ARCHIVUM MATHEMATICUM (BRNO)
Tomus 40 (2004), 359 – 366

OSCILLATION OF SOLUTIONS OF NON-LINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS OF HIGHER ORDER FOR $p(t) = \pm 1$

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Abstract. In this paper, the oscillation criteria for solutions of the neutral delay differential equation (NDDE)

$$(y(t) - p(t)\, y(t-\tau))^{(n)} + \alpha\, Q(t)\; G\,(y(t-\sigma)) = f(t)$$

has been studied where $p(t) = 1$ or $p(t) \le 0$, $\alpha = \pm 1$, $Q \in C\left([0,\infty), R^{+}\right)$, $f \in C([0,\infty), R)$, $G \in C(R,R)$. This work improves and generalizes some recent results and answer some questions that are raised in [1].

1. Introduction

In this paper sufficient conditions for oscillation and non-oscillation of solutions of NDDE

$$(y\,(t) - p\,(t)\; y\,(t-\tau))^{(n)} + \;\alpha\; Q\,(t) \;\; G\,(y\,(t-\sigma)) = f\,(t) \tag{1}$$

have been obtained where $\alpha = \pm 1$, $\tau > 0$, $\sigma \ge 0$, $p \in C^{(n)}([0,\infty), R)$, $Q \in C([0,\infty), R^{+})$, $f \in C([0,\infty), R)$ and $G \in C(R,R)$. Further the following assumptions are made for its use in the sequel.

(H_1) G is non-decreasing and $xG(x) > 0$ for $x \neq 0$.

(H_2) $\liminf\limits_{|u|\to\infty} \frac{G(u)}{u} > \beta > 0$.

(H_3) For $u > 0$, $\nu > 0$, $G(u) + G(\nu) > \delta G(u+\nu)$ and $G(u)G(\nu) \ge G(u\nu)$.

(H_4) $G(-u) = -G(u)$.

(H_5) There exists $F \in C^{(n)}([0,\infty), R)$ such that $F^{(n)}(t) = f(t)$ and $\lim\limits_{t\to\infty} F(t) = 0$.

(H_6) $\sum\limits_{i=0}^{\infty} \int\limits_{t_0+i\tau}^{\infty} (s - t_0 - i\tau)^{n-1}\; Q(s)\, ds < \infty$.

(H_7) $f(t) \le 0$ and $\sum\limits_{i=0}^{\infty} \int\limits_{t_0+i\tau}^{\infty} (s - t_0 - i\tau)^{n-1} f(s)\, ds > -\infty$.

2000 *Mathematics Subject Classification*: 34C10, 34 C15, 34K40.
Key words and phrases: oscillation, non-oscillation, neutral equations, asymptotic-behaviour.
Received November 19, 2002.

(H_8) $\int\limits_{\tau}^{\infty} t^{n-2} Q^*(t)\,dt = \infty$, where $Q^*(t) = \min\{Q(t), Q(t-\tau)\}$ and $n \geq 2$.

Remark 1. The prototype of G satisfying (H_1)–(H_4) is $G(u) = (\beta + |u|^{\mu})|u|^{\lambda}\operatorname{sgn} u$ where $\lambda > 0$, $\mu > 0$, $\lambda + \mu \geq 1$, $\beta \geq 1$. For verification we may take the help of the well known inequality (see [2, p. 292])

$$u^p + v^p \geq \begin{cases} (u+v)^p\,, & 0 \leq p < 1 \\ 2^{1-p}(u+v)^p\,, & p \geq 1 \end{cases}$$

During the last two decades many authors (see [1-10]) have taken active interest to study the oscillation and non-oscillation of solutions of NDDEs and many open problems have appeared in the literature (see [1]). Some of these have been proved and some have been disproved with appropriate counter examples (see [10]). In [1, p. 287], the authors have proposed the following open problems.

(10.10.2) Extend the results of section 10.4 to equations where the coefficient function $p(t)$ lies in different ranges.

(10.10.3) Obtain sufficient condition for the existence of a positive solution of the NDDE

$$\text{(E)} \qquad \big(y(t) - p(t)\,y(t-\tau)\big)^{(n)} + Q(t)\,y(t-\sigma) = 0\,.$$

This paper provides answer to both the problems (10.10.2) and (10.10.3) for the equation (1) with $\alpha = 1$, which is more general than (E). In [6], the authors have given an example to justify their assumption

$$\text{(2)} \qquad \int_{\tau}^{\infty} Q^*(t)\,dt = \infty$$

which is stronger than

$$\text{(3)} \qquad \int_{0}^{\infty} Q(t)\,dt = \infty$$

in order to find sufficient condition for oscillation of solutions of Eq. (1) with $\alpha = 1$ and $p(t) \equiv -1$. It may be noted that in [7, 9], the author has assumed $Q(t)$ is decreasing in addition to (3) and both these imply (2). The condition (H_8) assumed in this paper is weaker than (2). Thus this paper improves some results of [7, 9].

It seems that oscillation of solutions of non linear NDDEs is not studied much. In particular, the critical cases that is for the range $p(t) = \pm 1$ are still less studied. Again with $p(t) < -1$, very few results on the oscillatory behaviour of solutions of Eq. (1) are available in the literature. The present work is an attempt in this direction to get some results for the non linear NDDE(1) in the range $p(t) = 1$ or $p(t) \leq 0$ and answer the above mentioned open problems.

By a solution of Eq. (1) we mean a real valued continuous function $y \in C^{(n)}\big((T_y - \rho, \infty), R\big)$ for some $T_y \geq 0$ where $\rho = \max\{\tau, \sigma\}$ such that $y(t) - p(t)\ y(t-\tau)$ is n-times continuously differentiable and Eq. (1) is satisfied for $t \geq T_y$. A solution of Eq. (1) is said to be oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory.

In the sequel for convenience, when a functional inequality is written with out specifying its domain of validity, it is assumed that it holds for all sufficiently large t.

2. Main Results

Lemma 2.1 ([5], p. 376)**.** *If f and g be two positive functions in $[a,t]$ and $\lim_{t\to\infty} \frac{f(t)}{g(t)} = \ell \in R$, where ℓ is non zero then $\int_a^\infty f(t)\,dt$ and $\int_a^\infty g(t)\,dt$ converge or diverge together. Also if $f/g \to 0$ and $\int_a^\infty g(t)\,dt$ converges then $\int_a^\infty f(t)\,dt$ converges and $f/g \to \infty$ and $\int_a^\infty g(t)\,dt$ diverges then $\int_a^\infty f(t)\,dt$ diverges.*

The proof is straight forward and can be found in any higher calculus book, containing Improper Integrals.

Lemma 2.2 ([3], p. 193)**.** *Let $y \in C^{(n)}([0,\infty), R)$ be of constant sign. Let $y^{(n)}(t)$ be also of constant sign and $\not\equiv 0$ in any interval $[T,\infty)$, $T \ge 0$ and $y^{(n)}(t)\; y(t) \le 0$. Then there exists a number $t_0 \ge 0$ such that the functions $y^{(j)}(t)$, $j = 1, 2, \ldots, n-1$ are of constant sign on $[t_0,\infty)$ and there exists a number $k \in \{1, 3, 5, \ldots, n-1\}$ when n is even or $k \in \{0, 2, 4, \ldots, n-1\}$ when n is odd such that*

$$y(t)y^{(j)}(t) > 0 \quad \textit{for} \quad j = 0, 1, 2, \ldots, k, t \ge t_0$$
$$(-1)^{n+j-1}y(t)y^{(j)}(t) > 0 \quad \textit{for} \quad j = k+1, k+2, \ldots, n-1, \quad t \ge t_0$$

Theorem 2.3. *Suppose that $n \ge 2$ and $-p \le p(t) \le 0$ and* (H$_1$)–(H$_5$) *and* (H$_8$) *hold. Then every solution of Eq.* (1) *with $\alpha = 1$ oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t) > 0$ be a non-oscillatory solution of Eq. (1) for $t \ge t_0 > 0$. Then setting

$$z(t) = y(t) - p(t)y(t-\tau) \tag{4}$$

and

$$w(t) = z(t) - F(t) \tag{5}$$

we obtain from Eq. (1)

$$w^{(n)}(t) = -q(t)G(y(t-\sigma)) \le 0\,, \tag{6}$$

for $t \ge t_0 + \rho$. Hence $w(t)$, $w'(t)$, $w''(t), \ldots, w^{(n-1)}(t)$ are monotonic and $\lim_{t\to\infty} w(t) = \ell$, where $-\infty \le \ell \le \infty$. Hence $\lim_{t\to\infty} z(t) = \ell$ by (H$_5$). If $-\infty \le \ell < 0$, then $z(t) < 0$ for large t, a contradiction. Hence $0 \le \ell \le \infty$. If $\ell = 0$, then $y(t) \le z(t)$ implies $\lim_{t\to\infty} y(t) = 0$. If $0 < \ell \le \infty$, then $w(t) > 0$ for large t. From Lemma 2.2 it follows that there exists an integer $k, 0 \le k \le n-1$ and $t_1 > t_0 + \rho$ such that $n-k$ is odd, $w^{(j)}(t) > 0$ for $j = 0, 1, \ldots, k$ and $(-1)^{n+j-1}w^{(j)}(t) > 0$ for $j = k+1, k+2, \ldots, n-1$. Hence $\lim_{t\to\infty} w^{(k)}(t)$ exists and $\lim_{t\to\infty} w^{(i)}(t) = 0$ for $i = k+1, k+2, \ldots, n-1$ and $t \ge t_3 > t_2$. It may be noted that $0 < \ell < \infty$ implies

$k = 0$ but $\ell = \infty$ implies $k > 0$ such that $n - k$ is odd. Integrating (6), $n - k$-times from t to ∞ we obtain, for some constant β

$$w^k(t) = \beta + \frac{1}{(n-k-1)!} \int_t^\infty (s-t)^{n-k-1} Q(s)\, G\big(y(s-\sigma)\big)\, ds\,. \tag{7}$$

Hence from Lemma 2.1 and (7) we obtain

$$\int_\rho^\infty t^{n-k-1} Q(t)\, G\big(y(t-\sigma)\big)\, dt < \infty\,. \tag{8}$$

Since $Q(t) \ge Q^*(t+\tau)$, it follows that

$$\int_\rho^\infty t^{n-k-1} Q^*(t+\tau)\, G\big(y(t-\sigma)\big)\, dt < \infty\,.$$

Consequently $G(p) \int\limits_{\rho-\tau}^{\infty} (t-\tau)^{n-k-1} Q^*(t)\, G\big(y(t-\tau-\sigma)\big)\, dt < \infty$, which implies (by Lemma 2.1, (H$_1$) and the fact that $p(t) \ge -p$)

$$\int_{T_1}^\infty t^{n-k-1} Q^*(t)\, G\big(-p(t-\sigma)\big)\, G\big(y(t-\tau-\sigma)\big)\, dt < \infty\,,$$

where $T_1 \ge \rho + \tau$. This with the use of (H$_3$) yields

$$\int_{T_1}^\infty t^{n-k-1} Q^*(t)\, G\big(-p(t-\sigma)\, y(t-\tau-\sigma)\big)\, dt < \infty\,. \tag{9}$$

From (8) and the fact $Q(t) \ge Q^*(t)$, we obtain

$$\int_{T_1}^\infty t^{n-k-1} Q^*(t)\, G\big(y(t-\sigma)\big)\, dt < \infty\,. \tag{10}$$

Further using (H$_3$), (9) and (10) one may get

$$\int_{T_1}^\infty t^{n-k-1} Q^*(t)\, G\big(z(t-\sigma)\big)\, dt < \infty\,. \tag{11}$$

If $k = 0$, then (H$_8$) and (11) yield $\liminf\limits_{t\to\infty} t\, G\big(z(t-\sigma)\big) = 0$, which with application of (H$_2$) yields $\lim\limits_{t\to\infty} z(t) = 0$, a contradiction. If $k > 0$ then in this case $\lim\limits_{t\to\infty} w(t) = \infty$. Hence there exists $M_0 > 0$ such that $w(t) > M_0 t^{k-1}$ and by (H$_5$) we can find $0 < M_1 < M_0$ such that

$$z(t) > M_1 t^{k-1} \quad \text{for large} \quad t\,. \tag{12}$$

Then further use of (12), (H$_2$) and Lemma 2.1 gives

$$\begin{aligned}\int_{T_1}^\infty Q^*(t) t^{n-k-1} G\big(z(t-\sigma)\big)\, dt &\ge \int_{T_1}^\infty Q^*(t) t^{n-k-1} G\big(M_1 (t-\sigma)^{k-1}\big)\, dt \\ &\ge \beta M_1 \int_{T_1}^\infty Q^*(t)(t-\sigma)^{n-2} dt = \infty\end{aligned}$$

by (H$_8$), a contradiction to (11). When $y(t) < 0$ for $t \ge t_0 > 0$, we use (H$_4$) and proceed as above to get the desired result.

Remark 2. Theorem 2.3 answers the open problem 10.10.2 of [1] since the range of $p(t)$ in this theorem is different from the range of $p(t)$ in the results of section 10.4 of [1]. The ranges used in that section are $1 \le p(t) \le p$, $0 \le p(t) \le p < 1$ and $-1 < -p < p(t) \le 0$.

Remark 3. Theorem 2.3 improves and generalizes Theorem 2.4 of [9], Theorem 2.5 of [7] and Theorem 2.1 of [6]. In [7, 9], $Q(t)$ is monotonic decreasing and satisfies (3), which implies (2). But in Theorem 2.3 (H_8) is assumed, which is weaker than (2) for $n \ge 2$. Theorem 2.3 is true for both n odd and even. It holds when G is linear or superlinear.

Corollary 2.4. *If all the conditions of Theorem 2.3 are satisfied then every unbounded solution of Eq.* (1) *with* $\alpha = 1$ *oscillates.*

Theorem 2.5. *Let* $p(t) \equiv 1$ *and* n *be odd. Suppose that* (H_1), (H_6) *and* (H_7) *hold. Then Eq.* (1) *with* $\alpha = 1$ *has a bounded positive solution.*

Proof. Since (H_6) and (H_7) hold therefore, we can find $T > t_0$ such that

$$\begin{aligned} &\sum_{i=0}^{\infty} \int_{T+i\tau}^{\infty} (s - T - i\tau)^{n-1} Q(s)\, ds < \frac{(n-1)!}{2G(1)} \quad \text{and} \\ &\left| \sum_{i=0}^{\infty} \int_{T+i\tau}^{\infty} (s - T - i\tau)^{n-1} f(s)\, ds \right| < \frac{(n-1)!}{2} \end{aligned} \tag{13}$$

Define

$$L(t) = \begin{cases} \dfrac{G(1)}{(n-1)!} \displaystyle\int_t^{\infty} (s-t)^{n-1} Q(s)\, ds - \dfrac{1}{(n-1)!} \displaystyle\int_t^{\infty} (s-t)^{n-1} f(s)\, ds & \text{for} \quad t \ge T \\ (t - T + \tau) \dfrac{L(T)}{\tau} \qquad \text{for} \quad T - \tau \le t \le T \\ 0 \qquad \text{for} \quad t \le T - \tau \end{cases}$$

Clearly $L(t)$ is continuous and nonnegative in R.

Set

$$u(t) = \sum_{i=0}^{\infty} L(t - i\tau) \quad \text{for} \quad t \ge T\,.$$

Then $u(t)$ is continuous in $[T, \infty)$, $0 < u(t) \le 1$ and $u(t) - u(t - \tau) = L(t)$ for $t \ge T + \tau$.

Next we define a sequence $\{v_k(t)\}_{k=0}^{\infty}$ on $[t_0, \infty)$ as follows:

$$v_0(t) = 1\,, \quad \text{for} \quad t \ge t_0$$

and

$$v_k(t) = \begin{cases} \dfrac{1}{u(t)}\Big[u(t-\tau)v_{k-1}(t-\tau) + \dfrac{1}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}Q(s)\,ds \\ \qquad \times G\big(u(s-\sigma)v_k(s-\sigma)\big)\,ds - \dfrac{1}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}f(s)\,ds\Big]\,, \\ \qquad \text{for} \quad t \ge T+\rho \\ \dfrac{t+h}{T+\rho+h}v_k(T+\rho) + \Big(1 - \dfrac{t+h}{T+\rho+h}\Big)\,, \quad t_0 \le t \le T+\rho \end{cases}$$

where $\rho = \max\{\tau, \sigma\}$, $k = 1, 2, \dots$ and h is a constant such that $t_0 + h > 0$. For $t \ge T + \rho$,

$$\begin{aligned} v_1(t) &= \frac{1}{u(t)}\Big[u(t-\tau)v_0(t-\tau) \\ &\quad + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)\,G\big(u(s-\sigma)v_0(s-\sigma)\big)\,ds \\ &\quad - \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}f(s)\,ds\Big] \\ &\le \frac{1}{u(t)}\Big[u(t-\tau) + \frac{1}{(n-1)!}\,G(1)\int_t^\infty (s-t)^{n-1}Q(s)\,ds \\ &\quad - \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}f(s)\,ds\Big] \\ &\le \frac{1}{u(t)}\big[u(t-\tau) + L(t)\big] = 1 = v_0(t)\,. \end{aligned}$$

For $t_0 \le t \le T + \rho$, we have

$$\begin{aligned} v_1(t) &= \frac{t+h}{T+\rho+h}\,v_1(T+\rho) + \Big(1 - \frac{t+h}{T+\rho+h}\Big) \\ &\le \frac{t+h}{T+\rho+h}\,v_0(T+\rho) + \Big(1 - \frac{t+h}{T+\rho+h}\Big) = 1 = v_0(t)\,. \end{aligned}$$

Hence $0 \le v_1(t) \le v_0(t)$ for $t \ge t_0$. By using a simple induction we can prove $0 \le v_k(t) \le v_{k-1}(t) \le 1$ for $t \ge t_0$ for $k = 1, 2 \dots$. Thus $\{v_k(t)\}$ has a pointwise limit function $v(t)$ which satisfies $\lim_{k\to\infty} v_k(t) = v(t) \le 1$ for $t \ge t_0$. By monotone convergence theorem we have

$$v(t) = \begin{cases} \dfrac{1}{u(t)}\Big[u(t-\tau)\,v(t-\tau) \\ \qquad + \dfrac{1}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}Q(s)\,G\big(u(s-\sigma)v(s-\sigma)\big)\,ds \\ \qquad - \dfrac{1}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}f(s)\,ds\Big]\,, \quad \text{for} \quad t \ge T+\rho \\ \dfrac{t+h}{T+\rho+h}\,v(T+\rho) + \Big(1 + \dfrac{t+h}{T+\rho+h}\Big) \quad \text{for} \quad t_0 \le t \le T+\rho \end{cases}$$

since for $t_0 \le t < T+\rho$, $v(t) = \frac{t+h}{T+\rho+h}\, v(T+\rho) + \Big(1 - \frac{t+h}{T+\rho+h}\Big) > 0$. It can be easily seen $v(t) > 0$ for $t \ge t_0$. Set $y(t) = u(t)\, v(t) > 0$ and $y(t)$ is the required positive bounded solution of (1) with $\alpha = 1$ on $[T+\rho, \infty)$. $\square$

Remark 4. The above Theorem extends Lemma 2.1 in [4].

Remark 5. By Lemma 2.2 of [4] and Lemma 2.1 of this paper it is clear that (H_6) is equivalent to

$$(H_9) \qquad \int_0^\infty t^n Q(t)\, dt < \infty\,.$$

Corollary 2.6. *Suppose that* $p(t) \equiv 1$, *n is odd and* (H_1), (H_7) *hold. Then* (H_9) *is the sufficient condition for Eq.* (1) *with* $\alpha = 1$ *to have a positive bounded solution.*

Corollary 2.7. *Suppose that n is even,* $p(t) \equiv 1$ *and* (H_1), (H_7) *hold. Then* (H_9) *is the sufficient condition for Eq.* (1) *with* $\alpha = -1$ *to have a positive bounded solution.*

The proof is similar to that of Corollary 2.6, hence omitted.

Remark 6. Corollary 2.6 answers the open problem 10.10.3 of [1].

3. Final comments

In this concluding section we give some remarks and enlist some unanswered questions of this paper for further research. In Theorem 3.1 of [8], it is proved that (H_9) is necessary for Eq. (1) (with $p(t) \equiv 1$, $\alpha = 1$ and n odd) to have a bounded positive solution. Hence in view of Corollary 2.6 of this paper (H_9) is both necessary and sufficient condition for Eq. (1) (with $p(t) \equiv 1$, $\alpha = 1$ and n odd) to have a bounded positive solution. Thus taking $f(t) \equiv 0$ we can conclude

Corollary 3.1. *Suppose that n is odd and* (H_1) *hold. Then every bounded solution of*

$$\big(y(t) - y(t-\tau)\big)^{(n)} + Q(t) G\big(y(t-\sigma)\big) = 0$$

oscillates if and only if (H_{10}) *holds where*

$$(H_{10}) \qquad \int_0^\infty t^n Q(t)\, dt = \infty\,.$$

Similarly in view of Corollary 2.7 we can have the following result.

Corollary 3.2. *Suppose that n is even and* (H_1) *hold. Then every bounded solution of Eq.* (1) (*with* $\alpha = -1$, $p(t) \equiv 1$ *and* $f \equiv 0$) *oscillates if and only if* (H_{10}) *holds.*

Further in Theorem 2.5, one may be tempted to drop $f(t) \le 0$ and still get the same result. Also Theorem 2.3 provides a sufficient condition for every solution of Eq. (1) (with $p(t) \equiv -1$ and $\alpha = 1$) to be oscillatory or tending to zero. It seems that there is no result so far in literature which shows some condition like (3) is necessary for every solution of Eq. (1) (with $p(t) \equiv -1$ and $\alpha = 1$) to be oscillatory or tending to zero. It looks very difficult to get the desired result if we assume n to be even in Corollary 2.6. and odd in Corollary 2.7. Further one may attempt Theorem 2.3 for $\alpha = -1$ and prove on similar lines and under same assumptions that every solution $y(t)$ of (1) oscillates or tends to zero as $t \to \infty$ or $\limsup_{t\to\infty} |y(t)| = \infty$. But this result needs improvement.

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