eta_1 + \alpha_4\eta_1') = 0$$

or

$$A + Bu - Dv = 0 = C + Du + Bv\,, \tag{5.27}$$

where

$$\begin{cases} A = \alpha_1 + \alpha_3\xi_1 + \alpha_4\xi_1'\,, & C = \alpha_3\xi_2 + \alpha_4\xi_2'\,, \\ B = \alpha_2 + \alpha_3\eta_1 + \alpha_4\eta_1'\,, & D = \alpha_3\eta_2 + \alpha_4\eta_2'\,. \end{cases} \tag{5.28}$$

Now, (5.2) can be rewritten in the form

$$DP - CQ + BR - AS = 0\,, \tag{5.29}$$

where

$$\begin{cases} P = \beta_1 + \beta_3\xi_1 + \beta_4\xi_1'\,, & R = \beta_3\xi_2 + \beta_4\xi_2'\,, \\ Q = \beta_2 + \beta_3\eta_1 + \beta_4\eta_1'\,, & S = \beta_3\eta_2 + \beta_4\eta_2'\,. \end{cases} \tag{5.30}$$

As dim $B(\psi) = 2$ or 3, these three equations (5.27) and (5.29) in A, B, C, D will have two / three linearly independent solutions. Hence the rank of the corresponding coefficient matrix is not greater than two.

Hence we have

$$R + Su + Qv = 0 = P + Qu - Sv\,. \tag{5.31}$$

Now $U_\beta[\psi] = [P + Qu - Sv] + i[R + Su + Qv]$. Hence $L[\psi] = 0$, $U_\alpha[\psi] = 0$ imply $U_\beta[\psi] = 0$, by (5.31).

Case (II) : $\boldsymbol{\psi = \eta}$.

We note that $U_\alpha[\eta] = B + iD = 0$ implies $B = 0 = D$. Then (5.2) implies $AS + CQ = 0$.

The three equations $B = 0 = D = AS + CQ$ treated as linear equations in α_1, α_2, α_3, α_4 must yield at least two solutions. This leads to $\eta_2 = \eta_2' = 0$. Then dim $B(\eta) = 3$. So, all second order minors of the coefficient matrix of the above system of linear equations must also vanish, from which we have

$$\xi_2 Q = 0 = \xi_2' Q\,.$$

If $(\xi_2, \xi_2') \neq (0, 0)$, we get $Q = 0$. Hence $U_\beta[\eta] = Q + iS = 0$. If $\xi_2 = \xi_2' = \eta_2 = \eta_2' = 0$, there are two linearly independent NBCs given by (3.5), (3.6).

As $U_\alpha[y] = 0$ is not a NBC, we have

$$\alpha_1\xi_1 + \alpha_2\eta_1 - \alpha_3 = 0 = \alpha_1\xi_1' + \alpha_2\eta_1' - \alpha_4\,.$$

In this case $AS + CQ = 0$ implies $\alpha_1 = 0$. Then $B = 0$ implies $\alpha_2 + \alpha_3\eta_1 + \alpha_4\eta_1' = \alpha_2(1 + \eta_1^2 + \eta_1'^2) = 0$ or $\alpha_2 = 0$, so that we get $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, contrary to our hypothesis. Hence this case cannot arise.

The proof of Theorem III is now complete.

5.3. Compatibility of three boundary conditions

We are to find conditions under which three linearly independent BCs $U_\alpha[y] = 0 = U_\beta[y] = U_\gamma[y]$ are compatible.

Theorem IV. *Three linearly independent BCs $U_\alpha[y] = 0 = U_\beta[y] = U_\gamma[y]$ are compatible with respect to the DE $L[y] = 0$ if and only if*

$$\frac{A}{B} = \frac{P}{Q} = \frac{L}{M}, \tag{5.32}$$

where A, B, P, Q are as defined in (5.28) *and* (5.30)*, and*

$$L = \gamma_1 + \gamma_3\xi_1 + \gamma_4\xi_1', \qquad M = \gamma_2 + \gamma_3\eta_1 + \gamma_4\eta_1'. \tag{5.33}$$

Proof. A solution ψ of the DE $L[y] = 0$ will satisfy three linearly independent BCs, i.e., dim $B(\psi) = 3$, if

either (1) $\xi_2\eta_2' - \xi_2'\eta_2 = 0$, $(\xi_2, \xi_2', \eta_2, \eta_2') \neq (0,0,0,0)$, $v = 0$
$\xi_2 + u\eta_2 = 0 = \xi_2' + u\eta_2'$, $\psi = \xi + u\eta$

or (2) $\xi_2 = \xi_2' = \eta_2 = \eta_2' = 0$, $v = 0$, $\psi = \xi + u\eta$,

or (3) $\eta_1\eta_2' - \eta_1'\eta_2 = 0$, $\psi = \eta$.

Suppose that the three given BCs are compatible. Then,

$$U_\alpha[\psi] = 0 = U_\beta[\psi] = U_\gamma[\psi],$$

which leads to

$$A + Bu = 0 = P + Qu = L + Mu, \tag{5.34}$$

if $\psi = \xi + u\eta$. In other words, (5.32) is satisfied.

If $\psi = \eta$, we have $U_\alpha[\eta] = 0 = U_\beta[\eta] = U_\gamma[\eta]$, and $\eta_1\eta_2' - \eta_1'\eta_2 = 0$. It will then follow that $\alpha_2 = \beta_2 = \gamma_2 = 0$, whence $A = P = L = 0$. Hence (5.32) holds.

Conversely suppose (5.32) holds. We are to show that $L[\psi] = 0 = U_\alpha[\psi]$ should imply $U_\beta[\psi] = 0 = U_\gamma[\psi]$.

If $\psi = \xi + u\eta$, $U_\alpha[\psi] = 0$ implies $A + Bu = 0$.

Using (5.32), we can immediately show that $P + Qu = 0 = L + Mu$, i.e., $U_\beta[\psi] = 0 = U_\gamma[\psi]$.

If $\psi = \eta$, $U_\alpha[\eta] = 0$ implies $B = 0 = D$.

Using (5.32) again, we deduce that $Q = M = 0$; in other words $U_\beta[\eta] = 0 = U_\gamma[\eta]$.

6. Remarks

The present paper takes notice of the following:

(A) *All the solutions of a DE of the form* (1.1) *may satisfy one or more non-trivial boundary condition of the form* $U_\alpha[y] = 0$. *Such a boundary condition has been named a natural boundary condition (NBC).*

(i) A necessary and sufficient condition for the nonexistence of such NBC has been derived.

(ii) The number of NBCs for a given DE has been determined.

(iii) The NBC/s for a given DE have been presented.

(B) *Every real second-order linear homogeneous DE possesses two linearly independent NBCs.*

(C) *Each boundary condition of the form* $U_\alpha[y] = 0$, *which is not a NBC for DE* (1.1), *determines a solution of DE* (1.1) *uniquely up to a constant multiplier.*

(D) *Two boundary conditions* $U_\alpha[y] = 0 = U_\beta[y]$ *are said to be compatible with respect to DE* (1.1) *if both of them determine the same solution of DE* (1.1).

(i) The necessary and sufficient condition for the boundary conditions $U_\alpha[y] = 0 = U_\beta[y]$ to be compatible with respect to DE (1.1) has been obtained.

(ii) The necessary and sufficient condition for the boundary conditions $U_\alpha[y] = 0 = U_\beta[y] = U_\gamma[y]$ to be compatible with respect to DE (1.1) has also been derived.

These observations urge one to rethink about *Sturm-Liouville Problems* regarding the number of boundary conditions to be taken into account.

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